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Nonstandard Methods in Stochastics and Applications to Mathematical Finance

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Summary

The first chapter of this thesis is concerned with measure and probability theory from a Nonstandard Analysis point of view. First we give an overview of general facts in Nonstandard Analysis, i.e. we give a summary about the construction of nonstandard sets and about the main properties and notations in nonstandard analysis. The second part of the first chapter is about measure spaces in the "nonstandard universe", so called internal measure spaces. In this part we show that internal measure spaces are in general not measure spaces but that for this internal measure spaces, there is always a measure space in the usual sense, the so called Loeb space, which is an canonical extension of the nonstandard measure space, and we show how to construct this Loeb space. Then we apply these results to the Brownian Motion and show that the Brownian Motion can be obtained as an infinitesimal random walk i.e. we prove that the standard part of an infinitesimal random walk on an internal probability space is a Brownian Motion on the appropriate Loeb space. We also give some more general construction for a Brownian Motion by using Nonstandard Analysis. In the next part we show for a big class of functions and processes on probability spaces in the nonstandard universe that they in a certain sense correspond to standard entities. We use the standard part map to show the connection between functions on internal probability spaces, and functions on the appropriate Loeb spaces i.e. the standard part of a function on an internal probability space is a function on the corresponding Loeb space, which has similar properties as the original function. Hence the function on the internal probability space is a lifting of a certain standard function on the Loeb space. We do the same for stochastic processes and stopping times on a hyperfinite time interval and show that the standard parts are stochastic processes and stopping times on a continuous time interval.

In the second chapter we give some applications of nonstandard stochastics in mathematical finance. First we look at European call options in the Cox-Ross-Rubinstein model and in the Black-Scholes model. We show important properties of these models and we determine the fair price of a European call option in both models. Then we introduce the hyperfinite Cox-Ross-Rubinstein model, where we use random walks with infinitesimal time steps. The hyperfinite Cox-Ross-Rubinstein model extends the ordinary Cox-Ross-Rubinstein model and inherits many of its properties. Then we show that the Black-Scholes model is precisely the standard part of the hyperfinite Cox-Ross-Rubinstein model and we show that the standard part of the fair price of a European call option in the hyperfinite Cox-Ross-Rubinstein model is equal to the fair price in the Black-Scholes model. Finally we consider American put options in the hyperfinite Cox-Ross-Rubinstein model and we show that the unique optimal stopping time for an American put option in the Black-Scholes model is given by the standard part of an optimal stopping time in the hyperfinite Cox-Ross-Rubinstein model.

Zusammenfassung

Im ersten Kapitel dieser Diplomarbeit werden Methoden aus der Nichtstandard Analysis auf Maß- und Wahrscheinlichkeitstheorie angewendet. Der erste Abschnitt dieses Kapitels ist eine Zusammenfassung über die Konstruktion von Nichtstandard Mengen und über deren grundlegende Eigenschaften. Im zweiten Teil des ersten Kapitels werden Maßräume im Nichtstandard Universum beschrieben, sogenannte interne Maßräume. Wir zeigen, dass diese internen Maßräume im Allgemeinen keinen Maßräume im üblichen Sinne sind, aber dass für jeden internen Maßraum ein Maßraum, der sogenannte Loeb Raum, existiert, der eine kanonische Erweiterung des internen Maßraumes ist und wir werden eine genaue Konstruktion des Loeb Raumes angeben. Im Folgenden wenden wir diese Ergebnisse auf die Brownsche Bewegung an und zeigen den Zusammenhang der Brownschen Bewegung mit infinitesimalen Zufallspfaden d.h. diskrete Zufallspfade mit unendlich kleinen Zeitschritten. Im nächsten Abschnitt zeigen wir für eine große Klasse von Funktionen und stochastischen Prozessen auf internen Wahrscheinlichkeitsräumen, dass sie mit Standard Funktionen bzw. stochastischen Prozessen auf den zugehörigen Loeb Räumen in einem bestimmten Sinne übereinstimmen. Wir verwenden die Standardteilabbildung um den Zusammenhang zwischen Funktionen auf internen Wahrscheinlichkeitsräumen und Funktionen auf Loeb Räumen zu zeigen. Das heißt für Funktionen auf internen Wahrscheinlichkeitsräumen existiert eine Funktion auf dem Loeb Raum mit analogen Eigenschaften. Wir zeigen diesen Zusammenhang ebenfalls für Stochastische Prozesse und für Stoppzeiten.

Im zweiten Kapitel werden Anwendungen der im ersten Teil entwickelten Theorie auf die Finanzmathematik beschrieben. Zunächst beschreiben wir Europäische Optionen im Cox-Ross-Rubinstein Modell und im Black-Scholes Modell. Wir zeigen wichtige Eigenschaften und bestimmen den fairen Optionspreis innerhalb beider Modelle. Dann führen wir das hyperendliche Cox-Ross-Rubinstein Modell ein, welches eine Erweiterung mit infinitesimalen Zeitschritten ist und ähnliche Eigenschaften wie das gewöhnliche Cox-Ross-Rubinstein Modell besitzt. Im Folgenden zeigen wir, dass das Black-Scholes Modell durch den Standardteil des hyperendlichen Cox-Ross-Rubinstein Modells gegeben ist und zeigen, dass die Optionspreise in diesen beiden Modellen übereinstimmen. Als letztes betrachten wir Amerikanische Put-Optionen im hyperendlichen Cox-Ross-Rubinstein Modell und zeigen, dass die optimale Stoppzeit einer Amerikanischen Put-Option im Black-Scholes Modell durch den Standardteil der optimalen Stoppzeit im hyperendlichen Cox-Ross-Rubinstein Modell gegeben ist.

Chapter 1

Non-Standard Stochastics

1.1 Some General Remarks on Non-Standard Analysis

Nonstandard analysis was introduced by Abraham Robinson in 1960. In [1] he developed a rigorous foundation of the theory of infinitesimals. He introduced for example, an extension of the real numbers which contains infinitely large and infinitely small numbers. The following is a summary about important facts from nonstandard analysis. For details see [1], [2], [3].

We start with a set S, which contains all real numbers and all the standard entities we need. The elements of S are urelements or atoms, this means the elements of S are not sets. Using this set S we construct the superstructure $V(S)$ over S i.e. the superstructure over S is the set $V(S) = \bigcup_{k=1}^{\infty} S_k$ where $S_0 = S$ and $S_{k+1} = S_k \cup \mathfrak{P}(S_k)$ for $k \in \mathbb{N}$. We consider a set W and a function $^* : V(S) \to V(W)$. If ϕ is a formula in $V(S)$, then $^*\phi$ denotes the formula in $V(W)$ where every element $s \in V(S)$ which appears in the formula ϕ is replaced by $^*s \in V(W)$, more formally $^*\phi$ is defined by induction over the complexity of ϕ. We call the set $\mathcal{S} := \bigcup_{A \in V(S) \setminus S} {}^*A$ the nonstandard universe.

For $A \in V(W)$ we say that A is *standard* if $A = {}^*B$ for some $B \in V(S)$ and A is called *internal* if $A \in {}^*B$ for some $B \in V(S)$, otherwise A is called *external*. In particular, every standard set is internal and elements of internal sets are internal. For example $^*\mathbb{N}$ and $^*\mathbb{R}$ are internal because $\mathbb{N}, \mathbb{R} \in V(S)$ and all elements of $^*\mathbb{N}$ and $^*\mathbb{R}$ are internal.

One can show that there is a set W and a function $^* : V(S) \to V(W)$ such that $^*S = W$ and $^*s = s$ for all $s \in S$ and that the following important properties hold:

i. **Extension Principle:** The set $W = {}^*S$ is an extension of S, i.e. there are elements $r \in {}^*S \setminus S$.

ii. **Transfer Principle:** For all elementary statements ϕ in $V(S)$, ϕ holds, if and only if $^*\phi$ holds in $V(W)$.

iii. **Saturation Principle:** $V(W)$ is κ-saturated, where κ is the cardinality from the superstructure $V(S)$. This means that for any set T with cardinality smaller than κ and for any family $(A_t)_{t\in T}$ of internal sets with the finite intersection property, we have $\bigcap_{t\in T} A_t \neq \emptyset$.

If for example $S = \mathbb{R}$ we see by the Extension Principle that there are $r \in {}^*\mathbb{R}\setminus\mathbb{R}$ i.e. there are more real numbers in the nonstandard universe. The Transfer Principle gives a relation between the properties of $\mathbb{R}$ and the properties of the extension ${}^*\mathbb{R}$ and implies that ${}^*\mathbb{R}$ is an ordered field. So, by the Extension Principle, ${}^*\mathbb{R}$ contains infinitely small and infinitely large numbers. We call the elements of ${}^*\mathbb{R}$ the *hyperreal numbers*.

A consequence of the Saturation principle is that every function $f : X \to Y$ where $X, Y \in V(S)$, has an internal extension, i.e. for all internal sets $A \supseteq X$ and $B \supseteq Y$ there is an internal function $F : A \to B$ such that for each $x \in X$ we have $F(x) = f(x)$. For example, for every function $f : \mathbb{R} \to \mathbb{R}$ there is an internal function $F : {}^*\mathbb{R} \to {}^*\mathbb{R}$ such that $F(x) = f(x)$ for all $x \in \mathbb{R}$ and for every sequence $(A_n)_{n\in\mathbb{N}}$ there is a sequence $(A_n)_{n\in{}^*\mathbb{N}}$ which extends the sequence $(A_n)_{n\in\mathbb{N}}$. Now we give some notations and properties about the nonstandard universe, which we use in the following sections. See for instance [2].

Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. The *monad* of x is defined by $\mathrm{m}(x) := \bigcap_{A:a\in A, A\in\mathcal{T}} {}^*A$. For $y \in {}^*X$ we say y is *infintely close* to x, if $y \in \mathrm{m}(x)$ and we write $x \approx y$. We call $y \in {}^*X$ *nearstandard* if there is some $x \in X$ with $y \approx x$ and $\mathrm{ns}(A)$ denotes the set of all nearstandard points in A, where A is any subset of ${}^*X$. For $A \subseteq {}^*X$ the *standard part* of A is defined by $\mathrm{st}(A) = \{x \in X : y \approx x \quad \text{for some} \quad y \in A\}$.

We say that an element $r \in {}^*\mathbb{R}$ is *infinitesimal*, if $|r| < s$ for all $s \in \mathbb{R}^+$, finite, if there is $s \in \mathbb{R}^+$ such that $|r| < s$ and *infinite* if $|r| > s$ for all $s \in \mathbb{R}^+$. If $r \in {}^*\mathbb{R}$ is finite, then there is a unique $s \in \mathbb{R}$ such that $r \approx s$ or equivalently $\mathrm{st}(r) = s$ i.e. the standard part of a finite hyperreal numbers is uniquely determined. An internal set A is called *hyperfinite*, if there is $N \in {}^*\mathbb{N}$ and an internal bijection $f : A \to \{0, \ldots, N\}$.

A function $F : {}^*X \supseteq A \to {}^*Y$ where X and Y are topological spaces, is called *S-continuous* on A if for all $x, y \in A$ we have $x \approx y \Rightarrow F(X) \approx F(Y)$. This is equivalent to the following definition. $F : {}^*X \supseteq A \to {}^*Y$ is S-continuous on A if for every $\epsilon \in {}^*\mathbb{R}^+$ there is some $\delta \in {}^*\mathbb{R}^+$ such that $|F(x) - F(y)| \leq \epsilon$ whenever $|x - y| \leq \delta$.

A consequence of the Transfer principle is that the set $\mathbb{N}$ is external because in the standard universe we have that any set of natural numbers which is bounded above has a maximum. Therefore, by the Transfer principle, we have that every internal subset of ${}^*\mathbb{N}$ which is bounded above has a maximum. The set $\mathbb{N}$ is bounded, because every infinite number is an upper bound of $\mathbb{N}$ but $\mathbb{N}$ has no maximum, therefore $\mathbb{N}$ is external. It can be shown in a similar way that the sets ${}^*\mathbb{N} \setminus \mathbb{N}$, $\mathbb{R}$, ${}^*\mathbb{R} \setminus \mathbb{R}$ and $\mathrm{m}(r)$ for $r \in \mathbb{R}$ are external. Also the following important properties for internal sets in ${}^*\mathbb{N}$ and ${}^*\mathbb{R}$ are direct consequences of the Transfer principle. See [4] for a proof.

i. If $A \subseteq {}^*\mathbb{N}$ is a nonempty internal set, then A has a least element.

ii. If $A \subseteq {}^*\mathbb{R}$ is a nonempty internal set with an upper bound, then A has a least upper bound.

iii. If A is an internal set, which contains $\mathbb{N}$ then, A contains some infinite natural number.

iv. If A is an internal set, which contains ${}^*\mathbb{N} \setminus \mathbb{N}$, then A contains some finite natural number.

v. If A is an internal set, which contains every positive infinitesimal real number, then A contains some standard real number.

vi. If A is an internal set, which contains $\mathbb{R}^*$ then A contains some positive infinitesimal real number.

The properties ii. and iii. are called *overflow*- respectively *underflow principle*.

By using nonstandard analysis we can represent integrals as standard parts of hyperfinite sums. The proofs of the following important theorems are found for example in [2]. Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Then for $N \in {}^*\mathbb{N} \setminus \mathbb{N}$

$$\int_0^1 f(t)\mathrm{d}t \approx \frac{1}{N}\sum_{i=1}^{N} {}^*f\left(\frac{i}{N}\right).$$

Or more general, if $(\Omega, \mathcal{A}, \mathcal{P})$ is a measure space and $f : \Omega \to \mathbb{R}$ is a $\mathcal{P}$-integrable function, then there is a hyperfinite set $A \subseteq {}^*\Omega$ such that

$$\int f \mathrm{d}\mathcal{P} \approx \frac{1}{|A|}\sum_{\omega \in A} {}^*f(\omega).$$

1.2 Loeb Measures

In this section we explain the concept of Loeb Measures, which was introduced by Peter Loeb in 1975, see [5]. We show how to convert an internal measure to a σ-additive measure in the usual sense, for more details see [6].

To make this more precise, let Ω be an internal set. The power set $\mathfrak{P}(\Omega)$ is the set of all internal subsets of Ω. An *internal Algebra* on Ω is an internal set $\mathcal{A} \subseteq {}^*\mathfrak{P}(\Omega)$ which contains $\emptyset$ and Ω, and which is closed under finite and hyperfinite unions and under complements.

Suppose that $\mathcal{A} \subseteq {}^*\mathfrak{P}(\Omega)$ is an internal Algebra and that $\mu : \mathcal{A} \to {}^*[0,\infty)$ is an internal function with $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \in \mathcal{A}$. Clearly, for $N \in {}^*\mathbb{N}\setminus\mathbb{N}$ and $A_1, \ldots, A_N \in \mathcal{A}$ disjoint, $\bigcup_{i=1}^N A_i \in \mathcal{A}$ and $\mu\left(\bigcup_{i=1}^N A_i\right) = \sum_{i=1}^N \mu(A_i)$. Hence μ is an internal finitely additive measure on $\mathcal{A}$. We call $(\Omega, \mathcal{A}, \mu)$ an *internal measure space* and if $\mu(\Omega) = 1$ we say that $(\Omega, \mathcal{A}, \mu)$ is an *internal probability space*.

Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space such that $\mu(\Omega)$ is finite. Then $\mu(A)$ is finite for each A in $\mathcal{A}$ and we may define the function $\mathrm{st}(\mu) : \mathcal{A} \to [0,\infty)$

by $\mathrm{st}(\mu)(A) = \mathrm{st}(\mu(A))$. Clearly, for disjoint $A, B \in \mathcal{A}$ we have $\mathrm{st}(\mu(\mathcal{A} \cup B)) = \mathrm{st}(\mu(A)) + \mathrm{st}(\mu(B))$. This means that $\mathrm{st}(\mu)$ is finitely additive. Therefore $(\Omega, \mathcal{A}, \mathrm{st}(\mu))$ is a finitely additive measure space in the usual sense. If $\mathcal{A}$ is finite, then for disjoint $(A_n)_{n\in\mathbb{N}}$ exists $m \in \mathbb{N}$ such that $A_i = \emptyset$ for $i > m$. Therefore $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{m} A_n \in \mathcal{A}$ and $\mathcal{A}$ is a σ-algebra. But in general $(\Omega, \mathcal{A}, \mathrm{st}(\mu))$ is not a measure space because $\mathcal{A}$ is not a σ-algebra if $\mathcal{A}$ is not finite. Because if $\mathcal{A}$ is not finite, there exists internal and disjoint A_n such that $A_n \neq \emptyset$ for each $n \in \mathbb{N}$. Assume that $\bigcup_{n\in\mathbb{N}} A_n$ is internal and let $B_k = \bigcup_{n\in\mathbb{N}} A_n \setminus A_k$ for $k \in \mathbb{N}$. Then all B_k are internal and they have the finite intersection property. It follows from the Saturation Principle that $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$, which is a contradiction. Thus $\bigcup_{n\in\mathbb{N}} A_n$ is not in $\mathcal{A}$, and $\mathcal{A}$ is not a σ-algebra. In the following theorem we show that there is an extension of $\mathrm{st}(\mu)$ that turns the standard finitely measure space, to a measure space.

Theorem 1. *Let $\sigma(\mathcal{A})$ be the σ-algebra generated by $\mathcal{A}$. Then there exists a unique σ-additive extension of* $\mathrm{st}(\mu)$ *on $\sigma(\mathcal{A})$.*

Proof. Suppose that A_n for $n \in \mathbb{N}$ are disjoint sets in $\mathcal{A}$ such that $A = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$. $\mathcal{A}$ is internal and $A \in \mathcal{A}$. Thus A is internal, but by the Saturation Principle this is only the case if there exists $m \in \mathbb{N}$ such that $A_n = \emptyset$ for $n > m$, hence

$$\Rightarrow \mathrm{st}(\mu)\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mathrm{st}(\mu)\left(\bigcup_{n=1}^{m} A_n\right) = \sum_{n=1}^{m} \mathrm{st}(\mu(A_n)) = \sum_{n\in\mathbb{N}} \mathrm{st}(\mu(A_n))$$

$\Rightarrow \mathrm{st}(\mu)$ is σ-additive. Now the result follows from Caratheodorys extension theorem, which says that if there is a σ-additive measure on an Algebra $\mathcal{A}$, then there exists an unique σ-additive extension on the σ-Algebra generated by $\mathcal{A}$. □

In the following, we show that the sets $\bigcup_{n\in\mathbb{N}} A_n$ where $A_n \in \mathcal{A}$ differs from a set $A \in \mathcal{A}$ by a so called Loeb null set, which we define now.

Definition 1.2.1. Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space. $B \subseteq \Omega$ is a Loeb null set if for all $\epsilon \in \mathbb{R}^+$ there is a set $A \in \mathcal{A}$ with $B \subseteq A$ and $\mu(A) < \epsilon$.

Lemma 1.2.2. *Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space and let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence of sets with A_n in $\mathcal{A}$ for each $n \in \mathbb{N}$. Then there exists a set $A \in \mathcal{A}$ such that*

i. $\bigcup_{n\in\mathbb{N}} A_n \subseteq A$

ii. $\mathrm{st}(\mu(A)) = \lim_{n\to\infty} \mathrm{st}(\mu(A_n))$

iii. $A \setminus \bigcup_{n\in\mathbb{N}} A_n$ *is a Loeb null set.*

Proof. $(A_n)_{n\in\mathbb{N}}$ is a sequence of internal subsets of an internal set $\mathcal{A}$. Thus, by the Transfer Principle, there is an internal sequence $(A_n)_{n\in{}^*\mathbb{N}}$ of sets in $\mathcal{A}$ that extends the sequence $(A_n)_{n\in\mathbb{N}}$. Because $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence we have for each finite n

$$\mu(A_n) \leq \mathrm{st}(\mu(A_n)) + \frac{1}{n} \leq a + \frac{1}{n}$$

where $a = \lim_{n\to\infty} \mathrm{st}(\mu(A_n))$. By overflow there is $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ with

$$\mu(A_N) \leq a + \frac{1}{N} \quad \text{and} \quad A_N \in \mathcal{A}.$$

Let $A = A_N$, then $A \supseteq A_n$ for each finite n. Therefore $A \supseteq \bigcup_{n\in\mathbb{N}} A_n$ and for each finite n we have $\mu(A_n) \leq \mu(A)$. Thus

$$\begin{aligned} \mathrm{st}(\mu(A_n)) &\leq \mathrm{st}(\mu(A)) \leq a \\ &\text{and} \\ \mathrm{st}(\mu(A)) &= \lim_{n\to\infty} \mathrm{st}(\mu(A_n)) \\ \Rightarrow \mathrm{st}(\mu(A \setminus A_n)) &= \mathrm{st}(\mu(A)) - \mathrm{st}(\mu(A_n)) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

For every $m \in \mathbb{N}$ we have $A \setminus \bigcup_{n\in\mathbb{N}} A_n \subseteq A \setminus A_m$. Thus for each real $\epsilon > 0$ there is a set $A \setminus A_m$ in $\mathcal{A}$ with $A \setminus \bigcup_{n\in\mathbb{N}} A_n \subseteq A \setminus A_m$ and $\mu(A \setminus A_m) < \epsilon$. By definition $A \setminus \bigcup_{n\in\mathbb{N}} A_n$ is a Loeb null set. □

Lemma 1.2.3. *A countable union of Loeb null sets is* Loeb null.

Proof. Suppose that A_n is a Loeb null set for all $n \in \mathbb{N}$, and let $\epsilon \in \mathbb{R}^+$. Then, for every n exists a set $B_n \in \mathcal{A}$ such that $A_n \subseteq B_n$ and $\mu(B_n) < \frac{\epsilon}{2^n}$. Let $C_n = \bigcup_{i=1}^n B_i$. Then $(C_n)_{n\in\mathbb{N}}$ is a decreasing sequence and for all $n \in \mathbb{N}$ we have $C_n \in \mathcal{A}$. By Lemma 1.2.2 there exists $C \in \mathcal{A}$ with $\bigcup_{n\in\mathbb{N}} C_n \subseteq C$, $C \setminus \bigcup_{n\in\mathbb{N}} C_n$ is Loeb null and

$$\mathrm{st}(\mu(C)) = \lim_{n\to\infty} \mathrm{st}(\mu(C_n)) = \lim_{n\to\infty} \mathrm{st}\left(\mu\left(\bigcup_{i=1}^n B_i\right)\right) < \lim_{n\to\infty}\left(\sum_{i=1}^n \frac{\epsilon}{2^i}\right) = \epsilon.$$

Because $\bigcup_{n\in\mathbb{N}} A_n \subseteq C$, $\bigcup_{n\in\mathbb{N}} A_n$ is a Loeb null set. □

Definition 1.2.4. Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space and let $B \subseteq \Omega$.

i. B is *Loeb measurable* if there is a set $A \in \mathcal{A}$ such that $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is a Loeb null set.

ii. For a Loeb measurable $B \subseteq \Omega$ define $\mu^{\mathcal{L}}(B) = \mathrm{st}(\mu(A))$ where $A \in \mathcal{A}$ is the set such that $A \triangle B$ is a Loeb null set. Call $\mu^{\mathcal{L}}(B)$ the Loeb measure of B.

iii. $\mathcal{L}(\mathcal{A})$ denotes the collection of all Loeb measurable sets.

It is clear that $\mathcal{L}(\mathcal{A})$ contains $\mathcal{A}$ and all Loeb null sets. In the following we show that $\mathcal{L}(\mathcal{A})$ is a σ-algebra and $\mu^{\mathcal{L}}$ is a σ-additive extension of $\mathrm{st}(\mu)$ on $\mathcal{L}(\mathcal{A})$.

Lemma 1.2.5. *For $n \in \mathbb{N}$ let $A_n \in \mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}(\mathcal{A})$. If the sets A_n are pairwise disjoint, then*

$$\mu^{\mathcal{L}}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu^{\mathcal{L}}(A_n).$$

Proof. Define $B_n = \bigcup_{i=1}^{n} A_n$, then $(B_n)_{n\in\mathbb{N}}$ is an increasing sequence with $B_n \in \mathcal{A}$ for all n. By Lemma 1.2.2 there exists $B \in \mathcal{A}$ such that $B \triangle \bigcup_{n\in\mathbb{N}} B_n$ is Loeb null. Because $\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n$ we have $B \triangle \bigcup_{n\in\mathbb{N}} A_n$ is Loeb null and therefore $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}(\mathcal{A})$. If the sets A_n are pairwise disjoint, we have by Definition 1.2.4

$$\mu^{\mathcal{L}}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \operatorname{st}(\mu(B)) = \lim_{n\to\infty} \operatorname{st}(\mu(B_n)) =$$
$$= \lim_{n\to\infty} \operatorname{st}\left(\mu\left(\bigcup_{i=1}^{n} A_i\right)\right) = \lim_{n\to\infty} \operatorname{st}\left(\sum_{i=1}^{n} \mu(A_i)\right) = \sum_{n\in\mathbb{N}} \operatorname{st}(\mu(A_n)).$$

□

Theorem 2. $\left(\Omega, \mathcal{L}(\mathcal{A}), \mu^{\mathcal{L}}\right)$ *is a measure space.*

Proof. We first show that $\mathcal{L}(\mathcal{A})$ is a σ-algebra. For $A \in \mathcal{L}(\mathcal{A})$ there exists $B \in \mathcal{L}(\mathcal{A})$ so that $A \triangle B$ is Loeb null. Thus $A^c \triangle B^c = A \triangle B$ is Loeb null and $A^c \in \mathcal{L}(\mathcal{A})$. To show that $\mathcal{L}(\mathcal{A})$ is closed under countable unions, let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{L}(\mathcal{A})$. For every $n \in \mathbb{N}$ there exists $B_n \in \mathcal{A}$ such that $A_n \triangle B_n$ is Loeb null. By Lemma 1.2.5

$$\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{L}(\mathcal{A})$$

and

$$\bigcup_{n\in\mathbb{N}} A_n \triangle \bigcup_{n\in\mathbb{N}} B_n \subseteq \bigcup_{n\in\mathbb{N}} (A_n \triangle B_n).$$

Lemma 1.2.3 says that the countable union of Loeb null sets is a Loeb null set, so $\bigcup_{n\in\mathbb{N}}(A_n \triangle B_n)$ is a Loeb null set and therefore $\bigcup_{n\in\mathbb{N}} A_n$ is Loeb measurable.

To show that $\mu^{\mathcal{L}}$ is σ-additive let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{L}(\mathcal{A})$ and let $(B_n)_{n\in\mathbb{N}}$ as above. Suppose that the B_n are pairwise disjoint, otherwise take $C_n = B_n \setminus \bigcup_{i=1}^{n-1} B_i$. Then all C_n are pairwise disjoint and $A_n \triangle C_n \subseteq \bigcup_{i=1}^{n} A_i \triangle B_i$ which is Loeb null. We know from above that $\bigcup_{n\in\mathbb{N}} A_n \triangle \bigcup_{n\in\mathbb{N}} B_n$ is Loeb null and that $\bigcup_{n\in\mathbb{N}} B_n$ is Loeb measurable. We see from Lemma 1.2.5 and by Definition 1.2.4 that

$$\mu^{\mathcal{L}}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu^{\mathcal{L}}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n\in\mathbb{N}} \mu^{\mathcal{L}}(B_n) = \sum_{n\in\mathbb{N}} \mu^{\mathcal{L}}(A_n).$$

□

$\mathcal{L}(\mathcal{A})$ is called the *Loeb algebra* and the measure space $\left(\Omega, \mathcal{L}(\mathcal{A}), \mu^{\mathcal{L}}\right)$ is called the *Loeb space* given by $(\Omega, \mathcal{A}, \mu)$.

There is another direct construction to obtain the Loeb space $\left(\Omega, \mathcal{L}(\mathcal{A}), \mu^{\mathcal{L}}\right)$: The Loeb algebra $\mathcal{L}(\mathcal{A})$ can be defined as the collection of all sets A with the property that the inner and the outer Loeb measure of A is equal.

Definition 1.2.6. Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space and let $A \subseteq \Omega$. Define

$$\underline{\mu}(A) = \sup_{B \in \mathcal{A}, B \subseteq A} \{\mathrm{st}(\mu(B))\}$$

$$\overline{\mu}(A) = \inf_{B \in \mathcal{A}, B \supseteq A} \{\mathrm{st}(\mu(B))\}.$$

$\underline{\mu}(A)$ and $\overline{\mu}(A)$ are called the *inner* and the *outer Loeb measure* of A. A is μ-*approximable* if for every $\epsilon \in \mathbb{R}^+$ there are sets $B, C \in \mathcal{A}$ such that $B \subseteq A \subseteq C$ and $\mu(C \setminus B) < \epsilon$.

Now we show some properties of $\underline{\mu}$ and $\overline{\mu}$.

Lemma 1.2.7. *Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space and let $A_n \subseteq \Omega$ for $n \in \mathbb{N}$. Then*

i. If $A_n \subseteq A_{n+1}$ for $n \in \mathbb{N}$ then $\lim_{n\to\infty} \overline{\mu}(A_n) = \overline{\mu}\left(\bigcup_{n\in\mathbb{N}} A_n\right)$

ii. If $A_n \subseteq A_{n-1}$ for $n \in \mathbb{N}$ then $\lim_{n\to\infty} \underline{\mu}(A_n) = \underline{\mu}\left(\bigcap_{n\in\mathbb{N}} A_n\right)$

iii. $\overline{\mu}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \overline{\mu}(A_n)$

iv. If A_n are disjoint for $n \in \mathbb{N}$ then $\underline{\mu}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \geq \sum_{n\in\mathbb{N}} \underline{\mu}(A_n)$.

Proof. To show i let $\epsilon \in \mathbb{R}^+$. By definition of $\overline{\mu}$, there is $B_n \in \mathcal{A}$ such that $A_n \subseteq B_n$ and $\mathrm{st}(\mu(B_n)) \leq \overline{\mu}(A_n) + \epsilon\frac{1}{2^n}$. We show with induction that

$$\mathrm{st}\left(\mu\left(\bigcup_{i=1}^{n} B_i\right)\right) \leq \overline{\mu}(A_n) + \epsilon \sum_{i=1}^{n} \frac{1}{2^i}.$$

Assume that the inequality holds for n. Because $\mathrm{st}(\mu)$ finitely additive and $B_n \in \mathcal{A}$ for $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathrm{st}\left(\mu\left(\bigcup_{i=1}^{n+1} B_i\right)\right) &= \mathrm{st}\left(\mu\left(\bigcup_{i=1}^{n} B_i\right)\right) + \mathrm{st}\left(\mu\left(B_{n+1}\right)\right) - \mathrm{st}\left(\mu\left(\bigcup_{i=1}^{n} B_i \cap B_{n+1}\right)\right) \leq \\ &\leq \overline{\mu}(A_n) + \epsilon \sum_{i=1}^{n} \frac{1}{2^i} + \overline{\mu}(A_{n+1}) + \epsilon\frac{1}{2^{n+1}} - \mathrm{st}\left(\mu\left(\bigcup_{i=1}^{n} B_i \cap B_{n+1}\right)\right) \leq \\ &\leq \overline{\mu}(A_{n+1}) + \epsilon \sum_{i=1}^{n+1} \frac{1}{2^i}\end{aligned}$$

where the last inequality follows from $A_n \subseteq \bigcup_{i=1}^{n} B_i \cap B_{n+1}$. Therefore we have

$$\mathrm{st}\left(\mu\left(\bigcup_{i=1}^{n} B_i\right)\right) \leq \lim_{n\to\infty}\left(\overline{\mu}(A_n) + \epsilon \sum_{i=1}^{n} \frac{1}{2^i}\right) = \lim_{n\to\infty} \overline{\mu}(A_n) + \epsilon.$$

Let

$$\mathcal{A}_n = \left\{A \in \mathcal{A} : \bigcup_{i=1}^{n} B_i \subseteq A \quad \text{and} \quad \mu(A) \leq \lim_{n\to\infty} (\overline{\mu}(A_n)) + \epsilon\right\}.$$

$\mathcal{A}_n$ are internal sets with the finite intersection property, therefore by the Saturation Principle exists $A \in \bigcap_{n\in\mathbb{N}} \mathcal{A}_n$. By the Definition of $\mathcal{A}_n$ we have $\bigcup_{n\in\mathbb{N}} B_n \subseteq A$ and therefore we have

$$\begin{aligned}\overline{\mu}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \overline{\mu}\left(\bigcup_{n\in\mathbb{N}} B_n\right) &= \\ &= \mathrm{st}\left(\mu\left(\bigcup_{n\in\mathbb{N}} B_n\right)\right) \leq \\ &\leq \mathrm{st}\,(\mu(A)) \leq \\ &\leq \lim_{n\to\infty} \overline{\mu}(A_n) + \epsilon.\end{aligned}$$

The result follows from the monotonicity of $\overline{\mu}$.

ii. Because $\mathrm{st}(\mu)$ is a finitely additative measure, we obtain for $n \in \mathbb{N}$:

$$\begin{aligned}\underline{\mu}(A_n) &= \sup_{A\subset A_n, A\in\mathcal{A}} (\mathrm{st}(\mu(A))) = \\ &= \sup_{A\subset A_n, A\in\mathcal{A}} (\mathrm{st}(\mu(\Omega)) - \mathrm{st}(\mu(\Omega \setminus A))) = \\ &= \mathrm{st}(\mu(\Omega)) - \inf_{A\subset A_n, A\in\mathcal{A}} \mathrm{st}(\mu(\Omega \setminus A)) = \\ &= \mathrm{st}(\mu(\Omega)) - \overline{\mu}(\Omega \setminus A_n).\end{aligned}$$

Because $\Omega \setminus A_n \subseteq \Omega \setminus A_{n+1}$ and because of i, we have

$$\begin{aligned}\lim_{n\to\infty} \underline{\mu}(A_n) = \mathrm{st}(\mu(\Omega)) - \lim_{n\to\infty} \overline{\mu}(\Omega \setminus A_n) &= \\ &= \mathrm{st}(\mu(\Omega)) - \overline{\mu}\left(\bigcup_{n\in\mathbb{N}} \Omega \setminus A_n\right) = \\ &= \mathrm{st}(\mu(\Omega)) - \overline{\mu}\left(\Omega \setminus \bigcap_{n\in\mathbb{N}} A_n\right) = \\ &= \underline{\mu}\left(\bigcap_{n\in\mathbb{N}} A_n\right).\end{aligned}$$

To show iii, let $\epsilon \in \mathbb{R}^+$. By the definition on $\overline{\mu}$ there is $B_n \in \mathcal{A}$ with $A_n \subseteq B_n$ and

$\operatorname{st}(\mu(B_n) \leq \overline{\mu}(A_n) + \epsilon\frac{1}{2^n}$ for $n \in \mathbb{N}$. Then, we have for $k \in \mathbb{N}$

$$\begin{aligned}\overline{\mu}\left(\bigcup_{n=1}^{k} A_n\right) &\leq \overline{\mu}\left(\bigcup_{n=1}^{k} B_n\right) = \\ &= \operatorname{st}\left(\mu\left(\bigcup_{n=1}^{k} B_n\right)\right) \leq \\ &\leq \sum_{n=1}^{k} \operatorname{st}(\mu(B_n)) \leq \\ &\leq \sum_{n=1}^{k} \overline{\mu}(A_n) + \epsilon\frac{1}{2^n}.\end{aligned}$$

Because $\overline{\mu}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{k\to\infty} \overline{\mu}\left(\bigcup_{n=1}^{k} A_n\right)$ the result follows from i.

iv. Let $\epsilon \in \mathbb{R}^+$. For $n \in \mathbb{N}$ let $B_n \in \mathcal{A}$ with $B_n \subset A_n$ and $\underline{\mu}(A_n) \leq \operatorname{st}(\mu(B_n)) + \epsilon\frac{1}{2^n}$. Because B_n are disjoint for $n \in \mathbb{N}$ we have for $k \in \mathbb{N}$

$$\begin{aligned}\sum_{n=1}^{k} \underline{\mu}(A_n) &\leq \sum_{n=1}^{k} \operatorname{st}(\mu(B_n)) + \epsilon = \\ &= \operatorname{st}\left(\mu\left(\bigcup_{n=1}^{k} B_n\right)\right) + \epsilon \leq \\ &\leq \underline{\mu}\left(\bigcup_{n\in\mathbb{N}} A_n\right) + \epsilon.\end{aligned}$$

□

Theorem 3. *Let $(\Omega, \mathcal{A}, \mu)$ be an internal measure space and let $A \subseteq \Omega$. Then the following are equivalent:*

i. A is Loeb measurable

ii. A is μ-approximable

iii. $\underline{\mu}(A) = \overline{\mu}(A)$.

Proof.

i.⇒ ii. A is Loeb measurable

$\Rightarrow \exists B \in \mathcal{A} : A \triangle B$ is Loeb null

$\Rightarrow \forall \epsilon \in \mathbb{R}^+ \exists C \in \mathcal{A} : A \triangle B \subseteq C$ and $\mu(C) < \epsilon$

moreover

$C^c \cap B \subseteq A \subseteq C \cup B$ and $\mu((C \cup B) \setminus (C^c \cap B)) = \mu(C) < \epsilon$

$\Rightarrow A$ is μ-approximable.

ii.⇒ iii. A is μ-approximable

$$\begin{aligned}
&\Rightarrow \forall \epsilon \in \mathbb{R}^+ \exists B, C \in \mathcal{A} : B \subseteq A \subseteq C \quad \text{and} \quad \mu(C \setminus B) < \epsilon \\
&\Rightarrow 0 \leq \overline{\mu}(A) - \underline{\mu}(A) \leq \mu(C) - \mu(B) = \mu(C \setminus B) < \epsilon \\
&\Rightarrow \overline{\mu}(A) = \underline{\mu}(A).
\end{aligned}$$

iii.⇒i.

$$\begin{aligned}
&\overline{\mu}(A) = \underline{\mu}(A) \\
&\Rightarrow \forall \epsilon \in \mathbb{R}^+ \exists B, C \in \mathcal{A} : B \subseteq A \subseteq C \quad \text{and} \\
&\underline{\mu}(A) \leq \mu(B) + \frac{\epsilon}{2}, \overline{\mu}(A) \geq \mu(C) - \frac{\epsilon}{2} \\
&\Rightarrow \mu(C \setminus B) = \mu(C) - \mu(B) \leq \epsilon \\
&\text{moreover} \\
&(A \setminus B) \cup (B \setminus A) = A \setminus B \subseteq C \setminus B \\
&\Rightarrow A \triangle B \quad \text{is Loeb null and } A \text{ is Loeb measurable.}
\end{aligned}$$

□

1.3 Brownian Motion

The first nonstandard construction of a Brownian Motion was given by Robert M. Anderson in 1976 [7]. He showed how to construct a Brownian Motion as the standard part of an hyperfinite random walk, by using Loeb Measures. In this section we make a more general construction of a Brownian Motion, which includes the case of an infinitesimal random walk.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. For an $\mathcal{A}$-measurable function $f : \Omega \to \mathbb{R}$ let $\mathcal{P}_f(B) := \mathcal{P}(f^{-1}(B))$ for all $B \in \mathcal{B}(\mathbb{R})$ be the distribution of f under $\mathcal{P}$ and for a function $g : T \times \Omega \to \mathbb{R}$ let $g(\cdot, \omega)$ be the function $t \to g(t, \omega)$ for $t \in T$ and let $g(t, \cdot)$ be the function $\omega \to g(t, \omega)$ for $\omega \in \Omega$. For $B \in \mathcal{B}(\mathbb{R})$ let $\epsilon_0(B) = 1$ if $0 \in B$ and $\epsilon(B) = 0$ if $0 \notin B$.

Definition 1.3.1. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. A *Brownian Motion* is a function $b : [0, 1] \times \Omega \to \mathbb{R}$ that satisfies:

i. $b(t, \cdot)$ is $\mathcal{A}$-measurable for all $t \in [0, 1]$.

ii. $\mathcal{P}_{b(0,\cdot)} = \epsilon_0$ and $\mathcal{P}_{b(t,\cdot)-b(s,\cdot)} = \mathcal{N}(0; t - s)$ for $0 \leq s < t \leq 1$.

iii. For $i = 1, \ldots, n$ the differences $b(t_i, \cdot) - b(t_{i-1}, \cdot)$ are independent with respect to $\mathcal{P}$ whenever $0 = t_0 < t_1 < \ldots < t_n \leq 1$.

iv. $b(\cdot, \omega)$ is continuous for all $\omega \in \Omega$.

Subsequently we present a nonstandard construction of a Brownian Motion on a Loeb Space. For this we need the following results. Lemma 1.3.3 and Lemma 1.3.4 are nonstandard versions of Levi's inequality and the central limit theorem.

Lemma 1.3.2. *Let $\mathcal{A}$ be an internal algebra on an internal Ω and let $\mathcal{P} : \mathcal{A} \to {}^*[0,\infty)$ be internal and finitely additive such that $\mathcal{P}(\Omega) = 1$. Let $g_1, \dots, g_n : \Omega \to {}^*\mathbb{R}$ such that*

i.
$$\{\omega \in \Omega : g_i(\omega) < r_i\} \in \mathcal{A}$$

ii.
$$\mathcal{P}\left(\bigcap_{i=1}^{n} \{\omega \in \Omega : g_i(\omega) < r_i\}\right) = \prod_{i=1}^{n} \mathcal{P}\left(\{\omega \in \Omega : g_i(\omega) < r_i\}\right)$$

where $r_1, \dots, r_n \in \mathbb{R}$. Then $\operatorname{st}(g_1), \dots, \operatorname{st}(g_n)$ are $\mathcal{P}^{\mathcal{L}}$- independent.

Proof. Remember from Section 1.2 that $\mathcal{L}(\mathcal{A})$ is a σ-algebra extending the internal algebra $\mathcal{A}$ and $\mathcal{P}^{\mathcal{L}}$ is the measure on $\mathcal{L}(\mathcal{A})$ extending $\operatorname{st}(\mathcal{P})$. Because $\mathcal{L}(\mathcal{A})$ is a σ-algebra we have

$$\{\omega \in \Omega : \operatorname{st}(g_i(\omega)) < r\} = \bigcup_{k=1}^{\infty} \underbrace{\left\{\omega \in \Omega : g_i(\omega) < r - \frac{1}{k}\right\}}_{\in \mathcal{A}} \in \mathcal{L}(\mathcal{A}).$$

$\Rightarrow \operatorname{st}(g_i)$ is $\mathcal{L}(\mathcal{A})$-measurable for all $i = 1, \dots, n$.

$$\begin{aligned}
\mathcal{P}^{\mathcal{L}}\left(\operatorname{st}(g_1) < r_1, \dots, \operatorname{st}(g_n) < r_n\right) &= \mathcal{P}^{\mathcal{L}}\left(\bigcup_{k=1}^{\infty}\left(g_1 < r_1 - \frac{1}{k}, \dots, g_n < r_n - \frac{1}{k}\right)\right) = \\
&= \lim_{k\to\infty} \mathcal{P}^{\mathcal{L}}\left(g_1 < r_1 - \frac{1}{k}, \dots, g_n < r_n - \frac{1}{k}\right) = \\
&= \lim_{k\to\infty} \operatorname{st}\left(\mathcal{P}\left(g_1 < r_1 - \frac{1}{k}, \dots, g_n < r_n - \frac{1}{k}\right)\right) = \\
&= \lim_{k\to\infty} \operatorname{st}\left(\prod_{i=1}^{n} \mathcal{P}\left(g_i < r_i - \frac{1}{k}\right)\right) = \\
&= \lim_{k\to\infty} \prod_{i=1}^{n} \mathcal{P}^{\mathcal{L}}\left(g_i < r_i - \frac{1}{k}\right) = \\
&= \prod_{i=1}^{n} \mathcal{P}^{\mathcal{L}}\left(g_i < r_i\right).
\end{aligned}$$

and therefore $\operatorname{st}(g_1), \dots, \operatorname{st}(g_n)$ are $\mathcal{P}^{\mathcal{L}}$-independent. ◻

Lemma 1.3.3. *For all $n \in \mathbb{N}$ let $(\Omega_n, \mathcal{A}_n, \mathcal{P}_n)$ be internal probability spaces and let $\delta_{1n}, \ldots, \delta_{nn} : \Omega_n \to \mathbb{R}$ be $\mathcal{P}_n$-independent random variables, such that δ_{in} have for all $i = 1, \ldots, n$ and for all $n \in \mathbb{N}$ the same distribution with mean 0 and variance 1. Then for $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, $k \leq N$ and $x \in {}^*\mathbb{R}$ with $x \geq 2\sqrt{2}$*

$$\mathcal{P}_N\left(\left\{\omega \in \Omega_N : \max_{1\leq l\leq k}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{l}\delta_{iN}(\omega)\right| \geq x\right\}\right) \leq$$
$$\leq 2\mathcal{P}_N\left(\left\{\omega \in \Omega_N : \left|\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\delta_{iN}(\omega)\right| \geq \frac{x}{2}\right\}\right).$$

Proof. Levi‘s inequality implies that for $n \in \mathbb{N}$, $k \leq n$ and $x \leq 2\sqrt{2}$

$$\mathcal{P}_n\left(\left\{\omega \in \Omega_n : \max_{1\leq l\leq k}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{l}\delta_{in}(\omega)\right| \geq x\right\}\right) \leq$$
$$\leq 2\mathcal{P}_n\left(\left\{\omega \in \Omega_n : \left|\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\delta_{in}(\omega)\right| \geq \frac{x}{2}\right\}\right).$$

The transfer principle implies that for $N \in {}^*\mathbb{N}$, $k \leq N$ and $x \in {}^*\mathbb{R}$ with $x \geq 2\sqrt{2}$

$$\mathcal{P}_N\left(\left\{\omega \in \Omega_N : \max_{1\leq l\leq k}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{l}\delta_{iN}(\omega)\right| \geq x\right\}\right) \leq$$
$$\leq 2\mathcal{P}_N\left(\left\{\omega \in \Omega_N : \left|\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\delta_{iN}(\omega)\right| \geq \frac{x}{2}\right\}\right).$$

□

Lemma 1.3.4. *For all $n \in \mathbb{N}$ let $(\Omega_n, \mathcal{A}_n, \mathcal{P}_n)$ be internal probability spaces and let $\delta_{1n}, \ldots, \delta_{nn}$ be $\mathcal{P}_n$-independent random variables with the same distribution with mean 0 and variance 1. Then for $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, $x \in {}^*\mathbb{R}$ and $k \in {}^*\mathbb{N}$ with $\sqrt{N} \leq k \leq N$*

$$\mathcal{P}_N\left(\left\{\omega \in \Omega_N : \frac{1}{\sqrt{k}}\sum_{i=1}^{k}\delta_{iN}(\omega) < x\right\}\right) \approx {}^*\Phi(x)$$

and

$$\mathcal{P}_N\left(\left\{\omega \in \Omega_N : \left|\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\delta_{iN}(\omega)\right| \geq x\right\}\right) \approx 2(1 - {}^*\Phi(x))$$

where Φ denotes the standard normal density function.

Proof. Let $Q : \mathcal{B}(\mathbb{R}) \to [0,1]$ be the distribution of δ_{in}, i.e. $Q(B) = \mathcal{P}_n(\delta_{in} \in B)$ for all $i = 1, \ldots, n$. Q^n denotes the product measure, i.e. $Q^n : \mathcal{B}(\mathbb{R}^n) \to [0,1]$ and

$Q^n(B) = \mathcal{P}_n(\{\omega \in \Omega_n : (\delta_{1n}(\omega), \dots, \delta_{nn}(\omega)) \in B\})$. Let $\mathrm{id}_i(y) = y_i$ for $y \in \mathbb{R}^n$ and for $i \in \{1, \dots, n\}$. Note that id_i are Q^n-independent random variables with mean 0 and variance 1. Thus we can use the central limit theorem which implies that

$$\sup_{x\in\mathbb{R},\ \sqrt{n}\le k\le n} \left| Q^n\left(\left\{y \in \mathbb{R}^n : \frac{1}{\sqrt{k}} \sum_{i=1}^{k} y_i < x\right\}\right) - \Phi(x)\right| \to 0 \quad \text{as} \quad n \to \infty.$$

Since

$$\mathcal{P}_n\left(\left\{\omega \in \Omega_n : \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \delta_{in}(\omega) < x\right\}\right) = Q_n\left(\left\{y \in \mathbb{R}^n : \frac{1}{\sqrt{k}} \sum_{i=1}^{k} y_i < x\right\}\right)$$

$$\Rightarrow \sup_{x\in\mathbb{R},\ \sqrt{n}\le k\le n} \underbrace{\left| \mathcal{P}_n\left(\left\{\omega \in \Omega_n : \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \delta_{in}(\omega) < x\right\}\right) - \Phi(x)\right|}_{=:a_n} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $a_N \approx 0$ for all $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. For $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, $x \in {}^*\mathbb{R}$ and $k \in {}^*\mathbb{N}$ with $\sqrt{N} \le k \le N$

$$\Rightarrow \mathcal{P}_N\left(\left\{\omega \in \Omega_N : \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \delta_{iN}(\omega) < x\right\}\right) \approx {}^*\Phi(x).$$

The same equality holds for $< x$ instead of $\le x$ and by the Transfer Principle we have that ${}^*\Phi(-x) = 1 - {}^*\Phi(x)$ and that $\delta_{iN} : \Omega_N \to {}^*\mathbb{R}$ are random variables with mean 0. Therefore

$$\begin{aligned} &\mathcal{P}_N\left(\left\{\omega \in \Omega_N : \left|\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \delta_{iN}(\omega)\right| \ge x\right\}\right) = \\ &= 2\mathcal{P}_N\left(\left\{\omega \in \Omega_N : \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \delta_{iN}(\omega) \ge x\right\}\right) \approx \\ &\approx 2\left(1 - {}^*\Phi(x)\right). \end{aligned}$$

□

Theorem 4. *For all $n \in \mathbb{N}$ let $(\Omega_n, \mathcal{A}_n, \mathcal{P}_n)$ be internal probability spaces and let $\delta_{1n}, \dots, \delta_{nn} : \Omega_n \to \mathbb{R}$ be $\mathcal{P}_n$-independent random variables, such that δ_{in} have for all $i = 1, \dots, n$ and for all $n \in \mathbb{N}$ the same distribution with mean 0 and variance 1. Then for $N \in {}^*\mathbb{N} \setminus \mathbb{N}$,*

i. there is a set $M \in \mathcal{L}(\mathcal{A}_N)$ with $\mathcal{P}_N^{\mathcal{L}}(M) = 0$ so that

$${}^*[0, 1] \ni t \to \frac{1}{\sqrt{N}} \sum_{i=1}^{[Nt]} \delta_{iN}(\omega)$$

is S-continuous for $\omega \notin M$.

ii.

$$b(t,\omega) = \begin{cases} \mathrm{st}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{[Nt]}\delta_{iN}(\omega)\right) & \text{if} \quad \omega \notin M, t \in [0,1] \\ 0 & \text{if} \quad \omega \in M, t \in [0,1] \end{cases}$$

is a Brownian Motion with respect to $(\Omega_N, \mathcal{L}(\mathcal{A}_N), \mathcal{P}_N^{\mathcal{L}})$.

Proof. To show i, we define

$$B(t,\omega) := \frac{1}{\sqrt{N}}\sum_{i=1}^{[Nt]}\delta_{iN}(\omega) \quad \text{for} \quad \omega \in \Omega_N, t \in {}^*[0,1]$$

where $[Nt]$ is the first integer smaller than Nt. Let

$$C_k(\epsilon) := \bigcup_{j=1}^{k}\left\{\omega \in \Omega_N : \sup_{\frac{j-1}{k}\leq s,t\leq \frac{j}{k}} |B(t,\omega) - B(s,\omega)| \geq \epsilon\right\} \quad \text{for} \quad k \in \mathbb{N}, \epsilon \in \mathbb{R}^+$$

$$M := \bigcup_{m=1}^{\infty}\bigcap_{k=1}^{\infty} C_k\left(\frac{1}{m}\right).$$

If $\omega \notin M$ then $\omega \notin \bigcap_{k=1}^{\infty} C_k(\frac{1}{m})$ for all $m \in \mathbb{N}$ and so, for all $m \in \mathbb{N}$ exists $k(m) \in \mathbb{N}$ such that $\omega \notin C_k(\frac{1}{m})$. Let $s,t \in {}^*[0,1]$ with $s \approx t$. Then there exists $j \in \{1,\ldots,k(m)\}$ such that $\frac{j-1}{k(m)} \leq s,t \leq \frac{j+1}{k(m)}$. For all $m \in \mathbb{N}$ we have

$$s,t \in {}^*[0,1], s \approx t \Rightarrow |B(t,\omega) - B(s,\omega)| \leq \frac{2}{m}.$$

Hence $B(\cdot,\omega)$ is S-continuous for all $\omega \notin M$. It remains to show that $M \in \mathcal{L}(\mathcal{A}_N)$ and that $\mathcal{P}_N^{\mathcal{L}}(M) = 0$.

Fix $k \in \mathbb{N}$ with $k \geq 2$. Define $N_j := [N\frac{j}{k}]$ for $j = 0,\ldots,k$. For $s,t \in {}^*[0,1]$ with $s < t$ and for $j \in \{0,\ldots,k\}$ with $\frac{j-1}{k} \leq s < t \leq \frac{j}{k}$ we have $N_{j-1} \leq [Ns] \leq [Nt] \leq N_j$

$$\begin{aligned}
\Rightarrow \sup_{\frac{j-1}{k}\leq s,t\leq\frac{j}{k}} |B(t,\omega) - B(s,\omega)| = \sup_{\frac{j-1}{k}\leq s,t\leq\frac{j}{k}} \left|\frac{1}{\sqrt{N}}\sum_{i=[Ns]+1}^{[Nt]}\delta_{iN}(\omega)\right| &\leq \\
\leq \max_{N_{j-1}\leq a<b\leq N_j}\left|\frac{1}{\sqrt{N}}\sum_{i=a+1}^{b}\delta_{iN}(\omega)\right| &\leq \\
\leq \max_{N_{j-1}\leq a<b\leq N_j}\left(\left|\frac{1}{\sqrt{N}}\sum_{i=N_{j-1}+1}^{b}\delta_{iN}(\omega)\right| + \left|\frac{1}{\sqrt{N}}\sum_{i=N_{j-1}+1}^{a}\delta_{iN}(\omega)\right|\right) &\leq \\
\leq 2\max_{N_{j-1}\leq a\leq N_j}\left|\frac{1}{\sqrt{N}}\sum_{i=N_{j-1}+1}^{a}\delta_{iN}(\omega)\right|&
\end{aligned}$$

$$\Rightarrow C_k(\epsilon) = \bigcup_{j=1}^{k} \left\{ \omega \in \Omega_N : \sup_{\frac{j-1}{k} \le s,t \le \frac{j}{k}} |B(t,\omega) - B(s,\omega)| \ge \epsilon \right\} \subseteq$$
$$\subseteq \bigcup_{j=1}^{k} \left\{ \omega \in \Omega_N : \max_{N_{l-1} \le a \le N_j} \left| \frac{1}{\sqrt{N}} \sum_{i=N_{j-1}+1}^{a} \delta_{iN}(\omega) \right| \ge \frac{\epsilon}{2} \right\}.$$

Recall from Section 1.2 that $\overline{\mathcal{P}}_N(A) := \inf_{B\in\mathcal{A}_N, B\supseteq A} \{\mathrm{st}\,(\mathcal{P}_N(B))\}$ and that $\mathrm{st}\left(\overline{\mathcal{P}}_N\right)$ is a finitely additative measure. Because $\delta_{1N}, \ldots, \delta_{NN}$ are $\mathcal{P}_N$-independent random variables with the same distribution and by Lemma 1.2.7 we have:

$$\overline{\mathcal{P}}_N\left(C_k(\epsilon)\right) \le \sum_{j=1}^{k} \overline{\mathcal{P}}_N \left(\left\{ \omega \in \Omega_N : \max_{N_{j-1} \le a \le N_j} \left| \frac{1}{\sqrt{N}} \sum_{i=N_{j-1}+1}^{a} \delta_{iN}(\omega) \right| \ge \frac{\epsilon}{2} \right\} \right) =$$
$$= \sum_{j=1}^{k} \overline{\mathcal{P}}_N \left(\left\{ \omega \in \Omega_N : \max_{1 \le a \le N_j - N_{j-1}} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{a} \delta_{iN}(\omega) \right| \ge \frac{\epsilon}{2} \right\} \right) \le$$
$$\le k \cdot \overline{\mathcal{P}}_N \left(\left\{ \omega \in \Omega_N : \max_{1 \le a \le N(k)} \left| \frac{1}{\sqrt{N(k)}} \sum_{i=1}^{a} \delta_{iN}(\omega) \right| \ge \frac{\epsilon}{2} \sqrt{\frac{N}{N(k)}} \right\} \right)$$

where $N(k) := \max_{1\le j\le k}(N_j - N_{j-1})$. For a sufficiently large k we have $\frac{\epsilon}{2}\sqrt{\frac{N}{N(k)}} \ge \frac{\epsilon}{2}\sqrt{\frac{k}{2}} \ge 2\sqrt{2}$. By the Transfer Principle

$$\left\{ \omega \in \Omega_N : \max_{1 \le a \le N(k)} \left| \frac{1}{\sqrt{N(k)}} \sum_{i=1}^{a} \delta_{iN}(\omega) \right| \ge \frac{\epsilon}{2} \sqrt{\frac{N}{N(k)}} \right\} \in \mathcal{A}_N$$

and therefore Lemma 1.3.3 and Lemma 1.3.4 imply:

$$\overline{\mathcal{P}}_N(C_k(\epsilon)) \le k \cdot \mathcal{P}_N \left(\left\{ \omega \in \Omega_N : \max_{1 \le a \le N(k)} \left| \frac{1}{\sqrt{N(k)}} \sum_{i=1}^{a} \delta_{iN}(\omega) \right| \ge \frac{\epsilon}{2} \sqrt{\frac{N}{N(k)}} \right\} \right) \le$$
$$\le 2k \cdot \mathcal{P}_N \left(\left\{ \omega \in \Omega_N : \left| \frac{1}{\sqrt{N(k)}} \sum_{i=1}^{N(k)} \delta_{iN}(\omega) \right| \ge \frac{\epsilon}{4} \sqrt{\frac{N}{N(k)}} \right\} \right) \approx$$
$$\approx 4k \left(1 - {}^*\Phi\left(\frac{\epsilon}{4} \sqrt{\frac{N}{N(k)}} \right) \right) \le$$
$$\le 4k \left(1 - \Phi\left(\frac{\epsilon}{4} \sqrt{\frac{k}{2}} \right) \right).$$

Note that for all $\epsilon \in \mathbb{R}^+$:

$$\begin{aligned}
&\lim_{k\to\infty} 4k\left(1-\Phi\left(\frac{\epsilon}{4}\sqrt{\frac{k}{2}}\right)\right)=0\\
&\Rightarrow \overline{\mathcal{P}}_N\left(C_k(\epsilon)\right)\to 0 \quad \text{as} \quad k\to\infty\\
&\Rightarrow \overline{\mathcal{P}}_N\left(\bigcap_{k=1}^{\infty} C_k(\epsilon)\right)=0.
\end{aligned}$$

By Lemma 1.2.7

$$\begin{aligned}
\Rightarrow 0 \le \underline{\mathcal{P}}_N(M) \le \overline{\mathcal{P}}_N(M) = \overline{\mathcal{P}}_N\left(\bigcup_{m=1}^{\infty}\bigcap_{k=1}^{\infty} C_k\left(\frac{1}{m}\right)\right) &\le\\
\le \sum_{m=1}^{\infty}\overline{\mathcal{P}}_N\left(\bigcap_{k=1}^{\infty} C_k\left(\frac{1}{m}\right)\right) &= 0\\
\Rightarrow M\in\mathcal{L}(\mathcal{A}_N) \quad \text{and} \quad \mathcal{P}_N^{\mathcal{L}}(M)=0.&
\end{aligned}$$

ii. We first show that $b(t,\cdot)$ is an $\mathcal{L}(\mathcal{A}_N)$-measurable function for all $t\in[0,1]$. Let $t\in[0,1]$. Because $\mathcal{P}_N^{\mathcal{L}}(M)=0$ we have to show that $\{\omega\in\Omega_N\setminus M : b(t,\omega)<r\}\in\mathcal{L}(\mathcal{A}_N)$ for all $r\in\mathbb{R}$. The Transfer Principle implies

$$\left\{\omega\in\Omega_N : \frac{1}{\sqrt{N}}\sum_{i=1}^{[Nt]}\delta_{iN}<r\right\}\in\mathcal{A}_N.$$

Therefore,

$$\begin{aligned}
&\{\omega\in\Omega_N\setminus M : b(t,\omega)<r\} =\\
&=\left\{\omega\in\Omega_N\setminus M : \mathrm{st}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{[Nt]}\delta_{iN}(\omega)\right)<r\right\}=\\
&=\Omega_N\setminus M\cap\bigcup_{k=1}^{\infty}\left\{\omega\in\Omega_N : \frac{1}{\sqrt{N}}\sum_{i=1}^{[Nt]}\delta_{iN}<r-\frac{1}{k}\right\}\in\mathcal{L}(\mathcal{A}_N).
\end{aligned}$$

Next we show that $\left(\mathcal{P}_N^{\mathcal{L}}\right)_{b(0,\cdot)}=\epsilon_0$ and that $\left(\mathcal{P}_N^{\mathcal{L}}\right)_{b(t,\cdot)-b(s,\cdot)}=\mathcal{N}(0;t-s)$ for $0\le s<t\le 1$. Because $b(0,\cdot)\equiv 0$ we have $(\mathcal{P}_N^{\mathcal{L}})_{b(0,\cdot)}=\epsilon_0$. Fix $0\le s<t\le 1$. For all $r\in\mathbb{R}$ we have:

$$\begin{aligned}
\mathcal{P}_N^{\mathcal{L}}\left(B(t,\cdot)-B(s,\cdot)\le r\right)=\mathcal{P}_N^{\mathcal{L}}\left(\frac{1}{\sqrt{N}}\sum_{i=[Ns]+1}^{[Nt]}\delta_{iN}\le r\right)&=\\
=\mathcal{P}_N^{\mathcal{L}}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{[Nt]-[Ns]}\delta_{iN}\le r\right)&.
\end{aligned}$$

Let $k = [Nt] - [Ns]$, then

$$\mathcal{P}_N^{\mathcal{L}}(B(t,\cdot) - B(s,\cdot) \le r) = \mathcal{P}_N^{\mathcal{L}}\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}\delta_{iN} \le r\sqrt{\frac{N}{k}}\right).$$

Because $\frac{k}{N} \approx t - s$ we have $\sqrt{N} \le k \le N$. Therefore we can use Lemma 1.3.4:

$$\begin{aligned}\mathcal{P}_N^{\mathcal{L}}(B(t,\cdot) - B(s,\cdot) \le r) &= \mathrm{st}\left({}^*\Phi\left(r\sqrt{\frac{N}{k}}\right)\right) = \\ &= \Phi\left(r\frac{1}{\sqrt{t-s}}\right) = \\ &= \mathcal{N}(0; t-s)((-\infty, r]).\end{aligned}$$

Therefore, for $r \in \mathbb{R}$, $\epsilon \in \mathbb{R}_+$:

$$\begin{aligned}\mathcal{N}(0; t-s)((-\infty, r]) &= \mathcal{P}_N^{\mathcal{L}}(B(t,\cdot) - B(s,\cdot) \le r) \le \\ &\le \mathcal{P}_N^{\mathcal{L}}(b(t,\cdot) - b(s,\cdot) \le r) \le \\ &\le \mathcal{P}_N^{\mathcal{L}}(B(t,\cdot) - B(s,\cdot) \le r + \epsilon) = \\ &= \mathcal{N}(0; t-s)((-\infty, r+\epsilon]) \\ \Rightarrow \mathcal{P}_N^{\mathcal{L}}(b(t,\cdot) - b(s,\cdot) \le r) &= \mathcal{N}(0; t-s)((-\infty, r]) \\ \Rightarrow \left(\mathcal{P}_N^{\mathcal{L}}\right)_{b(t,\cdot)-b(s,\cdot)} &= \mathcal{N}(0; t-s).\end{aligned}$$

Next, we use Lemma 1.3.2 to show that $b(t_i,\cdot) - b(t_{i-1},\cdot)$ are $\mathcal{P}_N^{\mathcal{L}}$- independent functions for $0 = t_0 < t_1 < \ldots < t_n \le 1$.

$\delta_{1n}, \ldots, \delta_{nn}$ are $\mathcal{A}_n$-measurable and $\mathcal{P}_n$-independent for all $n \in \mathbb{N}$. The Transfer Principle implies that $\delta_{1N}, \ldots, \delta_{NN}$ are $\mathcal{A}_N$-measurable and $\mathcal{P}_N$-independent.

$$\Rightarrow \left\{\omega \in \Omega_N : B(t_j, \omega) - B(t_{j-1}, \omega) < r\right\} \in \mathcal{A}_N \quad \forall r \in \mathbb{R}$$

$$\begin{aligned}&\mathcal{P}_N\left(\bigcap_{j=1}^{n}\left\{\omega \in \Omega_N : B(t_j,\omega) - B(t_{j-1},\omega) < r_j\right\}\right) = \\ &= \prod_{j=1}^{n}\mathcal{P}_N\left(\left\{\omega \in \Omega_N : B(t_j,\omega) - B(t_{j-1},\omega) < r_j\right\}\right) \quad \forall r_j \in \mathbb{R}.\end{aligned}$$

Lemma 1.3.2 says that $\mathrm{st}\left(B(t_j,\omega) - B(t_{j-1},\omega)\right)$ for $j = 1, \ldots, n$ are $\mathcal{P}_N^{\mathcal{L}}$-independent random variables. Because $\mathrm{st}\left(B(t_j,\cdot) - B(t_{j-1},\cdot)\right) = b(t_j,\cdot) - b(t_{j-1},\cdot)$ $\mathcal{P}_N^{\mathcal{L}}$-almost everywhere, $b(t_j,\cdot) - b(t_{j-1},\cdot)$ are $\mathcal{P}_N^{\mathcal{L}}$- independent for $j = 1, \ldots, n$.

Finally, we show that the function $b(\cdot,\omega)$ is continuous for all $\omega \in \Omega_N$. If $\omega \in M$, then $b(\cdot,\omega) \equiv 0$ and therefore $b(\cdot,\omega)$ is continuous for all $\omega \in M$. Recall from i. that $B(\cdot,\omega)$ is S-continuous for all $\omega \in \Omega_N \setminus M$. Standard parts of S-continuous functions are continuous, therefore $b(\cdot,\omega) = \mathrm{st}(B(\cdot,\omega))$ is continuous for all $\omega \in \Omega_N$. □

Corollary 1.3.5. *Let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let Ω be the set of all internal $(\xi_i)_{i \leq N}$, $\xi_i \in \{-1, 1\}$ and C the counting measure on ${}^*P(\Omega)$, i.e. $C(A) = \frac{|A|}{|\Omega|}$ for all $A \in {}^*\mathfrak{P}(\Omega)$. Then there exists a Brownian Motion $b(t, \xi)$ with respect to the Loeb space $(\Omega, \mathcal{L}({}^*\mathfrak{P}(\Omega)), C^{\mathcal{L}})$ such that for $C^{\mathcal{L}}$- nearly all $\xi \in \Omega$ and for all $t \in [0, 1]$:*

$$b(t, \xi) = \mathrm{st}\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{[Nt]} \xi_i \right).$$

Proof. For $n \in \mathbb{N}$ let Ω_n be the set of all $(\xi_i)_{i<n}$ such that $\omega_i \in \{-1, 1\}$ for $i = 1, \ldots, n$, $\mathcal{A}_n = \mathfrak{P}(\Omega_n)$, $\delta_{in}(\xi) = \xi_i$ and let C_n be the counting measure on Ω_n. Then $\delta_{1n}, \ldots, \delta_{nn}$ are C_n-independent random variables and they have all the same distribution with mean 0 and variance 1. For $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ the transfer principle implies that Ω_N is the set of all internal $(\xi_i)_{i \leq N}$ such that $\xi_i \in \{-1, 1\}$ for $i = 1, \ldots, N$, $\mathcal{A}_N = {}^*\mathfrak{P}(\Omega_N)$ and C_N is the internal counting measure on ${}^*\mathfrak{P}(\Omega_N)$. $\square$

1.4 Lifting Theorems

One of the important tools in nonstandard analysis is the lifting construction. There, measurable functions on standard probability spaces are approximated by internal functions with similar properties. See for instance [8], [9], [10] for details.

Definition 1.4.1. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a hyperfinite probability space with Loeb space $(\Omega, \mathcal{L}(\mathcal{A}), \mathcal{P}^{\mathcal{L}})$. Let $f : \Omega \to \mathbb{R}$. A function $F : \Omega \to {}^*\mathbb{R}$ is called a *lifting* of f if F is internal and

$$\mathrm{st}(F(\omega)) = f(\omega)$$

for $\mathcal{P}^{\mathcal{L}}$-almost all $\omega \in \Omega$.

Theorem 5. *Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a hyperfinite probability space with Loeb space $(\Omega, \mathcal{L}(\mathcal{A}), \mathcal{P}^{\mathcal{L}})$. A function $f : \Omega \to \mathbb{R}$ is Loeb measurable if and only if f has an $\mathcal{A}$-measurable lifting $F : \Omega \to {}^*\mathbb{R}$.*

Proof. $(\Rightarrow)$ Let $(q_i)_{i \in \mathbb{N}}$ be a numeration of all rational numbers and let

$$U_i = \{\omega \in \Omega : f(\omega) \leq q_i\}.$$

Because f is Loeb measurable by assumption, we have $U_i \in \mathcal{L}(\mathcal{A})$ and $U_i \subseteq U_j$ for $q_i \leq q_j$. Therefore, by Definition 1.2.4 there is for all $i \in \mathbb{N}$ a set $A_i \in \mathcal{A}$ such that $\mathcal{P}^{\mathcal{L}}(A_i \triangle U_i) = 0$ and $A_i \subseteq A_j$ for $q_n \leq q_j$. By the Transfer Principle there is an internal sequence $(A_i)_{i \in {}^*\mathbb{N}}$ which extend the sequence $(A_i)_{i \in \mathbb{N}}$ and by Overflow there is $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that for all $i, j \leq N$ with $q_i \leq q_j$ we have $A_i \subseteq A_j$. The set $\{q_i : i = 1, \ldots, N\}$ is hyperfinite, so we can use a new numeration such that $q_{i_1} < q_{i_2} < \ldots < q_{i_N}$. Let

$$F(\omega) = \begin{cases} q_{i_k} & \text{if} \quad \omega \in A_{i_k} \setminus A_{i_{k-1}} \\ q_{i_N} + 1 & \text{if} \quad \omega \notin A_{i_N} \end{cases}$$

$$U = \bigcup_{i\in\mathbb{N}} (A_i \triangle U_i).$$

Then F is $\mathcal{A}$-measurable and for $\omega \notin U$ we have $F(\omega) \le q_i$ if and only if $f(\omega) \le q_i$ for all $i \in \mathbb{N}$. Therefore $\mathrm{st}(F(\omega)) = f(\omega)$ for all $\omega \notin U$ which is a Loeb null set. So F is a lifting of f.

($\Leftarrow$) Let F be an $\mathcal{A}$-measurable lifting of f. Define for every $r \in \mathbb{R}$ a neighborhood $U_{\frac{1}{n}}(r) = \{s \in \mathbb{R} : |r - s| < \frac{1}{n}\}$. We show that $f^{-1}\left(U_{\frac{1}{n}}(r)\right)$ is Loeb measurable. Because F is a lifting of f the set $U = \{\omega \in \Omega : \mathrm{st}(F(\omega)) = f(\omega)\}$ has Loeb measure 1. Let $\omega \in U$, then $f(\omega) \in U_{\frac{1}{n}}(r)$ if and only if $|r - \mathrm{st}(F(\omega))| < \frac{1}{n}$. Because the absolute value function is continuous, the conditions above are equivalent to the condition $\mathrm{st}(|r - F(\omega)|) < \frac{1}{n}$. Therefore for all $\omega \in U$

$$\left\{\omega \in U : f(\omega) \in U_{\frac{1}{n}}(r)\right\} = \left\{\omega \in U : \mathrm{st}(|r - F(\omega)|) < \frac{1}{n}\right\} = \\ = \bigcup_{k\in\mathbb{N}} \left\{\omega \in U : |r - F(\omega)| \le \frac{1}{n} - \frac{1}{k}\right\}.$$

Because

$$\left\{\omega \in U : |r - F(\omega)| \le \frac{1}{n} - \frac{1}{k}\right\} \in \mathcal{A} \\ \Rightarrow \left\{\omega \in U : f(\omega) \in U_{\frac{1}{n}}(r)\right\} \in \mathcal{L}(\mathcal{A}).$$

The set U is Loeb measurable with Loeb Measure 1, therefore f is Loeb measurable. □

In the following, we denote the elements of $[0,1]$ with s and t and the elements of ${}^*[0,1]$ with $\underline{s}$ and $\underline{t}$.

Definition 1.4.2. Let D be the space of all functions $f : [0,1] \times \Omega \to \mathbb{R}$ which are right continuous with left limits. Let $F \in {}^*D$ such that $F(\underline{t}) \in \mathrm{ns}({}^*\mathbb{R})$ for all $\underline{t} \in {}^*[0,1]$.

i. We say that F is *of class SD* if for each $t \in [0,1]$ there are $\underline{t}_1 \approx \underline{t}_2 \approx t$ such that if $\underline{s}_1 \approx \underline{s}_2 \approx t$ with $\underline{s}_1 < \underline{t}_1$ and $\underline{s}_2 \ge \underline{t}_2$ then $F(\underline{s}_1) \approx \lim_{\underline{s}\nearrow\underline{t}_1} F(\underline{s})$ and $F(\underline{s}_2) \approx F(\underline{t}_2)$.

ii. We say that F is *of class SDJ* if if for every $t \in [0,1]$ there is $\underline{t} \approx t$ such that if $\underline{s}_1 \approx t$ and $\underline{s}_1 < \underline{t}$ then $F(\underline{s}_1) \approx \lim_{\underline{s}\nearrow\underline{t}} F(\underline{s})$ and if $\underline{s}_2 \approx t$ and $\underline{s}_2 \ge \underline{t}$ then $F(\underline{s}_2) \approx F(\underline{t})$ and if for all $\underline{t} \approx 0$ in ${}^*[0,1]$ we have $F(\underline{t}) \approx F(0)$.

iii. If F is of class SD, then the *standard part* of F is the function $\mathrm{st}(F) : [0,1] \to \mathbb{R}$ defined by

$$\mathrm{st}(F)(t) = \lim_{\mathrm{st}(\underline{t})\searrow t} \mathrm{st}(F(\underline{t})).$$

Proposition 1.4.3. *Let $F : {}^*[0,1] \to {}^*\mathbb{R}$ be a function in ${}^*D$ such that $F(\underline{t}) \in \mathrm{ns}({}^*\mathbb{R})$ for all $\underline{t} \in {}^*[0,1]$. Then F is SD if and only if* $\mathrm{st}(F)$ *exists and belongs to D.*

Proof. ($\Rightarrow$) Suppose F is SD. Let $\epsilon \in \mathbb{R}^+$ and let $t \in [0,1]$. Because F is SD there is a $\underline{t} \approx t$ such that $F(\underline{s}) \approx F(\underline{t})$ for all $\underline{s} \approx \underline{t}$ that satisfies $\underline{s} \geq \underline{t}$. By overflow there is some $\delta \in \mathbb{R}^+$, such that $\left|F(\underline{s}) - F(\underline{t})\right| < \epsilon$ for all $\underline{s} \in [\underline{t}, \underline{t} + \delta)$. Therefore $\mathrm{st}(F)(t)$ exists and by the definition of the standard part, it is clear that $\mathrm{st}(F)$ is right continuous.

To show that the left limits of $\mathrm{st}(F)$ exists let $t \in [0,1]$ and $\epsilon \in \mathbb{R}^+$. Because F is SD there is $\underline{t} \approx t$ such that for all $\underline{s} \approx \underline{t}$ with $\underline{s} < \underline{t}$ we have $F(\underline{s}) = \lim_{\underline{r} \nearrow \underline{t}} F(\underline{r})$. By Overflow, there is some $\delta \in \mathbb{R}^+$ and some $\underline{t}' \approx t$ such that $\left|F(\underline{s}) - F(\underline{t}')\right| < \epsilon$ whenever $\underline{s} \in (\underline{t}' - \delta, \underline{t}']$. Therefore $\lim_{s \nearrow t} \mathrm{st}(F)(s) = \mathrm{st}(F(\underline{t}'))$. Therefore, the left limits exists and $\mathrm{st}(F)$ belongs to D.

($\Leftarrow$) Suppose that $\mathrm{st}(F)$ exists and belongs to D. By Definition 1.4.2 for the standard part, we have for $t \in [0,1]$ and for all $\epsilon \in \mathbb{R}^+$ that

$$|\mathrm{st}(F)(s) - \mathrm{st}(F)(t))| < \epsilon$$

whenever $s \in [t, t + \delta)$. Therefore, there exists some $\underline{t} \approx t$ such that for all $\underline{s} \geq \underline{t}$ with $\underline{s} \approx t$

$$F(\underline{s}) \approx F(\underline{t}).$$

On the other side, there is some $\underline{t} \approx t$ such that for all $\epsilon \in \mathbb{R}^+$

$$\left|\mathrm{st}(F)(s) - \lim_{r \nearrow t} \mathrm{st}(F)(r)\right| < \epsilon$$

whenever $s \in (t - \delta, t)$. Therefore, there is some $\underline{t} \approx t$ such that for all $\underline{s} < \underline{t}$ with $\underline{s} \approx t$

$$F(\underline{s}) \approx \lim_{\underline{r} \nearrow \underline{t}} \mathrm{st}(F)(\underline{r})$$

and therefore F is SD. □

Proposition 1.4.4. *If $F : {}^*[0,1] \to {}^*\mathbb{R}$ is SDJ then for all $t \in (0,1)$ there is $\underline{t} \approx t$ such that if $\underline{s} \approx t$ and $\underline{s} < \underline{t}$ then*

$$F(\underline{s}) \approx \lim_{r \nearrow t} \mathrm{st}(F)(r)$$

and if $\underline{s} \approx t$ and $\underline{s} \geq \underline{t}$ then

$$F(\underline{s}) \approx \mathrm{st}(F)(t).$$

Proof. Because F id SDJ, F is SD and therefore by Proposition 1.4.3 $\mathrm{st}(F)$ exists and belongs to D. Then the result follows from the definition of SDJ functions. □

Let $\mathcal{J}$ be the topology on D where a sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathcal{J}$-convergent to x if there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ where $\lambda_n : [0,1] \to [0,1]$ is continuous for all $n \in \mathbb{N}$ and

$$\sup_{0\le t\le 1} |\lambda_n(t) - t| \to 0 \quad \text{as} \quad n \to \infty$$
$$\sup_{0\le t\le 1} |x_n(t) - x(\lambda_n(t))| \to 0 \quad \text{as} \quad n \to \infty.$$

The topology $\mathcal{J}$ is called the Skorokhod topology on D, see [11], and a function in D is called a Cadlag function.

Proposition 1.4.5. *Let $F : {}^*[0,1] \to {}^*\mathbb{R}$ be a function in ${}^*D$ which is nearstandard in the $\mathcal{J}$-topology. Then F is SDJ and* st $|_{\mathrm{SDJ}}$ *is the standard part map for the $\mathcal{J}$-topology.*

Proof. Let $F : {}^*[0,1] \to {}^*\mathbb{R}$ be a function in ${}^*D$ such that F is nearstandard with respect to the $\mathcal{J}$-topology. Let $\mathrm{st}_{\mathcal{J}}$ be the standard part for the $\mathcal{J}$-topology. Because F is nearstandard $\mathrm{st}_{\mathcal{J}}(F) = f$ exists and is in D and by the Transfer principle ${}^*f$ is SDJ. By the definition of the $\mathcal{J}$-topology, there is a sequence $(\lambda_n)_{n\in{}^*\mathbb{N}}$ where $\lambda_n : {}^*[0,1] \to {}^*[0,1]$ are S-continuous internal functions such that for $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ we have $\lambda_N(\underline{t}) \approx \underline{t}$ and $F(\lambda_N(\underline{t})) \approx {}^*f(\underline{t})$ for all $\underline{t} \in {}^*[0,1]$ and therefore F is SDJ.

Suppose that F is a nearstandard SDJ function with $\mathrm{st}(F) = f$. Let $0 = t_0 < t_1 < \ldots < t_k = 1$ be the points in the interval $[0,1]$ where f is not continuous. By Proposition 1.4.4 there are points $\underline{t}_i \approx t_i$ such that whenever $\underline{t} \approx t_i$ and $\underline{t} \geq \underline{t}_i$ then $F(\underline{t}) \approx f(t_i)$ and whenever $\underline{t} \approx t_i$ and $\underline{t} < \underline{t}_i$ then $F(\underline{t}) = \lim_{s\nearrow t_i} f(s)$. Define $\lambda : {}^*[0,1] \to {}^*[0,1]$ by $\lambda(t_i) = \underline{t}_i$ and interpolating linearly. Then $\lambda(\underline{t}) \approx \underline{t}$ for all $\underline{t} \in {}^*[0,1]$ and if $\underline{t} \approx t_i$ and $\underline{t} \geq t_i$ then $F(\lambda(\underline{t})) \approx f(t_i) \approx {}^*f(\underline{t})$ and if $\underline{t} \approx t_i$ with $\underline{t} < t_i$ then $F(\lambda(\underline{t})) \approx \lim_{s\nearrow t_i} f(s) \approx {}^*f(\underline{t})$. Because f is continuous between the points t_i we have

$$\left| f(\mathrm{st}(\underline{t})) - \lim_{\underline{s}\nearrow\underline{t}} f(\mathrm{st}(\underline{s})) \right| = 0$$

and therefore by Proposition 1.4.4

$$\left| F(\lambda(\underline{t})) - {}^*f(\underline{t}) \right| \approx 0.$$

Therefore $F \in \mathrm{m}(f)$ and $\mathrm{st}_{\mathcal{J}}(F) = f$. □

Definition 1.4.6. Let T be an internal subset of ${}^*[0,1]$ such that $\{\mathrm{st}(\underline{t}) : \underline{t} \in T\} = [0,1]$. A stochastic process $X : T \times \Omega \to {}^*\mathbb{R}$ is of class SDJ if $\underline{t} \mapsto X(\underline{t}, \omega)$ is *of class SDJ* for almost all $\omega \in \Omega$.

Definition 1.4.7. i. Let $X : \Omega \times T \to {}^*\mathbb{R}$ be an internal stochastic process of class SDJ. The process $\mathrm{st}(X)$ is defined by

$$\mathrm{st}(X)(t) = \begin{cases} \mathrm{st}(X(\cdot,\omega))(t) & \text{if} \quad X(\cdot,\omega) \quad \text{is SDJ} \\ x_0 & \text{otherwise} \end{cases}$$

for some $x_0 \in \mathbb{R}$.

ii. A *SDJ lifting* of a stochastic process $x : [0,1]\times\Omega \to \mathbb{R}$ is a stochastic process X such that X is internal, SDJ and

$$x(t,\cdot) = \mathrm{st}(X)(t)$$

for almost all $\omega \in \Omega$ and for all $t \in [0,1]$.

Theorem 6. *A stochastic process $x : [0,1]\times\Omega \to \mathbb{R}$ has sample paths in D if and only if x has a SDJ lifting $X : T\times\Omega \to {}^*\mathbb{R}$ where $T \subset {}^*[0,1]$ such that* $\{\mathrm{st}(\underline{t}) : \underline{t} \in T\} = [0,1]$.

For the proof we use the following result, which is proved in [7]:
Let $(\Omega_1, \mathcal{A}_1, \mathcal{P}_1)$ and $(\Omega_2, \mathcal{A}_2, \mathcal{P}_2)$ be internal probability spaces, with corresponding Loeb spaces $\left(\Omega_1, \mathcal{L}(\mathcal{A}_1), \mathcal{P}_1^{\mathcal{L}}\right)$ and $\left(\Omega_2, \mathcal{L}(\mathcal{A}_2), \mathcal{P}_2^{\mathcal{L}}\right)$.

Let $\left(\Omega_1\times\Omega_2, \mathcal{L}(\mathcal{A}_1)\times\mathcal{L}(\mathcal{A}_2), \mathcal{P}_1^{\mathcal{L}}\times\mathcal{P}_2^{\mathcal{L}}\right)$ be the product space of the two Loeb spaces and let $\mathcal{A}_1\times\mathcal{A}_2$ be the internal algebra generated by the Cartesian product of $\mathcal{A}_1$ and $\mathcal{A}_2$. Then

$$\mathcal{L}(\mathcal{A}_1)\times\mathcal{L}(\mathcal{A}_2) \subseteq \mathcal{L}(\mathcal{A}_1\times\mathcal{A}_2)$$
$$(\mathcal{P}_1\times\mathcal{P}_2)^{\mathcal{L}}\,|_{\mathcal{L}(\mathcal{A}_1)\times\mathcal{L}(\mathcal{A}_2)} = \mathcal{P}_1^{\mathcal{L}}\times\mathcal{P}_2^{\mathcal{L}}.$$

Proof. ($\Rightarrow$) Let $N \in {}^*\mathbb{N}\setminus\mathbb{N}$ and let C be the internal algebra generated by internal unions of $\left[\frac{i}{N}, \frac{i+1}{N}\right)$ for $i \in \{0,\dots,N\}$. Let λ be the additive set function defined by

$$\lambda\left(\left[\frac{i}{N}, \frac{i+1}{N}\right)\right) = \frac{1}{N}.$$

Then $({}^*[0,1], C, \lambda)$ is an internal probability space and let $\left({}^*[0,1], \mathcal{L}(C), \lambda^{\mathcal{L}}\right)$ its Loeb space.

Define $x_1 : {}^*[0,1]\times\Omega \to \mathbb{R}$ by $x_1(\underline{t},\omega) = x(\mathrm{st}(\underline{t}),\omega)$. Then for $a \in \mathbb{R}$

$$\left\{(\underline{t},\omega) \in {}^*[0,1]\times\Omega : x_1(\mathrm{st}(\underline{t}),\omega) \le a\right\} =$$
$$= \left\{(\underline{t},\omega) \in {}^*[0,1]\times\Omega : x(\underline{t},\omega) \le a\right\} \in \mathcal{L}(\mathcal{A})\times\mathcal{L}(C) \subseteq \mathcal{L}(\mathcal{A}\times C).$$

Therefore x_1 is a Loeb measurable random variable with respect to the Loeb space $\left({}^*[0,1]\times\Omega, \mathcal{L}(\mathcal{A}\times C), (\mathcal{P}\times\lambda)^{\mathcal{L}}\right)$. By Theorem 5 the function x_1 has a lifting $X : {}^*[0,1]\times\Omega \to {}^*\mathbb{R}$. Because X is a lifting of x_1 we have

$$\mathrm{st}(X(\underline{t},\omega)) = x_1(\underline{t},\omega)$$

for $(\mathcal{P}\times\lambda)^{\mathcal{L}}$ almost all $(\underline{t},\omega) \in {}^*[0,1]\times\Omega$. For all $t \in [0,1]$ there is a $\underline{t} \in {}^*[0,1]$ such that $t \approx \underline{t}$ and $\mathrm{st}(X(\underline{t},\omega)) = x_1(\underline{t},\omega)$ otherwise $\mathrm{st}(X(\underline{t},\omega)) \ne x_1(\underline{t},\omega)$ for all $\underline{t} \in \mathrm{m}(t)$ and by overflow for all $\underline{t} \in {}^*[t-\delta, t+\delta]$ for some $\delta \in \mathbb{R}^+$, which is not a $\lambda^{\mathcal{L}}$ null set. Therefore

$$\mathrm{st}(X(\cdot,\omega))(t) = x(t,\omega)$$

for all $t \in [0,1]$ and for almost all $\omega \in \Omega$.

It remains to show that X is SDJ. $\mathrm{st}(X(\cdot,\omega))$ is in D for almost all ω, therefore $X(\cdot,\omega)$ is in ${}^*D$ for almost all ω and nearstandard with respect to the $\mathcal{J}$-topology. By Proposition 1.4.5 the stochastic process $X(\cdot,\omega)$ is SDJ and $X' = X\mid_{T\times\Omega}$ is the desired lifting of x.

($\Leftarrow$) Let $X : T\times\Omega \to {}^*\mathbb{R}$ be a SDJ lifting of x. Because X is SDJ we have for almost all $\omega \in \Omega$ that $X(\cdot,\omega) : T \to {}^*\mathbb{R}$ is SDJ. Then for all $t \in [0,1]$ and for almost all ω we have $\mathrm{st}(X(\cdot,\omega))(t) = x(t,\omega)$ and by Proposition 1.4.3 the process $x(\cdot,\omega)$ is in D for almost all $\omega \in \Omega$. □

Definition 1.4.8. Let $(\Omega,\mathcal{A},\mathcal{P})$ be an internal probability space, with Loeb space $(\Omega,\mathcal{L}(\mathcal{A}),\mathcal{P}^{\mathcal{L}})$. Let $\mathcal{N}$ be the set of all Loeb null sets, and let $T \subset {}^*[0,1]$ such that $\{\mathrm{st}(\underline{t}) : \underline{t} \in T\} = [0,1]$. Let $(\mathcal{A}_{\underline{t}})_{\underline{t}\in T}$ be an internal filtration on $(\Omega,\mathcal{A},\mathcal{P})$. Then, the standard part of the filtration $(\mathcal{A}_{\underline{t}})_{\underline{t}\in T}$ is defined by the filtration $(\mathcal{F}_t)_{t\in[0,1]}$ where

$$\mathcal{F}_t = \sigma\left(\bigcup_{\underline{s}\approx t,\underline{s}\in T} \mathcal{L}(\mathcal{A}_{\underline{s}}) \cup \mathcal{N}\right).$$

Lemma 1.4.9. *Let $(\mathcal{F}_t)_{t\in[0,1]}$ be the standard part of an internal filtration $(\mathcal{A}_{\underline{t}})_{\underline{t}\in T}$. Then for all $t \in [0,1]$*

$$\mathcal{F}_t = \bigcup_{\underline{s}\approx t} \sigma(\mathcal{L}(\mathcal{A}_{\underline{s}}) \cup \mathcal{N}).$$

Proof. For $n \in \mathbb{N}$ let $A_n \in \bigcup_{\underline{s}\approx t}\sigma(\mathcal{L}(A_{\underline{s}})\cup\mathcal{N})$ and assume that $A_n \in \sigma(\mathcal{L}(A_{\underline{s}_n})\cup\mathcal{N})$. Let $S_n = [\underline{s}_n, \underline{s}_n + \frac{1}{n}] \cap T$. Because $\underline{s}_n \approx t$ for all $n \in \mathbb{N}$ the family $(S_n)_{n\in\mathbb{N}}$ has the finite intersection property and therefore $\bigcap_{n\in\mathbb{N}} S_n \neq \emptyset$. Let $\underline{t} \in \bigcap_{n\in\mathbb{N}} S_n$, then $\underline{t} \approx t$ and $\underline{t} \geq \underline{s}_n$ for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned}\bigcup_{n\in\mathbb{N}} A_n \in \sigma(\mathcal{L}(\mathcal{A}_{\underline{t}}) \cup \mathcal{N}) \subseteq \\ \subseteq \bigcup_{\underline{s}\approx t}\sigma(\mathcal{L}(\mathcal{A}_{\underline{s}}) \cup \mathcal{N})\end{aligned}$$

and therefore $\bigcup_{\underline{s}\approx t}\sigma(\mathcal{L}(\mathcal{A}_{\underline{s}}) \cup \mathcal{N})$ is a σ-algebra for all $t \in [0,1]$, and the result follows from

$$\bigcup_{\underline{s}\approx t}\sigma(\mathcal{L}(\mathcal{A}_{\underline{s}}) \cup \mathcal{N}) \subseteq \mathcal{F}_t \subseteq \sigma\left(\bigcup_{\underline{s}\approx t}\sigma(\mathcal{L}(\mathcal{A}_{\underline{s}}) \cup \mathcal{N})\right).$$

□

By Definition 1.4.8 and Lemma 1.4.9 it is easy to see that $(\mathcal{F}_t)_{t\in[0,1]}$ is a filtration and that $(\mathcal{F}_t)_{t\in[0,1]}$ satisfies the usual conditions of a filtration, i.e. $(\mathcal{F}_t)_{t\in[0,1]}$ is right continuous and $\mathcal{F}_t$ contains all the null sets of $\mathcal{F}_1$ for every $t \in [0,1]$.

Proposition 1.4.10. *Let $(\mathcal{A}_{\underline{t}})_{\underline{t}\in T}$ be an internal filtration on an internal probability space $(\Omega, \mathcal{A}, \mathcal{P})$ and let $(\mathcal{F}_t)_{t\in[0,1]}$ be its standard part. Let $x : \Omega \to \mathbb{R}$ be an $\mathcal{F}_t$-measurable random variable. Then there is $\underline{t} \in T$ with $\underline{t} \approx t$ and an internal random variable $X : \Omega \to {}^*\mathbb{R}$ such that x is $\mathcal{A}_{\underline{t}}$-measurable and $\mathrm{st}(X) = x$ for almost all $\omega \in \Omega$.*

Proof. Let $(\underline{t}_n)_{n\in\mathbb{N}}$ be a decreasing sequence in T with $0 < \mathrm{st}(\underline{t}_n) - t < \frac{1}{n}$. Because x is $\mathcal{F}_t$-measurable and $\mathcal{F}_t \subseteq \sigma\left(\mathcal{L}(\mathcal{A}_{\underline{s}}) \cup \mathcal{N}\right)$ for $\mathrm{st}(\underline{s}) > t$ we have that x is $\sigma\left(\mathcal{L}(\mathcal{A}_{\underline{t}_n}) \cup \mathcal{N}\right)$-measurable, for all $n \in \mathbb{N}$. By Theorem 5 there is an $\mathcal{A}_{\underline{t}_n}$-measurable random variable $X_n : \Omega \to {}^*\mathbb{R}$ such that $\mathrm{st}(X) = x$ for almost all $\omega \in \Omega$. By Saturation, there is $\underline{t}_N \approx t$ for some infinite $N \in {}^*\mathbb{N}$ and a $\mathcal{A}_{\underline{t}_N}$-measurable random variable X such that $\mathrm{st}(X) = x$ for almost all $\omega \in \Omega$. □

Theorem 7. *Let $(\mathcal{A}_{\underline{t}})_{\underline{t}\in T}$ be an internal filtration with standard part $(\mathcal{F}_t)_{t\in[0,1]}$. Let $x : \Omega \times [0, 1] \to \mathbb{R}$ be a stochastic process. x is $(\mathcal{F}_t)_{t\in[0,1]}$-adapted and has almost all sample path in D if and only if x has an SDJ lifting $X : T \times \Omega \to {}^*\mathbb{R}$ that is $(\mathcal{A}_{\max\{\underline{t},\underline{\delta}\}})_{\underline{t}\in T}$-adapted for some positive infinitesimal $\underline{\delta} \in T$.*

Proof. ($\Rightarrow$) By Theorem 6 the stochastic process x has a SDJ lifting $X : T \times \Omega \to {}^*\mathbb{R}$. Let $\{t_i : i \in \mathbb{N}\}$ be a dense set of $[0, 1]$ such that $t_0 = 0$ and $x(t_i) = \lim_{s\searrow t_i} x(s)$ almost surely. For $i \in \mathbb{N}$ let $\underline{t}_i \in T$ such that $\underline{t}_0 = 0$ and $\underline{t}_i \approx t_i$. By the Saturation Principle, we can extend the set $\{\underline{t}_i : i \in \mathbb{N}\}$ to $\{\underline{t}_i : i \in {}^*\mathbb{N}\} \subseteq T$. Because x is continuous at t_i for almost all $\omega \in \Omega$ we have $\mathrm{st}(X(\underline{t}_i)) = x(t_i)$ for almost all ω and for all $i \in \mathbb{N}$. Define

$$\underline{\delta}_n = \max\left\{\underline{t} \in T : \underline{t} \leq \frac{1}{2^n}\right\}.$$

Because x is $(\mathcal{F}_t)_{t\in[0,1]}$-adapted, $\mathrm{st}(X(\underline{t}_i))$ is $\sigma(\mathcal{L}(\mathcal{A}_{\underline{t}_i+\underline{\delta}_n}) \cup \mathcal{N})$-measurable and therefore by the proof of Proposition 1.4.10 there is an $\mathcal{A}_{\underline{t}_i+\underline{\delta}_n}$-measurable random variable $Y^n(\underline{t}_i)$ such that $\mathrm{st}(Y^n(\underline{t}_i)) = \mathrm{st}(X(\underline{t}_i))$ almost surely. Let $0 = \underline{t}^n_0 \leq \underline{t}^n_1 \leq \ldots \leq \underline{t}^n_n$ be the elements of $\{\underline{t}_i : i \leq n\}$ and let

$$X^n(\underline{t}) = \begin{cases} Y^n(\underline{t}^n_i) & \text{if} \quad \underline{t} \in [\underline{t}^n_i, \underline{t}^{n+1}_i) \\ Y^n(\underline{t}^n_n) & \text{if} \quad \underline{t} \geq \underline{t}^n_n \end{cases} .$$

Then $X^n(\underline{t})$ is $\mathcal{A}_{\underline{t}+\underline{\delta}_n}$-measurable for all $\underline{t} \in T$ and for all $n \in \mathbb{N}$ and X^n is a SDJ lifting of x for each $n \in \mathbb{N}$. For $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ we have $X^N(\underline{t})$ is $\mathcal{A}_{\underline{t}+\underline{\delta}_N}$-measurable for all $\underline{t} \in T$ and $\mathrm{st}(X^N)(t) = x(t)$ for almost all $\omega \in \Omega$. Let

$$X^{'}(\underline{t}) = X^N(\max(\underline{t} - \underline{\delta}_N, 0))$$

then $X^{'}$ is $\mathcal{A}_{\max\{\underline{t},\underline{\delta}\}}$-measurable where $\underline{\delta} = \underline{\delta}_N \approx 0$ and $\underline{\delta} \in T$.

($\Leftarrow$) Let $X : T \times \Omega \to {}^*\mathbb{R}$ be an $(\mathcal{A}_{\max\{\underline{t},\underline{\delta}\}})_{\underline{t}\in T}$-adapted SDJ lifting of x. By Theorem 6 the process x has sample paths in D for almost all $\omega \in \Omega$. Let $t \in [0, 1]$. Because

$$\mathrm{st}(X(\cdot, \omega))(t) = \lim_{\mathrm{st}(\underline{t})\searrow t} \mathrm{st}(X(\underline{t}, \omega))$$

there is $\underline{t} \in T$ with $\underline{t} \approx t$ and $\underline{t} > t$ such that

$$\mathrm{st}(X(\cdot, \omega))(t) = \mathrm{st}(X(\underline{t}, \omega))$$

and therefore we have

$$x(t, \omega) = \mathrm{st}(X(\underline{t}, \omega))$$

for almost all ω. So for each $r \in \mathbb{R}$ we have

$$\begin{aligned}
\{\omega \in \Omega : \mathrm{st}(X(\underline{t}, \omega)) < r\} &= \bigcup_{k \in \mathbb{N}} \left\{\omega \in \Omega : X(\underline{t}, \omega) < a - \frac{1}{k}\right\} \\
&\Rightarrow \{\omega \in \Omega : \mathrm{st}(X(\underline{t}, \omega)) < r\} \in \mathcal{L}(\mathcal{A}_{\max\{\underline{t}, \underline{\delta}\}}) \\
&\Rightarrow \{\omega \in \Omega : x(t, \omega) < r\} \in \mathcal{F}_t.
\end{aligned}$$

□

Definition 1.4.11. An *internal stopping time* with respect to an internal filtration $(\mathcal{A}_{\underline{t}})_{\underline{t} \in T}$ is an internal function $\rho : \Omega \to T$ such that $\{\omega \in \Omega : \rho(\omega) \leq \underline{t}\} \in \mathcal{A}_{\underline{t}}$.

Theorem 8. *Let $(\mathcal{A}_t)_{\underline{t} \in T}$ be an internal filtration with standard part $(\mathcal{F}_t)_{t \in [0,1]}$. Then $\tau : \Omega \to [0, 1]$ is a stopping time with respect to $(\mathcal{F}_t)_{t \in [0,1]}$ if and only if $\tau = \mathrm{st}(\rho)$ a.s. for some internal $(\mathcal{A}_{\underline{t}})_{\underline{t} \in T}$-stopping time ρ.*

Proof. ($\Rightarrow$) Let $\tau : \Omega \to [0, 1]$ be an $(\mathcal{F}_t)_{t \in [0,1]}$-stopping time. Define $z : (0, 1) \times \Omega \to \{0, 1\}$ by

$$z(t, \omega) = \begin{cases} 1 & \text{if} \quad t \geq \tau(\omega) \\ 0 & \text{if} \quad t < \tau(\omega) \end{cases}.$$

Then z is $(\mathcal{F}_t)_{t \in [0,1]}$-adapted and $z(\cdot, \omega) \in D$ for all $t \in [0, 1]$. By Theorem 7 the function z has a SDJ lifting Z such that $Z(\underline{t}, \cdot)$ is $\mathcal{A}_{\max\{\underline{t}, \underline{\delta}\}}$-measurable for some infinitesimal $\underline{\delta} \in T$. Let

$$\rho'(\omega) = \min\{\underline{t} : Z(\underline{t}, \omega) = 1\}.$$

Then $\mathrm{st}(\rho') = \tau$ for almost all $\omega \in \Omega$ and for $\underline{t} \in T$ we have

$$\{\omega \in \Omega : \rho'(\omega) \leq \underline{t}\} = \{\omega \in \Omega : Z(\underline{t}, \omega) = 1\} \in \mathcal{A}_{\max\{\underline{t}, \underline{\delta}\}}.$$

Therefore ρ' is an $(\mathcal{A}_{\max\{\underline{t}, \underline{\delta}\}})_{\underline{t} \in T}$-stopping time. Let $\rho(\omega) = \max\{\rho'(\omega), \underline{\delta}\}$, then ρ is an $(\mathcal{A}_{\underline{t}})_{\underline{t} \in t}$-stopping time with $\mathrm{st}(\rho) = \tau$ for almost all ω.

($\Leftarrow$) Let $\rho : \Omega \to T$ be an $(\mathcal{A}_{\underline{t}})_{\underline{t} \in T}$-stopping time, such that $\tau = \mathrm{st}(\rho)$ for almost all $\omega \in \Omega$. Let

$$Z(\underline{t}, \omega) = \begin{cases} 1 & \text{if} \quad \underline{t} \geq \rho(\omega) \\ 0 & \text{if} \quad \underline{t} < \rho(\omega) \end{cases}.$$

Then Z is SDJ and $(\mathcal{A}_{\underline{t}})_{\underline{t} \in T}$-adapted and Z is a lifting of

$$z(t, \omega) = \begin{cases} 1 & \text{if} \quad t \geq \tau(\omega) \\ 0 & \text{if} \quad t < \tau(\omega) \end{cases}.$$

By Theorem 7 the function z is $(\mathcal{F}_t)_{t\in[0,1]}$-adapted. Then

$$\{\omega \in \Omega : \rho(\omega) \leq t\} = \{\omega \in \Omega : z(t,\omega) = 1\} \in \mathcal{F}_t$$

and ρ is a $(\mathcal{F}_t)_{t\in[0,1]}$ stopping time. □

Chapter 2

Mathematical Finance

2.1 The Cox-Ross-Rubinstein Model

The Cox-Ross-Rubinstein model or binomial model is a mathematical description of financial markets in discrete time and was first proposed by Cox, Ross and Rubinstein in 1979, [12]. Let $n \in \mathbb{N}$, $\Delta_n = \frac{1}{n}$ and $T_n = \{k\Delta_n : k = 0, \ldots, n\}$. Let Ω_n be the set of all ω with:

$$\begin{aligned} \omega(0) &= 0 \\ \omega(t + \Delta_n) &= \omega(t) \pm \sqrt{\Delta_n} \quad \text{for} \quad t \in T_n \setminus \{1\}. \end{aligned}$$

Let $\mathcal{A}_n = \mathfrak{P}(\Omega_n)$ and let C_n be the *normalized counting measure* on Ω_n defined by

$$C_n(A) = \frac{|A|}{2^n} \quad \text{for} \quad A \in \mathcal{A}_n.$$

So we have for every $n \in \mathbb{N}$ a probability space $(\Omega_n, \mathcal{A}_n, C_n)$. For $\omega \in \Omega_n$ and $t \in T_n$ define $B(t, \omega) := \omega(t)$ and let $\Delta B(t, \omega) = B(t + \Delta_N, \omega) - B(t, \omega) = \Delta\omega(t)$. In the Cox-Ross-Rubinstein (CRR) model we describe the evolution of a stock $S_n(t)$ with $t \in T_n$. A *stock* $S_n : T_n \times \Omega_n \to \mathbb{R}^+$ is a price process defined by the initial value $S_n(0) = s_0$ which is assumed to be given and by the process:

$$\begin{aligned} S_n(t + \Delta_n, \omega) &= S_n(t, \omega)(1 + \mu\Delta_n + \sigma\Delta B(t, \omega)) \\ \Delta S_n(t, \omega) &= S_n(t + \Delta_n, \omega) - S_n(t, \omega) = \\ &= S_n(t, \omega)(\mu\Delta_n + \sigma\Delta B(t, \omega)) \end{aligned}$$

where we assume that $\sigma > 0$ and $1 + \mu\Delta_n \pm \sigma\sqrt{\Delta_n} > 0$. We call μ the drift and σ the volatility of the price process. Next, we introduce another component in the CRR-model: A *bond* is a price process with an initial value and with a fixed interest rate r. We take 1 for the initial value of the bond and we can assume without loss of generality that $r = 0$, by discounting to current prices.

A *portfolio-process* is a predictable process $\Theta = (\Theta^b(t), \Theta^s(t)) : T_n \to \mathbb{R}^2$ where Θ^s denotes the number of units of a stock and Θ^b the number of units of the bond at time t. The value of a portfolio at time t is given by

$$V(t,\omega) = \Theta^b(t,\omega) + \Theta^s(t,\omega)S_n(t,\omega).$$

The function $\Theta : T_n \to \mathbb{R}^2$ gives a *trading strategy*, which is called *self-financing* if

$$\Delta V(t,\omega) = \Theta^s(t,\omega)\Delta S_n(t,\omega)$$

for all $t \in T_n$ and for all $\omega \in \Omega_n$. Self financing means that all changes of the value of the portfolio come from changes of the stock i.e. no money is added or removed during the whole time $[0,1]$.

A *European call option* with strike price K and exercise time T is the right to buy one unit of a stock at a fixed date T, for the fixed price K. We can describe the European call option in the CRR model by the random variable

$$C(\omega) = (S_n(1,\omega) - K)^+$$

where $T = 1$. The question is to determine the fair price at which such an option should be traded at time $t = 0$.

Theorem 9. *Let $C(\omega) = (S_n(1,\omega) - K)^+$. Then there exists a unique self-financing trading strategy Θ such that for all $\omega \in \Omega_n$ we have $C(\omega) = V(1,\omega)$, where $V(1,\omega)$ is the value of Θ at time 1.*

Proof.

$$C(\omega) = V(1,\omega) = V(0) + \sum_{t<1} \Theta^s(t,\omega)\Delta S_n(t,\omega).$$

For $t = k\Delta_n$ we have 2^k unknown values for Θ^s. Together with $V(0)$ there are 2^n unknown values in 2^n independent equations. We show that there is a unique solution:. For the values $\Theta^s(1-\Delta_n,\omega)$ we obtain that for all $\omega \in \Omega_n$ there is $\hat{\omega} \in \Omega_n$ such that $\omega(t) = \hat{\omega}(t)$ for all $t \in T_n \setminus \{1\}$, $\Delta S_n(t,\omega) = \Delta S_n(t,\hat{\omega})$ for all $t \in T_n \setminus \{1, 1-\Delta_n\}$ and $\Delta S_n(1-\Delta_n,\omega) \neq \Delta S_n(1-\Delta_n,\hat{\omega})$. So the 2^{n-1} values for $\Theta^s(1-\Delta_n,\omega)$ are uniquely determined. For the values of $\Theta^s(1-2\Delta_n,\omega)$ we have that for all $\omega \in \Omega_n$ there is $\tilde{\omega} \neq \hat{\omega}$ such that $\omega(t) = \tilde{\omega}(t)$ for all $t \in T_n \setminus \{1, 1-\Delta_n\}$, $\Delta S_n(t,\omega) = \Delta S_n(t,\tilde{\omega})$ for all $t \in T_n \setminus \{1, 1-\Delta_n, 1-2\Delta_n\}$ and $\Delta S_n(1-2\Delta_n,\omega) \neq \Delta S_n(1-2\Delta_n,\tilde{\omega})$. Hence the 2^{n-2} values for $\Theta^s(1-2\Delta_n,\omega)$ are uniquely determined. Because the same scheme works for all values of $t \in T_n$ the $2^n - 1$ values of Θ^s are uniquely determined and therefore $V(0)$ is uniquely determined. □

With $\Pi(C) := V(0)$ we define the *fair price* of an option $C(\omega)$, where V is the value of the self-financing trading strategy Θ, which generates C.

Theorem 10. *There exists a unique probability measure Q_n on Ω_n such that $S_n(t,\omega)$ is a martingale with respect to Q_n.*

Proof. For $t \in T_n$ let $\mathcal{A}_t$ be the σ-algebra, generated by the sets of the form $\mathcal{M}_\omega^t = \{\omega' \in \Omega_n : \omega'(s) = \omega(s) \quad \forall s < t\}$. For $s < t$ we have $\mathcal{A}_s \subset \mathcal{A}_t \subset \mathcal{A}_n$ and therefore $(\mathcal{A}_t)_{t\in T}$ is a filtration on $(\Omega_n, \mathcal{A}_n, C_n)$. For $\omega \in \Omega_n$ let

$$Q_n(\omega) = \prod_{t\in T_n} \frac{1}{2}\left(1 - \Delta\omega(t)\frac{\mu}{\sigma}\right).$$

Since

$$\begin{aligned}
Q_n(\Omega_n) &= \sum_{\omega\in\Omega_n} \prod_{t\in T_n} \frac{1}{2}\left(1 - \Delta\omega(t)\frac{\mu}{\sigma}\right) = \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta_n}\right)^k \left(\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta_n}\right)^{n-k} = \\
&= \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta_n} + \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta_n}\right)^n = 1
\end{aligned}$$

and for $A_i \in \mathcal{A}_n$ with $A_i \cap A_j = \emptyset$ we have

$$\begin{aligned}
Q_n\left(\bigcup_{i\in\mathbb{N}} A_i\right) &= \sum_{\omega\in\bigcup_{i\in\mathbb{N}} A_i} Q_n(\omega) = \\
&= \sum_{i\in\mathbb{N}} Q_n(A_i)
\end{aligned}$$

Q_n is a probability measure on Ω_n. For $t \in T_N$ let $S_n(t)$ be the function $\omega \to S_n(t,\omega)$, let B_t be the function $\omega \to B(t,\omega)$ and let id_t be the function $\omega \to \frac{1}{\sqrt{\Delta_N}}\Delta\omega(t)$. Note that the σ-algebra $\mathcal{A}_t$ and the function id_t are independent. This implies that

$$\mathbb{E}_{Q_n}(id_t|\mathcal{A}_t) = \mathbb{E}_{Q_n}(id_t) = \frac{1}{2}(1 - \frac{\mu}{\sigma}\sqrt{\Delta_n}) - \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta_n})$$

and so for $t \in T_n$ we have

$$\begin{aligned}
\mathbb{E}_{Q_n}(S_n(t+\Delta_n)|\mathcal{A}_t) &= S_n(t)\mathbb{E}_{Q_n}(1 + \mu\Delta_n + \sigma\Delta B_t|\mathcal{A}_t) = \\
&= S_n(t)(1 + \mu\Delta_n + \sigma\sqrt{\Delta_n}\mathbb{E}_{Q_n}(id_t|\mathcal{A}_t)) = \\
&= S_n(t)(1 + \mu\Delta_n + \sigma\sqrt{\Delta_n}\mathbb{E}_{Q_n}(id_t)) = \\
&= S_n(t)(1 + \mu\Delta_n - \mu\Delta_n) = \\
&= S_n(t).
\end{aligned}$$

This shows that $S_n(t)$ is a martingale with respect to Q_n.

To show that Q_n is the only measure with this property, let Q'_n be another probability measure such that S_n is a martingale with respect to Q'_n. Then we have

$$\begin{aligned}
&\mathbb{E}_{Q'_n}(S_n(t+\Delta_n)|\mathcal{A}_t) = \mathbb{E}_{Q_n}(S_n(t+\Delta_n)|\mathcal{A}_t) \\
&\Rightarrow \mathbb{E}_{Q'_n}(id_t) = \mathbb{E}_{Q_n}(id_t) \\
&\Rightarrow Q_n(\omega) = Q'_n(\omega) \quad \forall\omega \in \Omega.
\end{aligned}$$

Therefore the probability measure Q_n is unique. $\square$

Because S_n is a martingale with respect to Q_n we have $\mathbb{E}_{Q_n}(S_n(t)) = s_0$ and $\mathbb{E}_{Q_n}(\Delta S_n(t)) = \mathbb{E}_{Q_n}(S_n(t+\Delta_n) - S_n(t)) = 0$ for all $t \in T_n$. Because $\Theta^s(t,\omega)$ and $\Delta S_n(t,\omega)$ are independent random variables, we obtain

$$\begin{aligned}
\mathbb{E}_{Q_n}(C(\cdot)) &= \mathbb{E}_{Q_n}\left(V(0) + \sum_{s<t} \Theta^s(t)\Delta S_n(t)\right) = \\
&= \mathbb{E}_{Q_n}(V(0)) = V(0) = \\
&= \Pi(C).
\end{aligned}$$

So the fair price $\Pi(C)$ of an option $C(\cdot)$ is given by $\mathbb{E}_{Q_n}(C(.))$. The next theorem gives an exact formula for $\Pi(C)$. For this we use the following notations:

$$\begin{aligned}
u &= 1 + \mu\Delta_n + \sigma\sqrt{\Delta_n} \\
v &= 1 + \mu\Delta_n - \sigma\sqrt{\Delta_n} \\
q &= \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta_n} \\
M &= \{\omega \in \Omega_n : S_n(1,\omega) > K\}
\end{aligned}$$

and let m be the last integer with $s_0 u^m v^{n-m} > K$

Theorem 11. *The fair price $\Pi(C)$ for a European call option $C(\omega) = (S_n(1,\omega) - K)^+$ in the CRR-model is given by*

$$\Pi(C) = s_0 \sum_{k=m}^{n} \binom{n}{k} (uq)^k (1-uq)^{n-k} - K \sum_{k=m}^{n} \binom{n}{k} q^k (1-q)^{n-k}.$$

Proof.

$$\begin{aligned}
\Pi(C) &= \mathbb{E}_{Q_n}\left((S_n(1) - K)^+\right) = \\
&= \sum_{\omega\in\Omega_n} (S_n(1) - K)\mathbf{1}_M(\omega) Q_n(\omega) = \\
&= \sum_{\omega\in\Omega_n} \left(s_0 \prod_{t\in T_n}(1 + \sigma\Delta\omega(t) + \mu\Delta_n) - K\right)\mathbf{1}_M(\omega) \prod_{t\in T_n} \frac{1}{2}(1 - \Delta\omega(t)\frac{\mu}{\sigma}\sqrt{\Delta_n}) = \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(S(0)u^k v^{n-k} - K\right) \mathbf{1}_{\{S(0)u^k v^{n-k} > K\}}(k) q^k (1-q)^{n-k} = \\
&= s_0 \sum_{k=m}^{n} \binom{n}{k} (uq)^k (1-uq)^{n-k} - K \sum_{k=m}^{n} \binom{n}{k} q^k (1-q)^{n-k}.
\end{aligned}$$

□

2.2 The Black-Scholes Model

The model was first articulated by Black and Scholes in 1973, see [13]. Let $b(t,\omega)$ be a Brownian motion on the interval $[0,1]$. Let $\Omega = C_0[0,1]$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and let C be the Wiener measure on Ω defined by

$$C(A) = \mathcal{P}(b(\cdot,\omega) \in A)$$

for all $A \in \mathcal{A}$, where $\mathcal{A}$ denotes the Borel σ-algebra. The price of a *stock* is assumed to follow the stochastic differential equation

$$\mathrm{d}s_t = \sigma s_t \mathrm{d}b_t + \mu s_t \mathrm{d}t \tag{2.1}$$

where $\sigma > 0$ is the volatility and $\mu \in \mathbb{R}$ is the drift of s. Let s_0, the initial value of the stock, be given. By using Ito's formula, the solution of this equation is given by

$$s_t = s_0 e^{\sigma b_t + (\mu - \frac{1}{2}\sigma^2)t}.$$

The other component in the Black-Scholes model is the price of a *bond* with a fixed interest rate r. By discounting to current prices we may as in the discrete case without loss of generality assume that $r = 0$ and let 1 be the initial price of the bond. As in the CRR-model, we denote a *trading strategy* by $\theta = (\theta^s, \theta^b)$, where $\theta^s(t,\cdot)$ denotes the number of units of the stock and $\theta^b(t,\cdot)$ denotes the number of units of the bond. We assume that the processes $\theta^s, \theta^b : [0,1] \times \Omega \to \mathbb{R}$ are adapted with respect to the filtration $(\mathcal{A}_t)_{t\in[0,1]}$ generated by the Brownian motion b_t. The value of the portfolio $\theta = (\theta^s, \theta^b)$ at time t is given by

$$v(t,\omega) = \theta^b(t,\omega) + \theta^s(t,\omega)s(t,\omega).$$

A trading strategy θ is called *self-financing* if

$$v(t,\omega) = v(0) + \int_0^t \theta^s(u,\omega)\mathrm{d}s(u,\omega)$$

for all $t \in [0,1]$. The random variable $c(\omega) = (s_1(\omega) - K)^+$ describes an European call option in the BS-model with strike price K and exercise time $T = 1$. To determine the fair price $\pi(c)$ we use a risk neutral measure, this means a measure that makes the price process $s(t,\omega)$ a martingale.

Theorem 12. *There exists a unique probability measure Q on Ω such that $s(t,\omega)$ is a martingale under Q.*

Proof. The proof is an application of Girsanovs Theorem, which implies that

$$\tilde{b}(t) = b(t) + \frac{\mu}{\sigma}t$$

is a Brownian motion on the probability space $(\Omega, \mathcal{A}, Q)$ where the probability measure Q is defined by

$$Q(A) = \int_A e^{-\frac{\mu}{\sigma}b_1(\omega)-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2} \mathrm{d}C(\omega) \quad \forall A \in \mathcal{A}.$$

Because

$$\mathrm{d}\tilde{b}_t = \frac{\mu}{\sigma}\mathrm{d}t + \mathrm{d}b_t$$

the stochastic differential equation (2.1) changes to

$$\begin{aligned}
\mathrm{d}s_t &= \sigma s_t \mathrm{d}b_t + \mu s_t \mathrm{d}t = \\
&= \sigma s_t \left(\mathrm{d}\tilde{b}_t - \frac{\mu}{\sigma}\mathrm{d}t\right) + \mu s_t \mathrm{d}t = \\
&= \sigma s_t \mathrm{d}\tilde{b}_t
\end{aligned}$$

and therefore s_t is a martingale with respect to Q. □

Theorem 13. *Let $c(\omega) = (s_1(\omega) - K)^+$. Then there exists a unique self-financing trading strategy θ such that for Q-almost all $\omega \in \Omega$ we have $C(\omega) = v(1, \omega)$, where $v(1, \omega)$ is the value of Θ at time 1.*

Proof. Let $w(1)$ be an $\mathcal{A}(1)$-measurable random variable and define

$$w(t) = \mathbb{E}_Q\left(w(1)|\mathcal{A}(t)\right)$$

for $0 \le t \le 1$. Then $(w(t))_{t\in[0,1]}$ is a Q-martingale because it is clear that $w(t)$ is $\mathcal{A}_t$-measurable and for $s < t$ we have

$$\begin{aligned}
\mathbb{E}_Q\left(w(s)|\mathcal{A}(t)\right) &= \mathbb{E}_Q\left(\mathbb{E}_Q\left(w(1)|\mathcal{A}(s)\right)|\mathcal{A}(t)\right) = \\
&= \mathbb{E}_Q\left(w(1)|\mathcal{A}(t)\right) = \\
&= w(t).
\end{aligned}$$

By the Martingale Representation Theorem, there is an adapted process $(\Gamma_t)_{t\in[0,1]}$ such that

$$w(t) = w(0) + \int_0^t \Gamma(u)\mathrm{d}\tilde{b}(u). \tag{2.2}$$

The value of a self-financing portfolio $\theta = (\theta^s, \theta^b)$ at time t is given by

$$v(t, \omega) = v(0) + \int_0^t \Theta^s(u, \omega)\mathrm{d}s(u, \omega)$$

and we know from the proof of Theorem 12 that

$$\begin{aligned}
\mathrm{d}s_t &= \sigma s_t \mathrm{d}\tilde{b}_t \\
\Rightarrow v(t, \omega) &= v(0) + \int_0^t \Theta^s(u, \omega)\sigma s_u \mathrm{d}\tilde{b}_u.
\end{aligned}$$

Let $v(t) = w(t)$ for all $t \in [0,1]$, then

$$\begin{aligned} \Gamma(t) &= \Theta^s(t)\sigma s(t) \\ \Rightarrow \Theta^s(t) &= \frac{\Gamma(t)}{\sigma s(t)} \end{aligned}$$

for all $t \in [0,1]$. $\Gamma(t)$ is a.s. unique because if there is an adapted process $(\overline{\Gamma}_t)_{t\in[0,1]}$, which satisfies (2.2), then

$$\begin{aligned} 0 = \mathbb{E}_Q\left(\int_0^t \Gamma(u)\mathrm{d}\tilde{b}_u - \int_0^t \overline{\Gamma}(u)\mathrm{d}\tilde{b}_u\right)^2 &= \mathbb{E}_Q\left(\int_0^t \Gamma(u) - \overline{\Gamma}(u)\mathrm{d}\tilde{b}_u\right)^2 = \\ &= \int_0^t \mathbb{E}_Q\left(\Gamma(u) - \overline{\Gamma}(u)\right)^2 \mathrm{d}u \end{aligned}$$

and therefore $\Gamma(u) = \overline{\Gamma}(u)$ for Q-almost all $\omega \in \Omega$ and Θ^s is a.s. unique. □

We use the self-financing trading strategy θ from Theorem 13 to define the fair price $\pi(c)$ of the option $c(\omega) = (s_1(\omega) - K)^+$ by the initial value $v(0)$ of this trading strategy. Because v_t is a martingale we have

$$\pi(c) = v(0) = E_Q(v(0)) = E_Q(v(1)) = E_Q((s_1 - K)^+).$$

To calculate $E_Q((s_1 - K)^+)$ we use that $\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\mathrm{d}x$, for a borel measurable function g and for a random variable X with density function f. Take $X = \tilde{b}_1$ and $g(X) = (s_0 e^{-\frac{1}{2}\sigma^2+\sigma X} - K)^+$ then

$$\begin{aligned} \mathbb{E}_Q((s_1 - K)^+) &= \int_{-\infty}^{\infty} (s_0 e^{-\frac{1}{2}\sigma^2+\sigma x} - K)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x = \\ &= \int_a^{\infty} (s_0 e^{-\frac{1}{2}\sigma^2+\sigma x} - K) \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \mathrm{d}x = \\ &= \int_a^{\infty} s_0 e^{-\frac{1}{2}\sigma^2+\sigma x} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \mathrm{d}x - K \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \mathrm{d}x \end{aligned}$$

where $a = \frac{\ln\left(\frac{K}{s_0}\right)+\frac{1}{2}\sigma^2}{\sigma}$ is the least number x such that $s_0 e^{-\frac{1}{2}\sigma^2+\sigma x} - K$ is non-negative.

$$\begin{aligned} \Rightarrow \mathbb{E}_Q((s_1 - K)^+) &= s_0\Phi(\sigma - a) - K\Phi(-a) = \\ &= s_0\Phi\left(\frac{-\ln\left(\frac{K}{s_0}\right) + \frac{1}{2}\sigma^2}{\sigma}\right) - K\Phi\left(\frac{-\ln\left(\frac{K}{s_0}\right) - \frac{1}{2}\sigma^2}{\sigma}\right) \end{aligned}$$

where Φ denotes the normal density function. Therefore the price $\pi(c)$ of a European call option in the Black-Scholes model is given by

$$\pi(c) = s_0\Phi\left(\frac{\ln\left(\frac{s_0}{K}\right) + \frac{1}{2}\sigma^2}{\sigma}\right) - K\Phi\left(\frac{\ln\left(\frac{s_0}{K}\right) - \frac{1}{2}\sigma^2}{\sigma}\right).$$

2.3 The hyperfinite CRR-Model

In Section 2.1 we defined the probability space $(\Omega_n, \mathcal{A}_n, C_n)$ for every $n \in \mathbb{N}$. Let N be an infinite natural number, then by the transfer principle there is an internal probability space $(\Omega_N, \mathcal{A}_N, C_N)$, where Ω_N is the set of all internal ω with

$$\begin{aligned} &\omega(0) = 0 \\ &\omega(t + \Delta_N) = \omega(t) \pm \sqrt{\Delta_N} \quad \text{for} \quad t \in T_N \setminus \{1\} \end{aligned}$$

where $\Delta_N = \frac{1}{N}$ and $T_N = \{k\Delta_N : k = 0, \ldots, N\}$. $\mathcal{A}_N = {}^*\mathfrak{P}(\Omega_N)$ is the set of all internal subsets of Ω_N and C_N is the counting measure on $\mathcal{A}_N$ defined by

$$C_N(A) = \frac{|A|}{2^N} \quad \forall A \in \mathcal{A}_N.$$

For $t \in T_N \setminus \{1\}$ define the *evolution of a stock* S_N by the *initial value* $S_N(0)$ and the price process

$$S_N(t + \Delta_N, \omega) = S_N(t, \omega)(1 + \mu\Delta_N + \sigma\Delta B(t, \omega))$$

where $\Delta B(t, \omega) = B(t + \Delta_N, \omega) - B(t, \omega)$ and $B(t, \omega)$ is given by $B(t, \omega) := \omega(t)$. An *European call option* with strike price K is given by the random variable

$$C(\omega) = (S_N(1, \omega) - K)^+.$$

To determine the fair price of such an option, we need again a measure Q_N such that the process S_N becomes a martingale under the measure Q_N.

Theorem 14. *There exists a unique internal probability measure Q_N on Ω_N such that $S_N(t, \omega)$ is an internal martingale with respect to Q_N.*

Proof. For $t \in T_N$ let $\mathcal{A}_t$ be the internal algebra generated by the sets $\mathcal{M}^t_\omega = \{\omega' \in \Omega_N : \omega'(s) = \omega(s) \quad \forall s < t\}$ for $\omega \in \Omega_N$. Then for $s < t$ we have $\mathcal{A}_s \subset \mathcal{A}_t \subset \mathcal{A}_N$ and therefore $(\mathcal{A}_t)_{t\in T}$ is a filtration on the probability space $(\Omega_N, \mathcal{A}_N, C_N)$. For $\omega \in \Omega_N$ let

$$Q_N(\omega) = \prod_{t\in T_N} \frac{1}{2}\left(1 - \Delta\omega(t)\frac{\mu}{\sigma}\right).$$

Then Q_N is an internal probability measure on $(\Omega_N, \mathcal{A}_N, C_N)$. For $t \in T_N$ let id_t given by $\omega \to \frac{1}{\sqrt{\Delta_N}}\Delta\omega(t)$. The random variable id_t and the internal algebra $\mathcal{A}_t$ are independent for $t \in T_N$ and therefore we have $\mathbb{E}_{Q_N}(id_t|\mathcal{A}_t) = E_{Q_N}(id_t)$ for all $t \in T_N$

$$\begin{aligned} \Rightarrow \mathbb{E}_{Q_N}(S_N(t + \Delta_N)|\mathcal{A}_t) &= S_N(t)\mathbb{E}_Q(1 + \mu\Delta_N + \sigma\Delta B_t|\mathcal{A}_t) = \\ &= S_N(t)\left(1 + \mu\Delta_N + \sigma\sqrt{\Delta_N}\mathbb{E}_{Q_N}(id_t|\mathcal{A}_t)\right) = \\ &= S_N(t)\left(1 + \mu\Delta_N + \sigma\sqrt{\Delta_N}\mathbb{E}_{Q_N}(id_t)\right) = \\ &= S_N(t)(1 + \mu\Delta_N - \mu\Delta_N) = \\ &= S_N(t). \end{aligned}$$

Therefore $(S_N(t))_{t\in T_N}$ is a martingale. To show that Q_N is unique, let Q'_N be another probability measure such that S_N is a martingale with respect to Q'_N. Then we have

$$\begin{aligned}
\mathbb{E}_{Q'_N}(S_N(t+\Delta_N)|\mathcal{A}_t) &= \mathbb{E}_{Q_N}(S_N(t+\Delta_N)|\mathcal{A}_t)\\
\Rightarrow \mathbb{E}_{Q'_N}(id_t) &= \mathbb{E}_{Q_N}(id_t)\\
\Rightarrow Q_N(\omega) &= Q'_N(\omega) \quad \forall\omega\in\Omega_N.
\end{aligned}$$

Therefore the probability measure Q_N is unique. □

As in Section 2.1 one can show that the fair price $\Pi(C)$ of the option $(S_N(1)-K)^+$ is given by $\mathbb{E}_{Q_N}((S_N(1)-K)^+)$ where Q_N is the probability measure from Theorem 14. Let

$$\begin{aligned}
u &= 1+\mu\Delta_N+\sigma\sqrt{\Delta_N}\\
v &= 1+\mu\Delta_N-\sigma\sqrt{\Delta_N}\\
q &= \frac{1}{2}-\frac{\mu}{2\sigma}\sqrt{\Delta_N}\\
A &= \{\omega\in\Omega_N : S_N(1,\omega)>K\}\\
B &= \{k=1,\dots,N : s_0u^kv^{N-k}>K\}.
\end{aligned}$$

Then

$$\begin{aligned}
\Pi(C) &= \mathbb{E}_{Q_N}((S_N(1)-K)^+) =\\
&= \sum_{\omega\in\Omega_N}(S_N(1)-K)\mathbf{1}_A(\omega)Q(\omega) =\\
&= \sum_{\omega\in\Omega_N}\left(S_N(0)\prod_{t<1}(1+\Delta\omega(t)\sigma+\mu\Delta_N)-K\right)\mathbf{1}_A(\omega)\prod_{t<1}\frac{1}{2}(1-\Delta\omega(t)\frac{\mu}{\sigma}) =\\
&= \sum_{k=0}^{N}\binom{N}{k}\left(S_N(0)u^kv^{N-k}-K\right)\mathbf{1}_B(k)q^k(1-q)^{N-k}
\end{aligned}$$

$$\Pi(C) = S_N(0)\sum_{k=M}^{N}\binom{N}{k}(uq)^k(1-uq)^{N-k} - K\sum_{k=M}^{N}\binom{N}{k}q^k(1-q)^{N-k} \tag{2.3}$$

where M is the least number with $S_N(0)u^Mv^{N-M}>K$. Subsequently, we want to show that the Black-Scholes Formula is given by the standard part $\Pi(C)$. For that we need the following result, which is a special case of Lemma 1.3.4, the nonstandard version of the central limit theorem. I.e. the following theorem is nonstandard version of the de Moivre-Laplace theorem, which describes the normal approximation to the binomial distribution.

Theorem 15. *Let $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ and $p \in {}^*[0,1]$ with $p \not\approx 1$ and $p \not\approx 0$ and let $q = 1-p$. Then for all $A \in \mathbb{R}$*

$$\mathrm{st}\left(\sum_{K \le A\sqrt{Npq}+Np} \binom{N}{K} p^K q^{N-K}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A} e^{-\frac{z^2}{2}} \,\mathrm{d}z$$

and for $B \in {}^*\mathbb{R}$

$$\mathrm{st}\left(\sum_{K \le B} \binom{N}{K} p^K q^{N-K}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\mathrm{st}\left(\frac{B-Np}{\sqrt{Npq}}\right)} e^{-\frac{z^2}{2}} \,\mathrm{d}z.$$

In the proof of Lemma 1.3.4 we used the central limit theorem. For this special case we now want to give a direct proof without using the central limit Theorem.

Proof. Let $A, B \in \mathbb{R}$ and let $K \in {}^*\mathbb{N}$ such that $Np + A\sqrt{Npq} \le K \le Np + B\sqrt{Npq}$. Then $K, N-K \in {}^*\mathbb{N}\backslash\mathbb{N}$. Stirlings formula implies that for $N \in {}^*\mathbb{N}\backslash\mathbb{N}$

$$\begin{aligned}
N! &\approx \sqrt{2\pi} e^{-N} N^{N+\frac{1}{2}} \\
\Rightarrow \binom{N}{K} p^K q^{N-K} &\approx \frac{N^{N+\frac{1}{2}}}{\sqrt{2\pi} K^{K+\frac{1}{2}} (N-K)^{N-K+\frac{1}{2}}} p^K q^{N-K} = \\
&= \frac{\left(\frac{K}{N}\right)^{-K} \left(\frac{N-K}{N}\right)^{-(N-K)}}{\sqrt{2\pi N \frac{K}{N} \frac{N-K}{N}}} p^K q^{N-K} =
\end{aligned}$$

$$= \frac{\exp\left[K\left(\ln(p) - \ln\left(p + z_K\sqrt{\frac{pq}{N}}\right)\right) + (N-K)\left(\ln(q) - \ln\left(q - z_K\sqrt{\frac{pq}{N}}\right)\right)\right]}{\sqrt{2\pi N \frac{K}{N} \frac{N-K}{N}}}$$

where $z_K = \frac{K-Np}{\sqrt{Npq}}$. By Taylors Formula there are $v, w \in {}^*[0,1]$ such that

$$\begin{aligned}
\ln\left(p + z_K\sqrt{\frac{pq}{N}}\right) &= \ln(p) + z_K\sqrt{\frac{q}{Np}} - \frac{z_K^2}{2}\frac{q}{Np} + \frac{1}{3}\left(\frac{1}{\frac{\sqrt{Np}}{z_K\sqrt{q}} + v}\right)^3 \\
\ln\left(q - z_K\sqrt{\frac{pq}{N}}\right) &= \ln(q) - z_K\sqrt{\frac{p}{Nq}} - \frac{z_K^2}{2}\frac{p}{Nq} - \frac{1}{3}\left(\frac{1}{\frac{\sqrt{Np}}{z_K\sqrt{q}} - w}\right)^3.
\end{aligned}$$

Because $Np + A\sqrt{Npq} \le K \le Np + B\sqrt{Npq}$, we have that z_K is finite and $\frac{K}{N} \approx p$, therefore

$$
\begin{aligned}
&K\left(\ln(p) - \ln\left(p + z_K\sqrt{\frac{pq}{N}}\right)\right) + (N-K)\left(\ln(q) - \ln\left(q - z_K\sqrt{\frac{pq}{N}}\right)\right) = \\
&= K\left(-z_K\sqrt{\frac{q}{Np}} + \frac{z_K^2}{2}\frac{q}{Np} - \frac{1}{3}\left(\frac{1}{\frac{\sqrt{Np}}{z_K\sqrt{q}} + v}\right)^3\right) + \\
&+ (N-K)\left(z_K\sqrt{\frac{p}{Nq}} + \frac{z_K^2}{2}\frac{p}{Nq} + \frac{1}{3}\left(\frac{1}{\frac{\sqrt{Np}}{z_K\sqrt{q}} - w}\right)^3\right) \approx \\
&\approx -Kz_K\sqrt{\frac{q}{Np}} + K\frac{z_K^2}{2}\frac{q}{Np} + (N-K)z_K\sqrt{\frac{p}{Nq}} + (N-K)\frac{z_K^2}{2}\frac{p}{Nq} \approx \\
&\approx -\frac{Np + z_k\sqrt{Npq}}{\sqrt{Np}}z_K\sqrt{q} + \frac{Nq - z_K\sqrt{Npq}}{\sqrt{Nq}}z_k\sqrt{p} + \frac{z_K^2}{2}p + \frac{z_k^2}{2}q \approx \\
&\approx -\sqrt{Npq}z_K - z_K^2 q + \sqrt{Npq}z_K - z_K^2 p + \frac{z_K^2}{2}p + \frac{z_k^2}{2}q = \\
&= -\frac{z_K^2}{2}
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi N\frac{K}{N}\frac{N-K}{N}}} \approx \frac{1}{\sqrt{2\pi Npq}} \\
&\Rightarrow \binom{N}{K}p^K q^{N-K} \approx \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{Npq}}\exp\left(-\frac{z_K^2}{2}\right).
\end{aligned}
$$

Let $T_{A,B} = \{K : Np + A\sqrt{Npq} \leq K \leq Np + B\sqrt{Npq}\}$. Then we have

$$
\begin{aligned}
\mathrm{st}\left(\sum_{K\in T_{A,B}}\binom{N}{K}p^K q^{N-K}\right) &= \mathrm{st}\left(\frac{1}{\sqrt{2\pi}}\sum_{K\in T_{A,B}} e^{-\frac{z_K^2}{2}}\frac{1}{\sqrt{Npq}}\right) = \\
&= \frac{1}{\sqrt{2\pi}}\int_A^B e^{-\frac{z^2}{2}}\mathrm{d}z. \qquad (2.4)
\end{aligned}
$$

Where the last equation follows directly from the Theorem described in Section 1.1 which allows to represent Integrals as standard parts of hyperfinite sums. We also have for infinite N

$$
\frac{1}{\sqrt{2\pi}}\int_{-N}^{N} e^{-\frac{z^2}{2}}\mathrm{d}z \approx \sum_{K=0}^{N}\binom{N}{K}p^K q^{N-K} \approx 1.
$$

Therefore, for all positive ϵ in $\mathbb{R}$ there exists $B \in \mathbb{R}$ such that

$$
\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} e^{-\frac{z^2}{2}}\mathrm{d}z \in [1-\epsilon, 1] \qquad (2.5)
$$

$$\mathrm{st}\left(\sum_{K\in T_{-B,B}}\binom{N}{K}p^Kq^{N-K}\right)\in[1-\epsilon,1]. \tag{2.6}$$

Let

$$\begin{aligned} B_N(K) &= \binom{N}{K}p^Kq^{N-K} \\ \phi(z) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} \\ M_A &= Np + A\sqrt{Npq}. \end{aligned}$$

Then by equation (2.4), (2.5) and (2.6) we have for all $A\in\mathbb{R}$

$$\Rightarrow\left|\mathrm{st}\left(\sum_{K\le M_A}B_N(K)\right)-\mathrm{st}\left(\int_{-N}^{A}\phi(z)\mathrm{d}z\right)\right|=$$

$$=\left|\underbrace{\mathrm{st}\left(\sum_{K\le M_{-B}}B_N(K)\right)}_{\le\epsilon}-\underbrace{\mathrm{st}\left(\int_{-N}^{-B}\phi(z)\mathrm{d}z\right)}_{\le\epsilon}+\underbrace{\mathrm{st}\left(\sum_{K\in T_{-B,A}}B_N(K)\right)-\mathrm{st}\left(\int_{-B}^{A}\phi(z)\mathrm{d}z\right)}_{=0}\right|\le\epsilon$$

$$\Rightarrow\mathrm{st}\left(\sum_{K\le M_A}\binom{N}{K}p^Kq^{N-K}\right)=\int_{-\infty}^{A}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\mathrm{d}z.$$

□

Corollary 2.3.1. *Let $Z_N=\{z_K : K=0,\dots,N\}$ with $N\in{}^*\mathbb{N}\backslash\mathbb{N}$ and $z_K=\frac{K-Np}{\sqrt{Npq}}$. Let $f : Z_N\to{}^*\mathbb{R}$ be a function such that for $\bar{f}:\mathbb{R}\to\mathbb{R}$ defined by $\bar{f}(\mathrm{st}(z_K))=\mathrm{st}(f(z_K))$ we have that $\bar{f}(z)e^{-\frac{z^2}{2}}$ is integrable and that $\int_{-\infty}^{\infty}\bar{f}(z)e^{-\frac{z^2}{2}}\mathrm{d}z$ exists. Then for $A\in\mathbb{R}$*

$$\mathrm{st}\left(\sum_{K\le Np+A\sqrt{Npq}}f(z_K)\binom{N}{K}p^Kq^{N-K}\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{A}\bar{f}(z)e^{-\frac{z^2}{2}}\mathrm{d}z.$$

Proof. Let $A,B\in\mathbb{R}$ and let $K\in{}^*\mathbb{N}$ such that $Np+A\sqrt{Npq}\le K\le Np+B\sqrt{Npq}$. Then by the proof of Theorem 15

$$\begin{aligned}\sum_{K\in T_{A,B}}f(z_K)\binom{N}{K}p^Kq^{N-K} &\approx \frac{1}{\sqrt{2\pi}}\sum_{K\in T_{A,B}}f(z_K)e^{-\frac{z_K^2}{2}}\frac{1}{\sqrt{Npq}}\approx \\ &\approx\frac{1}{\sqrt{2\pi}}\int_A^B\bar{f}(z)e^{-\frac{z^2}{2}}\mathrm{d}z.\end{aligned}$$

Because

$$\mathrm{st}\left(\sum_{K=0}^{N} f(z_K)\binom{N}{K}p^K q^{N-K}\right) = \mathrm{st}\left(\frac{1}{\sqrt{2\pi}}\int_{-N}^{N} \bar{f}(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z\right) =: G \quad \in \mathbb{R}$$

for all positive ϵ in $\mathbb{R}$ there exists B in $\mathbb{R}$ with

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} \bar{f}(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z &\in [G-\epsilon, G]\\ \sum_{K\in T_{-B,B}} f(z_K)\binom{N}{K}p^K q^{N-K} &\in [G-\epsilon, G].\end{aligned}$$

Let

$$\begin{aligned}B_N(K) &= \binom{N}{K}p^K q^{N-K}\\ \phi(z) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\\ M_A &= Np + A\sqrt{Npq}.\end{aligned}$$

Then for all $A \in \mathbb{R}$,

$$\Rightarrow \left|\mathrm{st}\left(\sum_{K\le M_A} f(z_K)B_N(K)\right) - \mathrm{st}\left(\int_{-N}^{A}\bar{f}(z)\phi(z)\mathrm{d}z\right)\right| =$$

$$= \Bigg|\underbrace{\mathrm{st}\left(\sum_{K\le M_{-B}} f(z_K)B_N(K)\right)}_{\le\epsilon} - \underbrace{\mathrm{st}\left(\int_{-N}^{B}\bar{f}(z)\phi(z)\mathrm{d}z\right)}_{\le\epsilon} +$$

$$+ \underbrace{\mathrm{st}\left(\sum_{K\in T_{-B,A}} f(z_K)B_N(K)\right) - \mathrm{st}\left(\int_{-B}^{A}\bar{f}(z)\phi(z)\mathrm{d}z\right)}_{=0}\Bigg| \le \epsilon$$

$$\Rightarrow \mathrm{st}\left(\sum_{K\le M_A} f(z_K)\binom{N}{K}p^K q^{N-K}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{A}\bar{f}(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z.$$

□

With Theorem 15 we can determine the standard part of Equation 2.3 and we will see that the standard part of the price of an European call option in the hyperfinite CRR-Model is exactly the price of an European call option in the BS-model i.e. we have

Corollary 2.3.2. *If s_0 is the initial value of the stock in the BS-model and if* $\mathrm{st}(S_N(0)) = s_0$ *then*

$$\pi(c) = \mathrm{st}(\Pi(C)).$$

Proof. By Equation (2.3) we obtain that

$$\begin{aligned}
\mathrm{st}(\Pi(C)) &= \mathrm{st}\left(S_N(0)\sum_{k=M}^{N}\binom{N}{k}(uq)^k(1-uq)^{N-k} - K\sum_{k=M}^{N}\binom{N}{k}q^k(1-q)^{N-k}\right) = \\
&= s_0\mathrm{st}\left(\sum_{k=M}^{N}\binom{N}{k}(uq)^k(1-uq)^{N-k}\right) - K\mathrm{st}\left(\sum_{k=M}^{N}\binom{N}{k}q^k(1-q)^{N-k}\right) = \\
&= s_0\frac{1}{\sqrt{2\pi}}\int_{\mathrm{st}\left(\frac{M-Nuq}{\sqrt{Nuq(1-uq)}}\right)}^{\infty} e^{-\frac{z^2}{2}}\,\mathrm{d}z - K\frac{1}{\sqrt{2\pi}}\int_{\mathrm{st}\left(\frac{M-Nq}{\sqrt{Nq(1-q)}}\right)}^{\infty} e^{-\frac{z^2}{2}}\,\mathrm{d}z.
\end{aligned}$$

So we have to determine the two standard parts $\mathrm{st}\left(\frac{M-Nq}{\sqrt{Nq(1-q)}}\right)$ and $\mathrm{st}\left(\frac{M-Nuq}{\sqrt{Nuq(1-uq)}}\right)$, where M is the least integer such that $S_0u^Mv^{N-M} > K$. For $u = 1 + \mu\Delta_N + \sigma\sqrt{\Delta_N}$ and $v = 1 + \mu\Delta_N - \sigma\sqrt{\Delta_N}$ we obtain that

$$\sqrt{N}\ln\left(\frac{u}{v}\right) = \sqrt{N}\ln\left(1 + \frac{1}{\sqrt{N}}\left(\frac{2\sigma}{1 + \frac{\mu}{N} - \frac{\sigma}{\sqrt{N}}}\right)\right) \approx 2\sigma$$

and

$$N\ln(uv) = N\ln\left(1 + \frac{1}{N}\left(2\mu + \frac{\mu^2}{N} - \sigma^2\right)\right) \approx 2\mu - \sigma^2.$$

Let m be the solution of the equation

$$S_0u^mv^{N-m} = K$$

and we write $m = \frac{1}{2}N + \beta\sqrt{N}$ for some $\beta \in {}^*\mathbb{R}$. Because

$$\begin{aligned}
\left(\frac{1}{2}N + \beta\sqrt{N}\right)\ln\left(\frac{u}{v}\right) &= \ln\left(\frac{K}{s_0}\right) - N\ln(v) \\
\Leftrightarrow \beta &= \frac{\ln\left(\frac{K}{s_0}\right) - \frac{1}{2}N\ln(uv)}{\sqrt{N}\ln\left(\frac{u}{v}\right)}
\end{aligned}$$

we have that $\beta \in \mathrm{ns}({}^*\mathbb{R})$ with

$$\mathrm{st}(\beta) = \frac{\ln\left(\frac{K}{s_0}\right) - \mu + \frac{1}{2}\sigma^2}{2\sigma}$$

and we have $M = m + \epsilon = \frac{1}{2}N + \beta\sqrt{N} + \epsilon$ for some $\epsilon \in {}^*[0,1)$. Hence for $q = \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta_N}$ and $p = 1 - q$

$$\mathrm{st}\left(\frac{M - Nq}{\sqrt{Npq}}\right) = \mathrm{st}\left(\frac{\beta\sqrt{N} + \left(\frac{1}{2} - q\right)N}{\sqrt{N}\sqrt{pq}} + \frac{\epsilon}{\sqrt{Npq}}\right) =$$
$$= \mathrm{st}\left(\frac{\beta + \left(\frac{1}{2} - q\right)\sqrt{N}}{\sqrt{pq}}\right) = \mathrm{st}\left(\frac{\beta + \frac{\mu}{2\sigma}}{\frac{1}{2}\sqrt{1 - \frac{\mu^2}{\sigma^2 N}}}\right) =$$
$$= 2\cdot\left(\frac{\ln\left(\frac{K}{s_0}\right) - \mu + \frac{1}{2}\sigma^2}{2\sigma} + \frac{\mu}{2\sigma}\right) = \frac{\ln\left(\frac{K}{s_0}\right) + \frac{1}{2}\sigma^2}{\sigma}.$$

For the expression $\mathrm{st}\left(\frac{M-Nuq}{\sqrt{Nuq(1-uq)}}\right)$ we use that

$$\mathrm{st}\left(\sqrt{uq(1-uq)}\right) = \frac{1}{2}$$
$$\mathrm{st}\left(\sqrt{N}\left(\frac{1}{2} - uq\right)\right) = -\frac{\sigma}{2} + \frac{\mu}{2\sigma}.$$

Thus

$$\mathrm{st}\left(\frac{M - Nuq}{\sqrt{Nuq(1-uq)}}\right) = \mathrm{st}\left(\frac{\beta\sqrt{N} + \left(\frac{1}{2} - uq\right)N}{\sqrt{Nuq(1-uq)}} + \frac{\epsilon}{\sqrt{Nuq(1-uq)}}\right) =$$
$$= \mathrm{st}\left(\frac{\beta + \left(\frac{1}{2} - uq\right)\sqrt{N}}{\sqrt{uq(1-uq)}}\right) = 2\cdot\left(\frac{\ln\left(\frac{K}{s_0}\right) - \mu + \frac{1}{2}\sigma^2 - \sigma^2 + \mu}{2\sigma}\right) =$$
$$= \frac{\ln\left(\frac{K}{s_0}\right) - \frac{1}{2}\sigma^2}{\sigma}.$$

So we obtain for the price $\Pi(C)$ that

$$\mathrm{st}(\Pi(C)) = s_0\frac{1}{\sqrt{2\pi}}\int_{\mathrm{st}\left(\frac{\ln\left(\frac{K}{s_0}\right) - \frac{1}{2}\sigma^2}{\sigma}\right)}^{\infty} e^{-\frac{z^2}{2}}\,\mathrm{d}z - K\frac{1}{\sqrt{2\pi}}\int_{\mathrm{st}\left(\frac{\ln\left(\frac{K}{s_0}\right) + \frac{1}{2}\sigma^2}{\sigma}\right)}^{\infty} e^{-\frac{z^2}{2}}\,\mathrm{d}z =$$
$$= s_0\Phi\left(\frac{-\ln\left(\frac{K}{s_0}\right) + \frac{1}{2}\sigma^2}{\sigma}\right) - K\Phi\left(\frac{-\ln\left(\frac{K}{s_0}\right) - \frac{1}{2}\sigma^2}{\sigma}\right) = \pi(c).$$

□

Therefore, the two option prices in the hyperfinite CRR-model and in the BS-model are equal. In the following we show that the Black-Scholes model is precisely the standard part of the Cox-Ross-Rubinstein model, see [14], [6].

We know from Corollary 1.3.5 that there exists a Brownian motion $b(t, \omega)$ on the Loeb space $\left(\Omega_N, \mathcal{L}(\mathcal{A}_N), C_N^{\mathcal{L}}\right)$ such that for $C_N^{\mathcal{L}}$-nearly all $\omega \in \Omega_N$ and for all $t \in [0,1]$ we have $b(t, \omega) = \mathrm{st}(\omega([Nt]\Delta_N))$ where $[Nt]$ is the least integer smaller than Nt.

Theorem 16. *For $t \in [0,1]$ let $s : [0,1] \times \Omega_N \to \mathbb{R}$ be the function defined by $s(t,\omega) = \mathrm{st}(S_N([Nt]\Delta_N, \omega))$ and let $s_0 = \mathrm{st}(S_N(0))$. Then for $C_N^{\mathcal{L}}$-nearly all $\omega \in \Omega_N$ and for all $t \in [0,1]$*

$$s(t,\omega) = s_0 \exp\left(b(t,\omega) + \left(\mu - \frac{\sigma^2}{2}\right)t\right).$$

Proof. Let $\bar{t} := [Nt]\Delta_N \in T_N$. The evolution of a stock S_N is given by the initial value $S_N(0)$ and the price process

$$S_N(\bar{t} + \Delta_N, \omega) = S_N(\bar{t}, \omega)(1 + \mu\Delta_N + \sigma\Delta B(\bar{t}, \omega)).$$

So, the value of the stock at time $\bar{t}$ is given by

$$S_N(\bar{t}, \omega) = S_N(0) \prod_{\bar{s}<\bar{t}} (1 + \Delta\omega(\bar{s})\sigma + \mu\Delta_N).$$

Let $\alpha(\bar{s}) = \sigma\Delta\omega(\bar{s}) + \mu\Delta_N$. By Taylors formula there is $\nu(\bar{s}) \in {}^*[0,1]$ such that

$$\begin{aligned}\ln\left(\prod_{\bar{s}<\bar{t}} (1 + \alpha(\bar{s}))\right) &= \sum_{\bar{s}<\bar{t}} \ln(1 + \alpha(\bar{s})) = \\ &= \sum_{\bar{s}<\bar{t}} \left(\alpha(\bar{s}) - \frac{1}{2}\alpha(\bar{s})^2 + \frac{1}{6}\left(\frac{\alpha(\bar{s})}{1 + \nu(\bar{s})\alpha(\bar{s})}\right)^3\right).\end{aligned}$$

Because $\Delta B(\omega, \bar{t}) = \Delta\omega(\bar{t})$ we have

$$\begin{aligned}\alpha(\bar{s}) - \frac{1}{2}\alpha(\bar{s})^2 &= \sigma\Delta\omega(\bar{s}) + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta_N - \sigma\Delta\omega(\bar{s})\mu\Delta_N - \frac{1}{2}\mu^2\Delta_N^2 = \\ &= \sigma\Delta B(\omega, \bar{s}) + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta_N - \sigma\mu\Delta_N\Delta B(\omega, \bar{s}) - \frac{1}{2}\mu^2\Delta_N^2.\end{aligned}$$

For $\bar{t} \in T_N$ and for all $\omega \in \Omega_N$ such that $B(\bar{t}, \omega)$ is finite, we obtain

$$\begin{aligned}&\sigma \sum_{\bar{s}<\bar{t}} \Delta B(\bar{s}, \omega) = \sigma B(\bar{t}, \omega) \\ &\sum_{\bar{s}<\bar{t}} \left(\mu - \frac{1}{2}\sigma^2\right)\Delta_N = \left(\mu - \frac{1}{2}\sigma^2\right)\bar{t} \\ &\sigma\mu\Delta_N \sum_{\bar{s}<\bar{t}} \Delta B(\bar{s}, \omega) \approx 0 \\ &\sum_{\bar{s}<\bar{t}} \frac{1}{2}\mu^2\Delta_N^2 \approx 0\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\bar{s}<\bar{t}} \frac{1}{6}\left(\frac{\alpha(\bar{s})}{1+\nu\nu(\bar{s})\alpha(\bar{s})}\right)^3 = \\
&= \sum_{\bar{s}<\bar{t}} \frac{1}{6}\left(\frac{\sigma\Delta\omega(\bar{s})+\mu\Delta_N}{1+\nu(\bar{s})(\sigma\Delta\omega(\bar{s})+\mu\Delta_N)}\right)^3 = \\
&= \frac{1}{6}\sqrt{\Delta_N}^3 \sum_{\bar{s}<\bar{t}} \left(\frac{\sigma\Delta\omega(\bar{s})\frac{1}{\sqrt{\Delta_N}}+\mu\sqrt{\Delta_N}}{1+\nu(\bar{s})(\sigma\Delta\omega(\bar{s})+\mu\Delta_N)}\right)^3 \approx 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad &\ln\left(\prod_{\bar{s}<\bar{t}}(1+\sigma\Delta\omega(\bar{s})+\mu\Delta_N)\right) = \sum_{\bar{s}<\bar{t}}\left(\alpha(\bar{s})-\frac{1}{2}\alpha(\bar{s})^2+\frac{1}{6}\left(\frac{\alpha(\bar{s})}{1+\nu(\bar{s})\alpha(\bar{s})}\right)^3\right) \approx \\
&\approx \sum_{\bar{s}<\bar{t}}\left(\sigma\Delta B(\bar{s},\omega)+\left(\mu-\frac{1}{2}\sigma^2\right)\Delta_N-\sigma\mu\Delta_N\Delta B(\bar{s},\omega)-\frac{1}{2}\mu^2\Delta_N^2\right) \approx \\
&\approx \sigma B(\omega,\bar{t})+(\mu-\frac{1}{2}\sigma^2)\bar{t}
\end{aligned}$$

$$\Rightarrow S_N(t,\omega) \approx S_N(0)\exp\left(\sigma B(\bar{t},\omega)+(\mu-\frac{1}{2}\sigma^2)\bar{t}\right).$$

Because $B(\bar{t},\omega)$ is finite for $C_N^{\mathcal{L}}$-nearly all $\omega \in \Omega_N$ we have

$$s(t,\omega) = s_0\exp\left(\sigma b(t,\omega)+(\mu-\frac{1}{2}\sigma^2)t\right)$$

for $C_N^{\mathcal{L}}$-nearly all $\omega \in \Omega_N$ and for all $t \in [0,1]$ □

Therefore, the Black-Scholes model is the standard part of the hyperfinite Cox-Ross-Rubinstein model. We now give an other proof of Corollary 2.3.2 by using Theorem 16, which is shown in [15].

Theorem 17. *Let $C(\omega) = (S_N(1,\omega)-K)^+$ be an option in the hyperfinite Cox-Ross-Rubinstein model and let $c(\omega) = (s(1,\omega)-K)^+$ be an option in the Black-Scholes model. Then*

$$\pi(c) = \mathrm{st}\,(\Pi(C)).$$

Proof. Let $z_k = \frac{k-Np}{\sqrt{Npq}}$ and $Z_N = \{z_k : k = 0,\dots,N\}$ where $p = \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta_N}$ and $q = 1-p$. For $z_k \in Z_N$ let

$$f(z_k) = 2\left(\sqrt{pq}z_k+\left(p-\frac{1}{2}\right)\frac{1}{\sqrt{\Delta_N}}\right).$$

Then

$$f : Z_N \to \{\omega(1) : \omega \in \Omega_N\} = \left\{ \frac{2k - N}{\sqrt{N}} : k = 0 \dots N \right\}$$

and

$$Q_N\{\omega \in \Omega_N : \omega(1) = f(z_k)\} = \binom{N}{k} p^k q^{N-k}.$$

Let $S_N(f(z_k)) := S_N(1, \omega)$ for an $\omega \in \Omega_N$ such that $f(z_k) = \omega(1)$. By Theorem 16 we have

$$S_N(f(z_k)) \approx s_0 \exp\left(\mu - \frac{1}{2}\sigma^2 + \sigma \mathrm{st}(\omega(1))\right)$$

for $\omega \in \Omega_N$ with $\omega(1) = f(z_k)$. Because $\mathrm{st}(f(z_k)) = \mathrm{st}(z_k) - \frac{\mu}{\sigma}$ and $\mathrm{st}(S_N(f(z_k)) - K)^+ = (S_N(\mathrm{st}(f(z_k))) - K)^+$ and by Corollary 2.3.1 (with $A = \sqrt{\frac{Nq}{p}}$)

$$\begin{aligned}
\mathbb{E}_{Q_N}(S_N(1,\omega) - K)^+ &= \sum_{\omega \in \Omega_N} (S_N(1,\omega) - K)^+ Q_N(\omega) = \\
&= \sum_{k=0}^{N} (S_N(f(z_k)) - K)^+ \binom{N}{k} p^k q^{N-k} \approx \\
&\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{st}\left((S_N(f(z_k)) - K)^+\right) e^{-\frac{z^2}{2}} \mathrm{d}z = \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(s_0 \exp\left(\sigma \mathrm{st}(f(z_k)) + \mu - \frac{1}{2}\sigma^2\right) - K\right)^+ e^{-\frac{z^2}{2}} \mathrm{d}z = \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s_0 (e^{\sigma z + \frac{1}{2}\sigma^2} - K)^+ e^{-\frac{z^2}{2}} \mathrm{d}z = \\
&= \pi(c).
\end{aligned}$$

□

2.4 American Options in the hyperfinite CRR-Model

An *American put option* allows the holder to sell a stock for a fixed price at any time over a finite time period. If S_N, defined in Section 2.3 denotes the evolution of a stock and $T_N = \{k\Delta_N : k = 0, \dots, N\}$ for $\Delta_N = \frac{1}{N}$ and for any infinite, natural number N then we have a *price process*

$$Y_t = (K - S_t)^+ \quad \text{for} \quad t \in T_N$$

for a American option with strike price K. Let $\mathcal{T}$ be the set of all stopping times $\tau : \Omega \to T_N$. Then an *exercise strategy* is given by

$$Y_\tau = (K - S_\tau)^+.$$

The following definitions and properties for American options in discrete time are shown in [16]. Let $t \in T_N$. We denote with U_t the minimal amount that the seller of an American option needs to hedge against the exercise strategies Y_τ for all $\tau \in \mathcal{T}$. Then we have the following conditions for U_t

$$\begin{aligned} U_t \geq Y_t &\quad \text{for all} \quad t \in T_N \\ \mathbb{E}_{Q_N}(Y_{t+\Delta_N} \mid \mathcal{A}_t) = \mathbb{E}_{Q_N}(U_{t+\Delta_N} \mid \mathcal{A}_t) &\quad \text{for all} \quad t \in T_N \setminus \{1\}, \end{aligned}$$

where $(\mathcal{A}_t)_{t\in T_N}$ is the filtration defined in the proof of Theorem 14 and Q_N is the probability measure from Theorem 14 under which the price process S_N is a martingale. The process $(U_t)_{t\in T_N}$ is called the *Snell envelope* of the process $(Y_t)_{t\in T_N}$ and $(U_t)_{t\in T_N}$ is given by the recursion

$$\begin{aligned} U_1 &= Y_1 \\ U_t &= \max\{Y(t), \mathbb{E}_{Q_N}(U_{t+\Delta_N} \mid \mathcal{A}_t)\} \qquad \text{for} \quad t \in T_N \setminus \{1\}. \end{aligned}$$

Theorem 18. *For $t \in T_N$ let τ_t be the stopping time defined by $\tau_t = \min\{r \geq t : U_r = Y_r\}$. Then*

$$U_t = \mathbb{E}_{Q_N}(Y_{\tau_t} \mid \mathcal{A}_t) = \text{ess sup}_{\tau\in\mathcal{T}_t} \mathbb{E}_{Q_N}(Y_\tau \mid \mathcal{A}_t)$$

where $\mathcal{T}_t = \{\tau \in \mathcal{T} : \tau \geq t\}$.

Proof. By Definition $(U_t)_{t\in T_N}$ is a supermartingale under Q_N. Therefore, for any stopping time $\tau \in \mathcal{T}_t$, we have

$$U_t \geq \mathbb{E}(U_\tau, \mathcal{A}_t) \geq \mathbb{E}_{Q_N}(Y_\tau, \mathcal{A}_t).$$

Because this condition holds for all $\tau \in \mathcal{T}_t$, we have

$$U_t \geq \text{ess sup}_{\tau\in\mathcal{T}_t} \mathbb{E}_{Q_N}(Y_\tau \mid \mathcal{A}_t). \tag{2.7}$$

For $s, t \in T_N$ and $s < t < 1$ let $U^{(t)}$ be the stopped process defined by

$$U_s^{(t)} = U_{\min\{s,\tau_t\}}.$$

Because $\tau_t = \min\{s \geq t : U_s = Y_s\}$ we have for all $\omega \in \{\omega \in \Omega : \tau_t(\omega) > s\}$ the condition $U_s > Y_s$. Therefore we have for Q_N-almost all $\omega \in \{\omega \in \Omega : \tau_t(\omega) > s\}$ that

$$\begin{aligned} U_s^{(t)} = U_s = \max\{Y_s, \mathbb{E}_{Q_N}(U_{s+1} \mid \mathcal{A}_s)\} = \\ = \mathbb{E}_{Q_N}(U_{s+1} \mid \mathcal{A}_s) = \mathbb{E}(U_{s+1}^{(t)} \mid \mathcal{A}_s). \end{aligned}$$

On the other side, we have for all $\omega \in \{\omega \in \Omega : \tau_t(\omega) > s\}$

$$U_{s+1}^{(t)} = U_{\tau_t} = U_s^{(t)}$$

and therefore also

$$\mathbb{E}_{Q_N}(U_{s+1}^{(t)} \mid \mathcal{A}_s) = \mathbb{E}_{Q_N}(U_s^{(t)} \mid \mathcal{A}_s) = U_s^{(t)}$$

and $(U_s^{(t)})_{s\geq t}$ is a martingale under Q. Hence we have

$$\mathbb{E}_{Q_N}(U_{\tau_t} \mid \mathcal{A}_t) = \mathbb{E}_{Q_N}(U_1^{(t)} \mid \mathcal{A}_t) = U_t^{(t)} = U_t.$$

By the definition of τ_t we have

$$\mathbb{E}_{Q_N}(U_{\tau_t} \mid \mathcal{A}_t) = \mathbb{E}_{Q_N}(Y_{\tau_t} \mid \mathcal{A}_t)$$

which together with condition (2.7) proves the theorem. □

Proposition 2.4.1. *U_t is the smallest $(\mathcal{A}_t)_{t\in T_N}$-supermartingale dominating Y_t.*

Proof. Let $(U_t^{'})_{t\in T_N}$ be another supermartingale dominating $(Y_t)_{t\in T_N}$. Then we have $U_1^{'} \geq Y_1$, therefore

$$U_t^{'} \geq \mathbb{E}_{Q_N}(U_1^{'} \mid \mathcal{A}_t) \geq \mathbb{E}_{Q_N}(Y_1 \mid \mathcal{A}_t) = \mathbb{E}_{Q_N}(U_1 \mid \mathcal{A}_t) = U_t.$$

□

Definition 2.4.2. A stopping time $\tau \in \mathcal{T}_t$ is called *optimal* if

$$U_t = \mathbb{E}_{Q_N}(Y_\tau \mid \mathcal{A}_t).$$

Proposition 2.4.3. *A stopping time $\tau \in \mathcal{T}$ is optimal if and only if $Y_\tau = U_\tau$ for Q_N-almost all $\omega \in \Omega$ and if the stopped process $(U_{\min\{t,\tau\}})_{t\in T_N}$ is a martingale.*

Proof. ($\Rightarrow$) Because $\tau \in \mathcal{T}$ is optimal and $U_\tau \geq Y_\tau$ by definition of $(U_t)_{t\in T_N}$, we have

$$\begin{aligned} U_0 = \mathbb{E}_{Q_N}(Y_\tau \mid \mathcal{A}_0) = \mathbb{E}_{Q_N}(Y_\tau) \leq \\ \leq \mathbb{E}_{Q_N}(U_\tau) = \mathbb{E}_{Q_N}(U_\tau \mid \mathcal{A}_0) \leq U_0 \end{aligned}$$

and therefore we have $\mathbb{E}_{Q_N}(Y_\tau) = \mathbb{E}_{Q_N}(U_\tau)$ and $Y_\tau \leq U_\tau$. It follows that $Y_\tau = U_\tau$. We also have $\mathbb{E}_Q(U_\tau) = U_0$ where U_0 is constant. Therefore the stopped process $(U_{\min\{t,\tau\}})_{t\in T_N}$ is a martingale, because $(U_{\min\{t,\tau\}})_{t\in T_N}$ is a supermartingale with constant expection.

($\Leftarrow$) Because $Y_\tau = U_\tau$ and $(U_{\min\{t,\tau\}})_{t\in T_N}$ is a martingale we have

$$U_0 = \mathbb{E}_{Q_N}(U_\tau) = \mathbb{E}_{Q_N}(Y_\tau \mid \mathcal{A}_0)$$

and therefore, the stopping time τ is optimal. □

In the following we want to show that any internal stopping time in the Cox-Ross-Rubinstein model for some infinite N is a lifting of an optimal stopping time in the Black-Scholes model. This was shown in [17].

In the Black-Scholes model there is a unique optimal stopping time on $[t, 1]$ for every $t \in [0, 1]$ and this stopping time is given by

$$\rho_t = \inf\{v \in [t, 1] : y_v = u_v\} \tag{2.8}$$

where

$$y_t = (K - s_t)^+$$

denotes the price process of the American put option and

$$u_t = \text{ess sup}_{\rho \in \mathcal{P}_t} \mathbb{E}_Q(y_\rho \mid \mathcal{B}_t)$$

where $\mathcal{P}_t$ denotes the set of all stopping times $\rho : \Omega \to [t, 1]$. That the stopping time ρ_t defined by (2.8) is the only optimal stopping time was shown in [18] or in [19].

In the following, we denote the elements of T_N with $\underline{s}$ and $\underline{t}$ and the elements of $[0, 1]$ with s and t. Let $B(\underline{t}, \omega)$ be the random walk defined in Section 2.3 and let $\tilde{B}(\underline{t}, \omega) = B_{\underline{t}} + \frac{\mu}{\sigma}\underline{t}$. For $\underline{t} = \frac{k}{N}$ we have

$$\tilde{B}\left(\frac{k}{N}, \omega\right) = \frac{1}{\sqrt{N}} \sum_{i=1}^{k} \sqrt{N}\left(\Delta\omega\left(\frac{i}{N}\right) + \frac{\mu}{\sigma N}\right)$$

and $\sqrt{N}\left(\Delta\omega\left(\frac{i}{N}\right) + \frac{\mu}{\sigma N}\right)$ are Q_N-independent random variables with mean 0 and variance 1. By Theorem 4 there is $M \in \mathcal{L}(\mathcal{A}_N)$ such that $Q_N^{\mathcal{L}}(M) = 0$ and

$$\tilde{b}(t, \omega) = \begin{cases} \operatorname{st}\left(\tilde{B}_{\frac{[Nt]}{N}}(\omega)\right) & \text{if} \quad \omega \notin M, t \in [0, 1] \\ 0 & \text{if} \quad \omega \in M, t \in [0, 1] \end{cases}$$

is a Brownian motion on the Loeb space $(\Omega_N, \mathcal{L}(\mathcal{A}_N), Q_N^{\mathcal{L}})$. Let $(\mathcal{B}_t)_{t\in[0,1]}$ be the Brownian filtration and let $(\mathcal{F}_t)_{t\in[0,1]}$ be the standard part of the filtration $(\mathcal{A}_{\underline{t}})_{\underline{t}\in T_N}$ defined by Definition 1.4.8. Then we have

$$\left\{\omega \in \Omega_N : \operatorname{st}\left(\tilde{B}_{\underline{t}}(\omega)\right) \le r\right\} = \bigcap_{k\in\mathbb{N}} \left\{\omega \in \Omega_N : \tilde{B}_{\underline{t}}(\omega) \le r + \frac{1}{k}\right\} \in \mathcal{L}\left(\mathcal{A}_{\underline{t}}\right)$$

and for $\operatorname{st}(\underline{t}) = t$ the sets

$$\left\{\omega \in \Omega_N : \tilde{b}_t(\omega) \le r\right\}$$

and

$$\left\{\omega \in \Omega_N : \operatorname{st}\left(\tilde{B}_{\underline{t}}(\omega)\right) \le r\right\}$$

differ only by a Loeb null set, and therefore $\tilde{b}_t$ is $(\mathcal{F}_t)_{t\in[0,1]}$-measurable. Because $(\mathcal{F}_t)_{t\in[0,1]}$ is a right continuous filtration, we have $(\mathcal{F}_t)_{t\in[0,1]} \supseteq (\mathcal{B}_t)_{t\in[0,1]}$.

If we use Theorem 16 for $\tilde{b}_t$ instead of b_t we see that

$$\operatorname{st}(S_N(\underline{t}, \omega)) = s_0 \exp\left(\sigma\tilde{b}(t, \omega) - \frac{1}{2}\sigma^2 t\right)$$

for $Q_N^{\mathcal{L}}$-almost all ω. We know from Theorem 8 that $\rho : \Omega_N \to [0, 1]$ is an $(\mathcal{F}_t)_{t\in[0,1]}$-stopping time if and only if $\rho = \operatorname{st}(\tau)$ for some $(\mathcal{A}_{\underline{t}})_{\underline{t}\in T_N}$-stopping time $\tau : \Omega_N \to T_N$. So we have the following condition

$$\operatorname{st}(S_N(\tau(\omega), \omega)) = s(\rho(\omega), \omega).$$

Theorem 19. *For any internal optimal stopping time τ in the hyperfinite Cox-Ross-Rubinstein model, we have*

$$\hat{\rho} = \mathrm{st}(\tau)$$

for Q_N almost all ω, where $\hat{\rho}$ is the unique optimal stopping time in the Black-Scholes model.

Proof. Let $\tau : \Omega_N \to T_N$ be an optimal stopping time. Because τ is bounded, there is some $\rho : \Omega_N \to [0,1]$ such that $\rho = st(\tau)$ for Q_N almost all ω. By Theorem 8 the stopping time ρ is an $(\mathcal{F}_t)_{t\in[0,1]}$-stopping time. If $\hat{\rho}$ is the optimal stopping time we have

$$\begin{aligned} & \mathbb{E}_{Q_N^{\mathcal{L}}}(y_{\hat{\rho}} \mid \mathcal{F}_0) = \text{ess sup}_{\rho\in\mathcal{P}} \mathbb{E}_{Q_N^{\mathcal{L}}}(y_\rho \mid \mathcal{F}_0) \\ \Rightarrow \quad & \mathbb{E}_{Q_N^{\mathcal{L}}}(y_{\hat{\rho}}) \geq \mathbb{E}_{Q_N^{\mathcal{L}}}(y_\rho). \end{aligned}$$

On the other hand, by Theorem 8 there is some $(\underline{\mathcal{A}_t})_{\underline{t\in T_N}}$-stopping time $\tau^{'} : \Omega_N \to T_N$ such that $\hat{\rho} = \mathrm{st}(\tau^{'})$ for Q_N-almost all $\omega \in \Omega_N$. Because of the conditions $\mathrm{st}\,(Y(\tau(\omega),\omega)) = y(\rho(\omega),\omega)$ and $\mathrm{st}\left(Y(\tau^{'}(\omega),\omega)\right) = y(\hat{\rho}(\omega),\omega)$ for almost all ω, and because the stopping time τ is optimal, we have

$$\begin{aligned} \mathbb{E}_{Q_N^{\mathcal{L}}}(y_\rho) = \mathbb{E}_{Q_N^{\mathcal{L}}}\left(\mathrm{st}(Y_\tau)\right) = \mathrm{st}\left(\mathbb{E}_{Q_N}(Y_\tau)\right) \geq \\ \geq \mathrm{st}\left(\mathbb{E}_{Q_N}(Y_{\tau'})\right) = \mathbb{E}_{Q_N^{\mathcal{L}}}\left(\mathrm{st}(Y_{\tau'})\right) = \mathbb{E}_{Q_N^{\mathcal{L}}}(y_{\hat{\rho}}) \end{aligned}$$

and ρ is an optimal stopping time in the BS-model. Because the optimal stopping time in the BS-model is unique, we have $\rho = \hat{\rho}$ and $\hat{\rho} = \tau$ for almost all ω. □

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