Theoretical Studies of Nonlinear Water Waves

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Introduction

The mathematical study of the propagation of water waves is an extremely challenging endeavor due to the nonlinear structure of the corresponding free boundary value problem. The dynamics of the flow beneath the water surface are governed by time-dependent nonlinear partial differential equations – the incompressible Euler equations of motion, with gravity acting as the only external force. Furthermore – and this makes this problem especially delicate – the fluid domain is unknown. The free water surface, which is part of the solution, is described by nonlinear boundary conditions. Besides Gerstner’s wave for deep water waves and variations thereof, there are no known explicit solutions for the water wave problem \[10, 2\]. Even the most regular observed wave patterns, such as unidirectional periodic traveling waves propagating on the water surface at constant speed while retaining their symmetric shape, can not be understood by means of linear approximations. These waves, which are observed for instance in coastal regions, show rather flat troughs and sharp crests, whereas the sinusoidal waves encountered in linear water wave theory do not capture this phenomenon.

This doctoral thesis deals with several aspects of nonlinear water wave theory and can be roughly divided into two parts, which follow different approaches. The first part, consisting of the paper \[16\] and the preprint \[11\], is concerned with a new highly nonlinear model equation, which we derive from the governing equations for unidirectional water waves over a flat bed. The study of model equations, which approximate the governing equations within certain asymptotic regimes, is a way to tackle the water wave problem. A very well-known model for shallow water waves is the Korteweg-de Vries equation (KdV). It has explicit solutions and features realistic wave profiles \[9\]. However, KdV is only valid for shallow water waves of small amplitude. All physically reasonable solutions of KdV are smooth and global in time, see the discussion in \[4\]. Hence the phenomenon of breaking waves is not covered by KdV, and solutions do not feature crests with sharp angles. In order to obtain solutions satisfying these properties, one has to consider asymptotic regimes that allow for larger waves and lead to model equations with higher nonlinearities, such as the shallow water regime for waves of moderate amplitude. A well-known model in this regime is the Camassa-Holm equation (CH) \[1, 12, 7\]. One of the many remarkable properties of CH is that certain solutions blow up after a finite time in the form of wave breaking \[5, 6\]. Moreover it has traveling wave solutions with singularities such as peaks or cusps at crests \[14\]. The model equation that we consider in this thesis belongs to a shallow water regime for waves of large amplitude and contains cubic third order terms. This equation describes, similar as CH, the evolution of the horizontal component of the flow field at a specific depth beneath the free surface. We study the corresponding Cauchy problem and establish local well-posedness in the natural Sobolev setting by using Kato’s semigroup approach for quasilinear hyperbolic evolution equations \[13\]. Additionally we prove that the regularity of the solution of the Cauchy problem inherits the regularity of the initial data in terms of the Sobolev exponent. This enables us to derive a blow-up criterion for solutions, which allows a meaningful interpretation from a physical point of view. In \[11\], we investigate the traveling wave solutions of this highly nonlinear model equation. By applying qualitative methods from the theory of dynamical systems, in particular tools from integrable planar systems, we establish a full classification of all traveling waves.
Thereby we discover completely new types of traveling waves, e.g. peaked solitary waves with compact support and periodic traveling waves with peaked crests and troughs, which do not appear as solutions of shallow water equations for waves of moderate amplitude such as CH.

The second part is concerned with the dynamics of certain ocean flows. In [17] we give a detailed qualitative analysis of the irrotational velocity field beneath smooth symmetric periodic traveling surface waves in the equatorial region and describe the pattern of the paths of particles beneath such waves. Our analysis is inspired by [3, 8] and makes use of methods from complex and harmonic analysis. It does not rely on approximations, but on certain simplifying assumption on the solutions. However, one does not need to impose restrictions on the amplitude of the wave. The existence of the assumed smooth symmetric periodic traveling wave solutions can be established by using bifurcation theory, similar as in [4]. The underlying governing equations differ slightly from the classical water wave problem: the Euler equations are extended by linear terms accounting for the effects of the earth’s rotation, referred to as Coriolis effect, near the Equator. This effect becomes relevant for large ocean waves. The aim of [17] is to find out whether this perturbation of the classical water wave problem affects the qualitative properties of the flow field and the particle paths. It turns out that both the velocity field and the pattern of the paths of particles have the same qualitative properties as it is known from flows for the classical problem beneath Stokes waves, where the Coriolis effect is not taken into account. Finally we discuss some aspects of a stratified wind-induced current field with eddy viscosity in the equatorial region in [15], where additional reasonable simplifying assumptions yield a static system of linear partial differential equations defined on a fixed strip. More precisely, we demonstrate that wind-stress forces at the water surface propagate down to the sea floor, and thus may generate so-called benthic storms.

References


A new highly nonlinear shallow water wave equation

Ronald Quirchmayr


Abstract

We derive a quasilinear shallow water equation directly from the governing equations for gravity water waves within a certain regime for large amplitude waves which has not been studied so far. Furthermore we demonstrate local well-posedness of the corresponding Cauchy problem and finally discuss some aspects of the blow-up behavior of solutions.

1 Introduction

The water wave problem for gravity waves is described by Euler’s equations of motion, the equation of mass conservation and dynamic and kinematic conditions on the boundary of the domain consisting of the free surface and the bed, which is in our case taken to be flat; see the beginning of Section 2 for the details.

One way to reduce the complexity of this problem is to study model equations with only one unknown instead of the before mentioned coupled system for two-dimensional flows with four unknowns: the horizontal and vertical component of the velocity field, the pressure and the free surface. Such equations can formally be derived from the governing equations via double asymptotic power series expansions in the two fundamental dimensionless positive parameters $\delta$, the shallowness parameter, and $\varepsilon$, the amplitude parameter; see [20] and [8]. When speaking about shallow water waves, one assumes $\delta$ to be small: $\delta \ll 1$. By relating $\varepsilon$ and $\delta$, one finds equations modeling water waves in particular asymptotic regimes.

Let us for example consider the so-called long wave regime for small-amplitude waves

$$\delta \ll 1, \quad \varepsilon = O(\delta^2).$$

Within this regime one derives weakly non-linear models such as the Korteweg-de Vries (KdV) equation

$$\eta_t + \eta_x + \frac{2\varepsilon}{3} \eta_{xx} + \frac{\delta^2}{6} \eta_{xxx} = 0.$$  

KdV describes both the evolution of the free surface and the horizontal component of the velocity field at any depth of water of a right moving wave. It is the simplest equation embodying soliton solutions (see the discussion in [13]), has an integrable structure and all physically relevant solutions exist globally in time (see [3]). KdV is contained in a larger family of asymptotically equivalent equations, known as the BBM-equations, named after Benjamin, Bona and Mahoney, who studied them in [1]:

$$\eta_t + \eta_x + \frac{2\varepsilon}{3} \eta_{xx} + \delta^2 \alpha \eta_{xxx} + \beta \eta_{xxt} = 0.$$
with $\beta \leq 0$, to prevent ill-posedness, and $\alpha = \frac{1}{6} + \beta$. The BBM equations possess - like KdV - global solutions for very general initial data, in particular all physically relevant waves; see [28] for details and also the discussion in [3]. Hence the phenomenon of breaking waves - i.e. solutions which remain bounded whereas their slopes become unbounded in finite time - is not captured by them. However it is known, that some shallow water waves break. In order to account for this aspect, one has to consider equations for larger waves, which behave more nonlinear then dispersive [3]. A possible way to derive such equations directly from the governing equations is to consider regimes that bring such higher order non-linear terms, e.g. the regime for shallow water waves of moderate amplitude:

$$\delta \ll 1, \quad \varepsilon = O(\delta).$$

The following two parameter family (with parameters $z_0$ and $p$) approximate the governing equations within this regime, see [9]:

$$u_t + u_x + \frac{3\varepsilon}{2} uu_x + \delta^2(\alpha u_{xxx} + \beta u_{xxt}) = \varepsilon\delta^2(\gamma u_{xxx} + \zeta u_x u_{xx}).$$

Here $u = u|_{z_0}$ with $z_0 \in [0, 1]$ and $p \in \mathbb{R}$ such that, with $\lambda = \frac{1}{2}(\varepsilon^2 - \frac{1}{4})$, the coefficients $\alpha, \beta, \gamma, \zeta$ satisfy

$$\alpha = p + \lambda, \quad \beta = p - \frac{1}{6} + \lambda, \quad \gamma = -\frac{3}{2}p - \frac{1}{6} - 3\frac{\lambda}{2}, \quad \zeta = -\frac{9}{2}p - \frac{23}{24} - \frac{3}{2}\lambda.$$

This family of equations models the evolution of the horizontal component of the velocity field $u$ at a specific depth $z_0$. The two parameters have to be chosen in such a way that $\beta < 0$, in order to avoid ill-posedness or a change of the type of equation. In [9] it has been proved that this family is indeed a good approximation of the governing equations; the derivation is due to formal asymptotic procedures in [20]. By means of a suitable choice for $z_0$ and $p$, and by imposing a transformation of variables (see [9] for the details), one recovers two prominent equations within this family, namely the Camassa-Holm (CH)

$$U_t + \kappa U_x + 3UU_x - U_{xxt} = 2U_xU_{xx} + UU_{xxx}$$

and the Degasperis-Procesi (DP) equation

$$U_t + \hat{\kappa} U_x + 4UU_x - U_{xxt} = 3U_xU_{xx} + UU_{xxx}.$$ 

CH was first considered in [15] as a bi-Hamiltonian equation. It was re-derived in [2] in the context of water waves (see also the alternative derivation in [20] and [9]; we refer to [10] for a variational derivation of CH). DP was first derived in [12]. The formal considerations in [11] show that it is, besides KdV and CH, the only candidate being asymptotically integrable within the above family of equations. We refer to [7] for a rigorous approach to integrability of DP; integrability of CH is addressed in [10]. Furthermore we refer to [17, 18] for the fact that CH and DP are the only completely integrable equations of the aforementioned type (besides KdV). In contrast to KdV, both CH and DP admit smooth solutions that develop singularities in finite time in the form of breaking waves, cf. [4, 5, 6, 14].

Let us emphasize at this point, that the evolution of the free surface is - unlike the earlier mentioned model equations in the regime for small amplitude waves - described by a different collection of asymptotically equivalent equations, see [9]. It contains the
following:

\[
\eta_t + \eta_x + \frac{3\varepsilon}{2} \eta_{xx} - \frac{3\varepsilon^2}{8} \eta^2 \eta_x + \frac{\delta^2}{12} \eta_{xxx} - \frac{\delta^2}{12} \eta_{xxt} = -\frac{3\varepsilon^3}{16} \eta^3 \eta_x - \frac{7\varepsilon \delta^2}{24} \left(2\eta_x \eta_{xx} + \eta_{xxx}\right).
\]

This surface equation for shallow water waves of moderate amplitude is locally well-posed and its structural properties can be exploited in order to prove wave breaking for initial data with sufficiently steep slope; see [9], and for the periodic case also [25].

A natural question is to ask for equations exhibiting additional terms with higher non-linearities with the aim to describe large amplitude waves. Such equations can be obtained by considering regimes, where \(\varepsilon\) may be much larger than \(\delta\). In Section 2 we derive equations which approximate the governing equations in the regime

\[\delta \ll 1, \quad \varepsilon = \mathcal{O}(\sqrt{\delta}),\]

which we call the shallow water regime for waves of large amplitude. More precisely we derive a two-parameter family of asymptotically equivalent equations for the horizontal component of the velocity field at various fixed depths of water. We pick one particular equation out of this family which we shall study in the sequel, namely

\[u_t + u_x + \frac{3\varepsilon}{2} u u_x - \frac{4\delta^2}{18} u_{xxx} - \frac{7\delta^2}{18} u_{xxt} = \frac{\varepsilon \delta^2}{6} \left(2u_x u_{xx} + uu_{xxx}\right) - \frac{\varepsilon^2 \delta^2}{96} \left(398u_x u_{xx} + 45u^2 u_{xxx} + 15u^3 u_x\right).\]  

(1.1)

The property which distinguishes (1.1) from other equations of the corresponding family (2.19), is that the \(\varepsilon^2 \delta^2\)-term can be written as an \(x\)-derivative, since it is of the special form

\[2(A + B)u_x u_{xx} + Au^2 u_{xxx} + Bu^3_x = (Au^2 u_{xx} + Bu^3 u_x)_x.\]

In Section 3 we demonstrate local well-posedness of the Cauchy problem for (1.1) on the real line, i.e. we show that (1.1) has unique local solutions for a large class of initial data (namely the space of \(H^3(\mathbb{R})\)-functions, which covers the physically relevant initial states) and that the solution depends continuously on the data. This result will be improved in Section 4 where we show that the solution possesses additional regularity, if the initial function is smoother than \(H^3\). In Section 5 we prove a blow-up criterion for solutions of (1.1) and provide an integral of motion.

2 Physical Derivation

The system under study are the governing equations for one-dimensional (or two-dimensional unidirectional) gravity water waves. They consist of Euler’s equations of motion and the equation of mass conservation (see e.g. [19]):

\[
\begin{cases}
  u_t + uu_x + wu_z = -\frac{1}{\rho} P_x \\
  w_t + uw_x + ww_z = -\frac{1}{\rho} P_z - g \\
  u_x + w_z = 0,
\end{cases}
\]
which shall hold in the domain $\Omega_h(t) := \{(x,z): 0 < z < h(x,t)\}$, together with the boundary conditions on the free surface

\[
\begin{align*}
P &= P_{\text{atm}} \\
w &= h_t + uh_x & \text{on} & \quad z = h(x,t),
\end{align*}
\]

and the boundary condition on the flat bed:

\[w = 0 \quad \text{on} \quad z = 0.\]

This system is written in physical variables. The horizontal and vertical directions are denoted by $x$ and $z$ respectively, we use $t$ as the time variable, $(u,w) = (u(x,z,t), w(x,z,t))$ denotes the velocity field of the fluid, the scalar field $P = P(x,z,t)$ expresses the pressure distribution, and $h = h(x,t)$ is a parametrization of the shape of the free surface. Furthermore the constants $P_{\text{atm}} = \text{const}$, $\rho$ and $g$ denote the atmospheric pressure, the density of the water and the constant of gravity acceleration respectively.

In the next step we bring this system into a non-dimensional form. Therefore we consider the following dimensionless variables (cf. [19], [8], or [21] for a more detailed discussion):

\[
x^* := \frac{x}{\lambda}, \quad z^* := \frac{z}{h_0}, \quad t^* := \frac{\sqrt{gh_0}}{\lambda} t, \quad u^* := \frac{u}{\sqrt{gh_0}}, \quad w^* := \frac{\lambda}{\h_0 \sqrt{gh_0}} w,
\]

where $h_0$ is the average depth of water, $\lambda$ is a typical wavelength, $\sqrt{gh_0}$ is a typical speed $u$ when a wave passes by, $(h_0 \sqrt{gh_0} \lambda)^{-1}$ is a typical speed $w$ when a wave passes by, and $\lambda/\sqrt{gh_0}$ is a time of reference. The dimensional surface elevation $h$ can be written as $h = h_0 + a \eta^*$, where $a$ is a typical height of a wave amplitude and $\eta^*$ denotes the non-dimensional deviation from the flat surface. Hence

\[
\frac{h}{h_0} = 1 + a \h_0 \eta^*
\]

is a non-dimensional expression for the surface elevation. Let $P^*$ be the deviation from the hydrostatic pressure distribution, then the dimensional pressure $P$ can be written as the sum of the hydrostatic pressure component and the hydrodynamic pressure component:

\[
P = P_{\text{atm}} + \rho g (h_0 - z) + \rho g h_0 P^*.
\]

From now on, we will exclusively consider the above mentioned dimensionless variables and thus omit the asterisks in our notation, i.e. we write $x, u, t$ etc. instead of $x^*, u^*, t^*$ etc.

At this point, we are able to rewrite the original system given in physical variables as a non-dimensional system by taking into account the dimensionless variables from above. This leads to a non-dimensional system with only two dimensionless parameters, namely

\[
\epsilon := \frac{a}{h_0} \quad \text{and} \quad \delta := \frac{h_0}{\lambda}
\]

The parameter $\epsilon$ is known as the amplitude parameter and $\delta$ is called the shallowness parameter. By imposing the additional transformation

\[(u, w, P) \mapsto \epsilon (u, w, P)\]

we obtain the following scaled system in non-dimensional form:

\[
\begin{align*}
\left\{ \begin{array}{l}
u_t + \epsilon (uu_x + wu_z) = -P_x \\
\delta^2 (w_t + \epsilon (uw_x + ww_z)) = -P_z \\
x_x + w_z = 0
\end{array} \right.
\]

where
for the domain $\Omega_\eta(t) = \{(x, z) : 1 < z < 1 + \varepsilon\eta(x, t)\}$, with the boundary conditions

\[
\begin{aligned}
P &= \eta \\
w &= \eta_t + \varepsilon u z \\
w &= 0
\end{aligned}
\text{ on } z = 1 + \varepsilon \eta
\text{ on } z = 0.
\]

We are still not ready to solve this system, since we encounter another difficulty: the boundary conditions on the free surface. These relations for $P, \eta, u$ and $w$ have to be satisfied on the surface $1 + \varepsilon \eta$, which is itself unknown. We overcome this issue by approximating the boundary conditions on the free surface about $z = 1$. This is done by using Taylor series expansions of $P, w$ and $u$ around this line. Since we will need at most a quadratic order approximation of the boundary conditions in our derivation, we write them as:

\[
\begin{aligned}
P + \varepsilon \eta P_z + \frac{1}{2} \varepsilon^2 \eta^2 P_{zz} &= \eta \\
w + \varepsilon \eta w_z + \frac{1}{2} \varepsilon^2 \eta^2 w_{zz} &= \eta_t + \varepsilon \eta_z \left( u + \varepsilon \eta u_z + \frac{1}{2} \varepsilon^2 \eta^2 u_{zz} \right)
\end{aligned}
\text{ on } z = 1.
\]

In order to proceed, we follow the approach in [20], and employ the far field variables

\[
\begin{aligned}
\xi := \sqrt{\varepsilon} (x - t), & \quad \tau := \sqrt{\varepsilon} t.
\end{aligned}
\]

It turns out that this is a suitable choice of coordinates. We will, however, transform back to the original variables $x$ and $t$ afterwards. Additionally to the above transformation of the time variable and the horizontal direction, we need to require the scaling $w \mapsto \sqrt{\varepsilon} w$, such that the conservation of mass remains correct.

We have that $\partial_\xi = \sqrt{\varepsilon} \partial_x$ and $\partial_\tau = \sqrt{\varepsilon} (-\partial_x + \varepsilon \partial_x)$, and thus our system becomes

\[
\begin{aligned}
\varepsilon u_\tau - u_\xi + \varepsilon (u u_\xi + w u_z) &= -P_\xi \\
\varepsilon \delta^2 (w_\tau - w_\xi + \varepsilon (u w_\xi + w w_z)) &= -P_z \\
u_\xi + w_z &= 0
\end{aligned}
\text{ in } 0 < z < 1
\]

\[
\begin{aligned}
P + \varepsilon \eta P_z + \frac{1}{2} \varepsilon^2 \eta^2 P_{zz} &= \eta \\
w + \varepsilon \eta w_z + \frac{1}{2} \varepsilon^2 \eta^2 w_{zz} &= \eta_\tau - \eta_x + \varepsilon \eta \left( u + \varepsilon \eta u_z + \frac{1}{2} \varepsilon^2 \eta^2 u_{zz} \right)
\end{aligned}
\text{ on } z = 1
\text{ and } w = 0 \text{ on } z = 0.
\]

Below we illustrate how to solve this system formally, by assuming that the involved functions can be written as double asymptotic expansions in $\varepsilon$ and $\delta$:

\[
q = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon^n \delta^m q_{nm}
\]

as $\varepsilon \to 0$ and $\delta \to 0$ independently. Here, $q$ (and correspondingly $q_{nm}$) stands for $u, w, P, \eta$.

For the leading order approximation (the $\varepsilon^0 \delta^0$-level), we obtain the following system

\[
\begin{aligned}
u_{00} \xi &= P_{00} \xi \\
P_{00} \tau &= 0 \\
w_{00} \tau &= -u_{00} \xi
\end{aligned}
\text{ in } 0 < z < 1
\]
\[ \begin{align*}
\left\{ \begin{array}{ll}
P_{00} &= \eta_{00} & \text{on } z = 1 \\
w_{00} &= -\eta_{00} \xi & \text{on } z = 0 \\
w_{00} &= 0 & \text{on } z = 0,
\end{array} \right.
\end{align*} \]

which has the solution
\[ \begin{align*}
P_{00} &= \eta_{00} = u_{00}, & w_{00} &= -\eta_{00} \xi & \text{for all } z \in [0, 1].
\end{align*} \]

Observe that \( P_{00} \) and \( u_{00} \) do not depend upon \( z \) at this level of approximation. Moreover, up to now, the solutions are arbitrary functions \( [0, \infty) \to \mathbb{R} \) when considered as functions of \( \tau \) (by fixing \( \xi \) and \( z \)).

Let us also solve the system that corresponds to the \( \varepsilon^1 \delta^0 \)-level approximation:
\[ \begin{align*}
\left\{ \begin{array}{ll}
\eta_{10} = & P_{10} = 0 \quad \text{(2.3)} \\
u_{10} = & u_{10} - u_{00} u_{00\xi} = -P_{10\xi} \quad \text{(2.2)} \\
u_{10\xi} = & -w_{10z}. \quad \text{(2.4)}
\end{array} \right.
\end{align*} \]

where the first three equations hold within \( 0 < z < 1 \), the first two boundary conditions are satisfied on the surface \( z = 1 \) and the last relation holds on the flat bed \( z = 0 \). From equation (2.3) and the boundary condition (2.5) we deduce that \( P_{10\xi} = \eta_{10} \xi \) for \( 0 \leq z \leq 1 \). Furthermore equation (2.2) and equation (2.4), taking into account the identities for the solutions of the leading order approximation, yield that
\[ -u_{10\xi} = -\left( \eta_{10} \xi + \eta_{00} \tau + \eta_{00} \eta_{00} \xi \right) = w_{10z}. \quad \text{(2.8)} \]

Next we integrate with respect to \( z \) in order to get
\[ w_{10} = -\left( \eta_{10} \xi + \eta_{00} \tau + \eta_{00} \eta_{00} \xi \right) z + \alpha(\xi, \tau), \quad \text{(2.9)} \]
where \( \alpha(\xi, \tau) \) is a constant of integration. From (2.7) we infer that \( \alpha \equiv 0 \) and hence on the boundary \( w_{10} \) must satisfy
\[ w_{10}\big|_{z=1} = -\left( \eta_{10} \xi + \eta_{00} \tau + \eta_{00} \eta_{00} \xi \right). \quad \text{(2.10)} \]

On the other hand, the boundary condition (2.6) tells us that
\[ w_{10}\big|_{z=1} = 2\eta_{00} \eta_{00} \xi + \eta_{00} \tau - \eta_{10} \xi. \quad \text{(2.11)} \]

Therefore, by subtracting (2.10) from (2.11), we infer that \( \eta_{00} \) satisfies
\[ 2\eta_{00} \tau + 3\eta_{00} \eta_{00} \xi = 0. \]

By exploiting this identity, we obtain from (2.9) that \( w_{10} \) satisfies
\[ w_{10} = -\eta_{10} \xi + \frac{1}{2} \eta_{00} \eta_{00} \xi. \]
and from (2.8) we infer that:

\[ u_{10} = \eta_{10} - \frac{1}{4} \eta_{60}. \]

This procedure can be continued successively up to any level in order to reveal the higher order terms of the corresponding equations for \( \eta, u, w \) and \( P \).

In this paper, we are interested in finding equations which approximate the governing equations in the regime

\[ \delta << 1, \quad \epsilon = O(\sqrt{\delta}). \tag{2.12} \]

We want to find model equations containing all terms up to the order \( O(\delta^{3}) \) in view of (2.12). Concerning the surface elevation \( \eta \), the following terms, which we obtained by continuing the above procedure, will be relevant:

\[
2\eta_{x} + 3\eta_{\xi} + \frac{1}{3} \delta^{2} \eta_{\xi\xi\xi\xi} - \frac{3}{4} \epsilon \eta_{z}^{2} \eta_{\xi} = - \frac{1}{12} \epsilon \delta^{2} \left( 23 \eta_{\xi} \eta_{\xi\xi} + 10 \eta_{\xi\xi\xi\xi} \right) \\
- \frac{3}{8} \eta_{z}^{2} \eta_{\xi} - \frac{1}{16} \epsilon^{2} \delta^{2} \left( 46 \eta_{\xi} \eta_{\xi\xi\xi} + 10 \eta_{\xi\xi\xi\xi\xi} + 19 \eta_{\xi}^{3} \right) + \frac{15}{64} \epsilon^{3} \eta_{z}^{4} \eta_{\xi} \\
- \frac{21}{128} \epsilon^{4} \eta_{z}^{6} \eta_{\xi} + \frac{27}{256} \epsilon^{5} \eta_{z}^{6} \eta_{\xi} + \ldots \tag{2.13}
\]

where \( \eta = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon^{n} \delta^{2m} \eta_{nm} \).

These are the terms that are needed in order to obtain approximate equations, written in \((x, t)\)-coordinates, with a remainder term of order \( O(\delta^{3}) \) in the regime (2.12). Note that the terms \( \epsilon \eta_{z}^{2} \eta_{\xi}, \epsilon \delta \eta_{\xi\xi\xi\xi\xi} \) and clearly any term of higher order will be contained in the remainder term after transforming back to the original \((x, t)\)-coordinates. Therefore these terms are not explicitly mentioned in (2.13).

The corresponding equation for the horizontal component \( u \) of the velocity field in terms of \( \eta \) and \( z \) is given by

\[
u = \eta - \frac{1}{4} \epsilon \eta_{z}^{2} + \frac{1}{8} \epsilon^{2} \eta_{z}^{3} - \frac{5}{64} \epsilon^{3} \eta_{z}^{4} + \eta \frac{1}{128} \epsilon^{4} \eta_{z}^{5} + \epsilon^{2} \left( \frac{1}{3} - \frac{1}{2} \epsilon^{2} \right) \eta_{\xi\xi} \\
- \frac{9}{256} \epsilon^{5} \eta_{z}^{6} + \epsilon^{2} \delta^{2} \left[ \left( \frac{1}{2} + \frac{1}{4} \epsilon^{2} \right) \eta \eta_{\xi\xi} + \left( \frac{3}{16} + \frac{1}{4} \epsilon^{2} \right) \eta_{\xi}^{2} \right] + \ldots \tag{2.14}
\]

with \( u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon^{n} \delta^{2m} u_{nm} \).

The further procedure is the following. We would like to derive an equation for the horizontal component of the velocity field \( u \) from (2.13) and (2.14), which purely consists of \( u \)-terms. Since \( u \) does not only depend upon the time variable and the horizontal space variable, but also upon the water depth, this can only be realized by considering \( u \) evaluated at a fixed depth of water \( z_{0} \in [0, 1] \). We are then able to invert (2.13), i.e. to write \( \eta \) in terms of \( u|_{z_{0}} \). This expression for \( \eta \) can be plugged into (2.13) afterwards, and the outcome will be an evolution equation for the unknown \( u|_{z_{0}} \). To this end we define \( \hat{u} := u|_{z_{0}} \) and write (2.14) as

\[
\hat{u} = \eta - \frac{1}{4} \epsilon \eta_{z}^{2} + \frac{1}{8} \epsilon^{2} \eta_{z}^{3} - \frac{5}{64} \epsilon^{3} \eta_{z}^{4} + \eta \frac{1}{128} \epsilon^{4} \eta_{z}^{5} - \frac{9}{256} \epsilon^{5} \eta_{z}^{6} \\
+ \epsilon^{2} \delta^{2} \lambda_{1} \eta_{\xi\xi} + \epsilon^{2} \delta^{2} \left( \lambda_{2} \eta \eta_{\xi\xi} + \lambda_{3} \eta_{\xi}^{2} \right) \tag{2.15}
\]

where \( \lambda_{1} := \frac{1}{4} - \frac{1}{2} z_{0}^{2}, \lambda_{2} := \frac{1}{2} + \frac{1}{2} z_{0}^{2} \) and \( \lambda_{3} := \frac{3}{16} + \frac{1}{4} z_{0}^{2} \) for a fixed depth of water \( z_{0} \).

In view of the computations below, it is convenient to distinguish between these three
parameters, despite the fact, that they actually depend only on one fixed value $z_0$, which will be determined later on. With this in mind, we rewrite (2.15) in order to get an expression for $\eta$:

$$\eta = \hat{u} + \varepsilon^2 \eta^2 - \varepsilon^2 \eta^3 + \frac{5}{64} \eta^4 - \frac{7 \varepsilon^4}{128} \eta^5 + \frac{9 \varepsilon^5}{256} \eta^6 - \varepsilon^2 \lambda_1 \eta \xi \xi - \varepsilon^2 \delta^2 \left( \lambda_2 \eta \xi \xi + \lambda_3 \eta \xi \right).$$

(2.16)

Next we express $\eta$, i.e. the right hand side of (2.16), in terms of $\hat{u}$. This can be computed recursively and one thereby obtains

$$\eta = \hat{u} + \frac{1}{4} \varepsilon \hat{u}^2 - \frac{3}{512} \varepsilon^5 \hat{u}^6 - \varepsilon^2 \lambda_1 \hat{u} \xi \xi - \varepsilon^2 \delta^2 \left[ \left( \lambda_1 + \lambda_2 \right) \hat{u} \xi \xi + \left( \frac{1}{2} \lambda_1 + \lambda_3 \right) \hat{u}^2 \xi \right].$$

Plugging this expression for $\eta$ into (2.13) gives

$$2 \hat{u}_t + 2 \hat{u}_x + 3 \varepsilon \hat{u} \hat{u}_x + \varepsilon \hat{u} \hat{u}_{xx} + \frac{1}{3} \delta^2 \hat{u} \hat{u}_{xxx} = -\varepsilon^2 \left[ \left( 6 \lambda_1 + \frac{29}{12} \right) \hat{u} \xi \xi + \frac{5}{6} \hat{u} \xi \xi \right]$$

$$- \varepsilon^2 \delta^2 \left( e \hat{u}_x \hat{u}_{xx} + f \hat{u} \hat{u}_{xxx} + g \hat{u}_x^3 \right)$$

(2.17)

where

$$e = \frac{31}{8} + 9 \lambda_1 + 6 \lambda_2, \quad f = \frac{5}{6} + \frac{3}{2} \lambda_1, \quad g = \frac{103}{48} + \frac{3}{2} \lambda_1 + 3 \lambda_3.$$

Next we transform back to the original variables $x$ and $t$. Recall that $\partial_\xi = \frac{1}{\sqrt{\varepsilon}} \partial_x$ and $\varepsilon \partial_\tau = \frac{1}{\sqrt{\varepsilon}} \left( \partial_t + \partial_x \right)$. It is therefore convenient to multiply (2.17) by $\varepsilon$ first, then apply the change of variables and multiply the outcome by $\sqrt{\varepsilon}$ in order to obtain

$$2 \hat{u}_t + 2 \hat{u}_x + 3 \varepsilon \hat{u} \hat{u}_x + \varepsilon \hat{u} \hat{u}_{xx} + \frac{1}{3} \delta^2 \hat{u} \hat{u}_{xxx} = -\varepsilon^2 \left[ \left( 6 \lambda_1 + \frac{29}{12} \right) \hat{u} \xi \xi + \frac{5}{6} \hat{u} \xi \xi \right]$$

$$- \varepsilon^2 \delta^2 \left( e \hat{u}_x \hat{u}_{xx} + f \hat{u} \hat{u}_{xxx} + g \hat{u}_x^3 \right)$$

(2.18)

with a remainder term of order $O(\delta^4)$. Equation (2.18) tells us that

$$\hat{u}_t + \hat{u}_x + \frac{3}{2} \varepsilon \hat{u} \hat{u}_x = O(\delta^2).$$

Therefore

$$\mu \delta^2 \left( \hat{u}_t + \hat{u}_x + \frac{3}{2} \varepsilon \hat{u} \hat{u}_x \right) = O(\delta^4)$$

for any real $\mu$. We differentiate this expression twice with respect to $x$:

$$\mu \delta^2 \left( \hat{u}_{xx} + \hat{u}_{xxx} + \varepsilon \left( \frac{9}{2} \hat{u}_x \hat{u}_{xx} + \frac{3}{2} \hat{u} \hat{u}_{xxx} \right) \right)$$

and get an $O(\delta^4)$-term, which we may subtract from the left hand side of (2.18). Due to this manipulation we obtain the following two-parameter family of asymptotically
We obtain equivalent equations
\[2 \hat{u}_t + 2 \hat{u}_x + 3 \varepsilon \hat{u}_x + \delta^2 (a \hat{u}_{xxx} - \mu \hat{u}_{xxt}) = - \varepsilon \delta^2 \left( c \hat{u}_x \hat{u}_{xx} + d \hat{u}_x \hat{u}_{xxx} \right) - \varepsilon^2 \delta^2 \left( e \hat{u}_x \hat{u}_{xx} + f \hat{u}_x^{2} \hat{u}_{xxx} + g \hat{u}_x^{3} \right)\] (2.19)
with a remainder term of order \(O(\delta^4)\), where
\[a = \frac{1}{3} - \mu, \quad c = \frac{9}{2} - \frac{29}{12} \mu, \quad d = \frac{5}{6} - \frac{3}{2} \mu.\]

Finally we make an appropriate choice for \(\mu\) and \(z_0\). In order to obtain a ratio 2:1 of the coefficients of \(\hat{u}_x \hat{u}_{xx}\) and \(\hat{u}_x \hat{u}_{xxx}\) respectively, cf. the Camassa-Holm equation, the real factor \(\mu\) has to satisfy
\[\mu = \frac{1}{2} + 4 \lambda_1.\]
Moreover we would like the \(\varepsilon^2 \delta^2\)-term in (2.19) to have a certain structural property by choosing \(z_0\) appropriately. We try to find some \(z_0 \in [0,1]\), such that this term can be written as an \(x\)-derivative. Indeed, there exists exactly one value \(z_0\) with this property, namely
\[z_0 = \sqrt{\frac{19}{6}}\]
With this choice we get \(\lambda_1 = \frac{5}{72}, \quad \mu = \frac{7}{5}, \quad \lambda_2 = \frac{15}{17}\) and \(\lambda_3 = \frac{7}{21}\), thus we obtain (1.1) from (2.19), with a remainder term of order \(O(\delta^4)\); note that we have set \(u := \hat{u} \) in (1.1) for the purpose of better readability.

The corresponding one-parameter family for the evolution of the free surface in the regime (2.12) reads
\[\eta_t + \eta_x + \frac{3}{2} \varepsilon \eta_x + \delta^2 (a \eta_{xxx} + \mu \eta_{xxt}) - \frac{3}{8} \varepsilon^2 \eta_x^2 \eta_x + \frac{3}{16} \varepsilon^3 \eta_x^3 \eta_x \]
\[= - \frac{15}{128} \varepsilon^4 \eta_x^4 \eta_x - \frac{1}{24} \varepsilon \delta^2 \left( c \eta_x \eta_{xx} + d \eta_{xxx} \right) - \frac{21}{256} \varepsilon^5 \eta_x^5 \eta_x + \frac{27}{512} \varepsilon^6 \eta_x^6 \eta_x - \frac{1}{32} \delta^2 \left( \varepsilon^4 \eta_x \eta_{xx} + f \varepsilon^2 \eta_x^2 \eta_{xxx} + g \varepsilon^3 \eta_x^3 \right)\] (2.20)
with a remainder term of order \(O(\delta^4)\) and coefficients
\[a = \frac{1}{6} + \mu, \quad c = 23 + 108 \mu, \quad d = 10 + 36 \mu,\]
\[e = 46 - 72 \mu, \quad f = 10 - 12 \mu, \quad g = 19 - 24 \mu.\]
We obtain (2.20) directly from (2.13) after the transformation to \((x,t)\)-coordinates and by inserting a term \(\mu \delta^2 \eta_{xxt}\) analogously as we did in order to get (2.19). By choosing \(\mu := -\frac{3}{12}\) we obtain the following equation for the surface elevation \(\eta\):
\[\eta_t + \eta_x + \frac{3}{2} \varepsilon \eta_x + \frac{6}{12} \delta^2 \eta_{xxx} - \frac{3}{8} \varepsilon^2 \eta_x^2 \eta_x + \frac{3}{16} \varepsilon^3 \eta_x^3 \eta_x - \frac{15}{128} \varepsilon^4 \eta_x^4 \eta_x - \frac{7}{24} \varepsilon \delta^2 \left( 2 \eta_x \eta_{xx} + \eta_{xxx} \right) - \frac{21}{256} \varepsilon^5 \eta_x^5 \eta_x + \frac{27}{512} \varepsilon^6 \eta_x^6 \eta_x - \frac{1}{32} \varepsilon^2 \delta^2 \left( 52 \eta_x \eta_{xx} + 11 \eta_x^2 \eta_{xxx} + 21 \eta_x^3 \right).\] (2.21)
cf. the equation for the evolution of the free surface for waves of moderate amplitude $(\delta \ll 1$ and $\varepsilon = \mathcal{O}(\delta))$ mentioned earlier in the introduction.

3 Local well-posedness

In this section we show local well-posedness of the Cauchy problem for (1.1) on the real line:

\begin{equation}
\begin{aligned}
\{ & 1.1 \} \text{ is satisfied for } x \in \mathbb{R}, \ t > 0 \\
& u(0, x) = u_0(x) \quad x \in \mathbb{R}.
\end{aligned}
\end{equation}

More precisely, we prove the following result:

**Theorem 3.1.** For every $u_0 \in H^3(\mathbb{R})$ there is a maximal time of existence $T \in (0, \infty]$ and a unique $u \in C([0, T); H^3(\mathbb{R})) \cap C^1((0, T); H^2(\mathbb{R}))$ which solves (1.1) in $H^2(\mathbb{R})$ for $t \in (0, T)$ and satisfies the initial condition $u(0, x) = u_0(x)$.

Moreover the mapping which maps initial data $u_0$ to unique local solutions $u$ of (1.1) is continuous as a mapping from $H^3$ to $C([0, \tilde{T}); H^3(\mathbb{R}))$ for sufficiently small $0 < \tilde{T} \leq T$.

This theorem is based on Kato’s semigroup approach for quasilinear evolution equations, see [22]. Below we state a simplified version of Kato’s theorem, which applies to our particular situation, see Theorem 3.2. Let us first introduce its assumptions.

We consider the following abstract quasilinear initial value problem in a Hilbert space $X$:

\begin{equation}
\begin{aligned}
y_t &= A(y)y + f(y) \quad t > 0 \\
y(0) &= y_0.
\end{aligned}
\end{equation}

Let $Y$ be another Hilbert space which is continuously and densely embedded into $X$, and let $S: Y \to X$ be a topological isomorphism. Assume furthermore that

(i) For any given $r > 0$ it holds that for all $y \in B_r(0) \subseteq Y$ (the ball around zero in $Y$ with radius $r$), the linear operator $A(y): X \to X$ generates a strongly continuous semigroup $T_y(t)$ in $X$ which satisfies

$$
\|T_y(t)\| \leq e^{\omega_r t} \quad \text{for all } t \in [0, \infty)
$$

for a uniform constant $\omega_r > 0$.

(ii) $A$ maps $Y$ into $\mathcal{L}(Y, X)$, more precisely the domain $D(A(y))$ contains $Y$ and the restriction $A(y)|_Y$ belongs to $\mathcal{L}(Y, X)$ for any $u \in Y$. Furthermore $A$ is Lipschitz continuous in the sense that for all $r > 0$ there exists a constant $C_1$ which only depends on $r$ such that

$$
\|A(y) - A(z)\|_{\mathcal{L}(Y, X)} \leq C_1 \|y - z\|_X
$$

for all $y, z$ inside $B_r(0) \subseteq Y$.

(iii) For any $y \in Y$ there exists a bounded linear operator $B(y) \in \mathcal{L}(X)$ satisfying $B(y) = S A(y) S^{-1} - A(y)$ and $B: Y \to \mathcal{L}(X)$ is uniformly bounded on bounded sets in $Y$. Furthermore for all $r > 0$ there exists a constant $C_2$ which depends only on $r$ such that

$$
\|B(y) - B(z)\|_{\mathcal{L}(X)} \leq C_2 \|y - z\|_Y
$$

for all $y, z \in B_r(0) \subseteq Y$. 


Theorem 3.2 (Kato [22]). Let the assumptions (i)-(iv) hold true. Then for any $y_0 \in Y$ there exists a maximal time of existence $T \in (0, \infty)$ and a unique solution $y \in C([0, T); Y) \cap C^1((0, T); X)$ of the abstract Cauchy problem (3.25) in $X$.

Moreover the solution depends continuously on the initial data, i.e. the mapping which maps initial data $y_0$ to unique local solutions $y$ of (3.23) is continuous from $Y$ to $C([0, T]; Y)$ for sufficiently small $0 < T \leq T$.

In the sequel we will, for notational convenience, demonstrate local well-posedness of the following initial value problem with different coefficients:

$$
\begin{aligned}
\begin{cases}
  u_t + u_x + 2uu_x - u_{xxx} - u_{xxt} \\
  w(x, 0) = w_0(x);
\end{cases}
\end{aligned}
$$

see Theorem 3.3. As a consequence we may infer local well-posedness of (3.22), i.e. Theorem 3.1. This is a valid maneuver, since the concrete values of the coefficients in (3.24) (except the sign of the term $u_{xxt}$) have no influence on the local well-posedness result; all constructions and proofs below are independent of the particular choice of coefficients. Note however, that a positive coefficient of the term $u_{xxt}$ would imply ill-posedness of the linear part of the the corresponding equation, which can be seen from passing to Fourier transforms. This would in turn imply ill-posedness of the whole equation. Since our framework relies on a quasilinear first order reformulation of the problem, we have to assume that the coefficient of $u_{xxt}$ is strictly negative; for convenience our choice is $-1$.

Next we introduce the framework that makes Theorem 3.2 applicable to (3.24). Let $Q := (1 - \partial_x^2)^\frac{3}{2}$ and set $y := Q^2 u$. Then $u = Q^{-2} y$ and equation (3.24) transforms into the quasilinear first order equation

$$
\begin{aligned}
  y_t &= -y_x - (Q^{-2} y)_x y + (Q^{-2} y)^2 y_x - (Q^{-2} y)_x y + (Q^{-2} y)(Q^{-2} y)_x y \\
  &\quad - 2(Q^{-2} y)^2 (Q^{-2} y)_x y - (Q^{-2} y)^3.
\end{aligned}
$$

In view of Theorem 3.2 we choose the spaces $X := L^2(\mathbb{R})$, $Y := H^1(\mathbb{R})$ and consider the isometric isomorphism $S := Q: H^1(\mathbb{R}) \to L^2(\mathbb{R})$. In the sequel we will for simplicity drop the domain $\mathbb{R}$ when speaking about function spaces and use the shorthand notation $L^2 = H^0$, $L^\infty$, $H^k$, etc. instead. Furthermore we will simply write $\int$ instead of $\int_{\mathbb{R}}$. The character $K$ stands for a generic constant throughout this paper. For $y \in H^1$ we define

$$
A(y)w := a_y \partial_x w + b_y w
$$

with

$$
a_y := -1 - (Q^{-2} y) + (Q^{-2} y)^2 \quad \text{and} \quad b_y := -(Q^{-2} y)_x + (Q^{-2} y)(Q^{-2} y)_x.
$$
Note that \(1 + a_y \in H^3\) and \(b_y \in H^2\) for all \(y \in H^1\), since \(H^k\) is closed under multiplication for integers \(k \geq 1\). The domain \(D(A(y))\) is the space \(H^1\). In addition to the definition for the operators \(A(y)\) we consider the function \(f : H^1 \to H^1\) given by
\[
f(y) := -2(Q^{-2}y)^2(Q^{-2}y)_x - (Q^{-2}y)_x^3.
\]
Now equation (3.25) takes the desired form of the operator equation in the abstract initial value problem (3.23):
\[
y_t = A(y)y + f(y).
\]

**Theorem 3.3.** For every \(y_0 \in H^1\) there exists a maximal time of existence \(T \in (0, \infty]\) and a unique \(y \in C([0,T); H^1) \cap C^1((0,T); L^2)\) which solves (3.25) in \(L^2\) for \(t \in (0,T)\) and satisfies \(y(0, x) = y_0(x)\).

Moreover the mapping which maps initial data \(u_0\) to unique local solutions of (3.24) is continuous as a mapping from \(H^1\) to \(C([0,T]; H^1)\) for sufficiently small \(0 < T \leq T^*\).

The subsequent lemmas verify the assumptions of Theorem 3.2 and thereby yield Theorem 3.3. As already mentioned above, we can then deduce the assertion of Theorem 3.1 i.e. the well-posedness result for (1.1).

The fact that the linear operators \(A(y) : L^2 \to L^2\), \(y \in H^1\), satisfy assumption (i) of Theorem 3.2 is - in light of Sobolev’s embedding theorem - an immediate consequence of the following lemma. Its proof is very similar to the reasoning in the proof of proposition 2.3 in [4].

**Lemma 3.4.** Let \(a \in C^1 \cap L^\infty\) with \(a' \in L^\infty\) and let \(b \in L^\infty\). Then the operator
\[
A = a(x)\partial_x + b(x)\text{id}\quad \text{with}\quad D(A) = \{u \in L^2 : au \in H^1\}
\]
lies in \(G(1, \omega, L^2)\), i.e. \(A\) generates a strongly continuous semigroup \(T(t)\) in \(L^2\) which satisfies
\[
\|T(t)\|_{L^2} \leq e^{\omega t}\quad \text{with}\quad \omega = \frac{1}{2} \sup_{x \in \mathbb{R}} \{ |a'(x)| \} + \sup_{x \in \mathbb{R}} \{ |b(x)| \}.
\]

**Proof.** The strategy of the proof is to recognize the operator \(A\) as some bounded perturbation of a skew-symmetric operator. Stone’s theorem and a standard perturbation result then yield the assertion. The proof is divided into four steps. In the first step we show that \(A\) is a well-defined linear operator in \(L^2\). Step two is an approximation result, that ensures that the domain of the formal adjoint of the before mentioned operator is large enough. Step three then reveals its skew-symmetry. Finally in step four we apply Stone’s theorem and a perturbation result.

**Step 1.** Let us write \(A = \tilde{A} + b(x)\text{id}\), where the action of the linear operator \(\tilde{A}\), the principal part of \(A\), with domain \(D(\tilde{A}) = D(A)\) is given by \(\tilde{A}u := (au)_x - axu\). The usual multiplication \(H^1 \times H^1 \to H^1\) has a unique continuous extension to a product \(H^1 \times H^{-1} \to H^{-1}\). The generalized Leibniz formula for distributions restricted to this product gives for any \(u \in L^2\)
\[
\partial_x(au) = axu + au_x \quad \text{in} \ H^{-1}.
\]
If \(u \in D(A)\), then both \((au)_x\) and \(axu\) belong to \(L^2\), thus
\[
\tilde{A}u = (au)_x - axu = au_x \in L^2.
\]
and we infer that \( \tilde{A} \) is a well-defined linear operator in \( L^2 \) and so is \( A \). Since the function \( a(x) \) in the definition of \( A \) can be zero on arbitrary subsets of \( \mathbb{R} \), \( A \) can be degenerate.

Step 2. We prove that \( C^\infty \cap L^2 \) is a core of \( \tilde{A} \) in \( L^2 \), so that we can approximate \( \tilde{A}v \) by a sequence \( (\tilde{A}v_n)_n \) in \( L^2 \), where each \( v_n \) is smooth and square integrable. Let \( v \in D(\tilde{A}) \), let \( \rho \in C_c^\infty \) be a mollifier with \( \text{supp}(\rho) \subseteq [-1, 1] \). We approximate \( v \) in \( L^2 \) by \( v_n := \rho_n \ast v \), where \( \rho_n := n\rho(nx) \). We show that \( \tilde{A}v_n \to \tilde{A}v \) in \( L^2 \). Note that we may write

\[
\tilde{A}(\rho_n \ast v) = a(\rho_n \ast v)_x - \rho_n \ast (av_x - a_x v) + \rho_n \ast (av_x - a_x v),
\]

hence the claim follows, once we show that

\[
P_n v \to 0 \quad \text{in} \ L^2.
\]

If \( v \in C_c^\infty \), then the limit in (3.26) obviously holds true. In what follows, we show that the family \( \{P_n\}_n \) extends to a family of uniformly bounded linear operators in \( L^2 \). Then the density \( C^\infty \cap L^2 \) yields (3.26) for arbitrary \( v \in D(\tilde{A}) \), and thus \( \tilde{A}v_n \to \tilde{A}v \) in \( L^2 \).

Every \( P_n \) is well-defined on \( L^2 \) via the extension

\[
P_n v = a((\rho_n)_x \ast v) - (\rho_n)_x \ast av + (\rho_n \ast (a_x v)).
\]

Therefore we obtain for \( x \in \mathbb{R} \) that

\[
P_n v(x) = n^2 \int_{-1}^{1} \rho_b(ny)(a(x) - a(x - y))v(x - y) \, dy + (\rho_n \ast (a_x v))(x)
\]

\[
= n \int_{-1}^{1} \rho_b(ny) \left(a(x) - a\left(x - \frac{y}{n}\right)\right)v\left(x - \frac{y}{n}\right) \, dy + (\rho_n \ast (a_x v))(x),
\]

and, by means of the mean value theorem, we estimate

\[
\left| \left( \frac{n^2}{\|a_x\|_L^\infty} \int_{-1}^{1} |\rho_b(ny)||y||v(x - \frac{y}{n})| \right. \right| \left. \, dy \right|
\]

\[
\leq \|a_x\|_L^\infty \int_{-1}^{1} \left| |\rho_b(ny)||y||v(x - \frac{y}{n})| \right. \left. \, dy \right|
\]

If we set \( C := \|a_x\|_L^\infty \int_{-1}^{1} |\rho_b(y)||y|^2 \, dy \), Cauchy Schwarz’s inequality and Fubini’s theorem allow us to estimate

\[
\left\| \frac{n^2}{\|a_x\|_L^\infty} \int_{-1}^{1} \rho_b(ny)(a(x) - a(x - y))v(x - y) \, dy \right\|_{L^2}^2
\]

\[
\leq C \int_{-1}^{1} \int_{-1}^{1} v^2 \left(x - \frac{y}{n}\right) \, dx \, dy = C \int_{-1}^{1} \int_{-1}^{1} v^2 \left(x - \frac{y}{n}\right) \, dx \, dy
\]

\[
= C \int_{-1}^{1} \|v\|_{L^2}^2 \, dy = 2C\|v\|_{L^2}^2.
\]

Since by Young’s inequality,

\[
\|\rho_n \ast (a_x v)\|_{L^2}^2 = \|\rho \ast (a_x v)\|_{L^2}^2 \leq \|\rho\|_{L^2}^2 \| (a_x v) \|_{L^2}^2
\]

\[
= \| (a_x v) \|_{L^2}^2 \leq \|a_x\|_L^\infty \|v\|_{L^2}^2,
\]

\footnote{This is not the case in our particular application: \( A(y) \) is nonzero away from bounded sets for any \( y \in H^1 \).}
we can conclude that $P_n \in \mathcal{L}(L^2)$ for every $n \in \mathbb{N}$ with a uniform bound
\[ \|P_n\|^2_{\mathcal{L}(L^2)} \leq M, \]
where $M$ can be taken to be $M = 2 \max\{2C, \|a_x\|^2\}$.

**Step 3.** We show that $A_0 := \bar{A} + \frac{1}{2}a_x \text{id}$ with domain $D(A_0) = D(\bar{A})$ is skew-adjoint in $L^2$.

Let us denote by $A_0^*$ the adjoint of $A_0$, which is uniquely defined since $A_0$ is densely defined. Fix $w \in D(A_0^*)$, then the linear functional $F_w : C_c^\infty \to \mathbb{R}$ given by
\[ F_w(\varphi) := (A_0, w)_L^2 = \int \left(a \varphi_x + \frac{1}{2}a_x \varphi\right) w \, dx = (\varphi, A_0^* w)_L^2 \]
is continuous with respect to $\| \cdot \|_{L^2}$. On the other hand, $F_w(\varphi)$ coincides with $-(aw_x + \frac{1}{2}a_x w, \varphi)$, the action of the distribution $-(aw_x + \frac{1}{2}a_x w)$ on the test functions $\varphi$. This shows that
\[ A_0^* w = -(aw_x + \frac{1}{2}a_x w) = -A_0 w \in L^2, \]
hence $aw \in H^1$ which in turn implies that $w \in D(A_0)$. This shows that $A_0^* \subseteq -A_0$.

Conversely, let $v \in D(A_0)$ and let $v_n := \rho_n * v$. Due to the approximation result in step 3, we obtain for any $w \in D(A_0)$ that
\[ (A_0 w, v)_L^2 = \lim_{n \to \infty} (A_0 w, v_n)_L^2 = \lim_{n \to \infty} \int \left(a \varphi_x + \frac{1}{2}a_x \varphi\right) v_n \]
\[ = -\lim_{n \to \infty} \int \left(a (v_n)_x + \frac{1}{2}a_x v_n\right) w \]
\[ = -\int \left(a v_x + \frac{1}{2}a_x v\right) w = -(w, A_0 v)_L^2, \]
which shows that $-A_0 \subseteq A_0^*$.

**Step 4.** Due to step 3, the operator $iA_0$ is self-adjoint:
\[ (iA_0 v, z)_L^2 = -i(A_0 v, z)_L^2 = i(v, A_0 z)_L^2 = (v, iA_0 z)_L^2. \]

Therefore, Stone’s theorem implies that $A_0$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2$.

Since $A_0 - \bar{A} = \frac{1}{2} \text{id}$ is bounded in $L^2$, a standard perturbation result in the theory of operator semigroups, see e.g. [20], implies that $\bar{A}$ is the generator of a strongly continuous semigroup of contractions $S(t)$, which satisfies
\[ \|S(t)\|_{L^2} \leq e^{\frac{1}{2}\|a_x\| \|t\|} \quad \text{for all} \quad t \geq 0. \]

Applying the same perturbation result once more, yields that $A$ is the generator of a strongly continuous semigroup $T(s)$ which satisfies
\[ \|T(s)\|_{L^2} \leq e^{\frac{1}{2}\|a_x\| + \|b\| \|s\|} \quad \text{for all} \quad s \geq 0. \]

The next lemma verifies assumption (ii) of Theorem 3.2, i.e. the continuity properties of the operators $A(y)$ and $A$. 

\[ \square \]
Lemma 3.5. The operators $A(y)$ belong to $\mathcal{L}(H^1, L^2)$ for all $y \in H^1$. Furthermore for all $r > 0$ there exists a constant $C_1$ which depends only on $r$ such that
\[
\|A(y) - A(z)\|_{\mathcal{L}(H^1, L^2)} \leq C_1 \|y - z\|.
\]
for all $y, z \in B_r(0)$.

Proof. It is clear that $A(y)$, a linear first order differential operator, lies in $\mathcal{L}(H^1, L^2)$ for all $y \in H^1$. Let $y, z, w \in H^1$. Exploiting the fact that $H^1$ is closed under multiplication and that $Q^{-2} \in \mathcal{L}(L^2, H^2)$ yields the estimate:
\[
\|(A(y) - A(z))w\|_{L^2} \\
= \|[-Q^{-2}(y - z) + Q^{-2}(y + z)Q^{-2}(y - z)]w_x \\
+ [-Q^{-2}(y - z) + \frac{1}{2}Q^{-2}(y - z)Q^{-2}(y + z)]w_x w\|_{L^2} \\
\leq K \|(Q^{-2}(y - z))\|_{H^2} \left(1 + \|Q^{-2}(y + z)\|_{H^2}\right)\|w\|_{H^1} \\
\leq K \|y - z\|_{L^2} \left(1 + \|y\|_{L^2} + \|z\|_{L^2}\right)\|w\|_{H^1},
\]
where $K$ stands for a generic constant. This shows that there exists always some constant $C_1$ depending upon the distance from $y$ and $z$ to the origin (in $H^1$) such that
\[
\|A(y) - A(z)\|_{\mathcal{L}(H^1, L^2)} \leq C_1 \|y - z\|_{L^2}.
\]

The lemma below validates assumption (iii) of Theorem 3.2 on the family of operators $B(y)$.

Lemma 3.6. The following assertions hold.

(i) The operator $B(y)$ given by $B(y) := QA(y)Q^{-1} - A(y)$ is bounded in $L^2$ for any $y \in H^1$ and the family $\{B(y)\}_{\|y\|_{H^1} \leq C}$ is uniformly bounded for every $C > 0$.

(ii) For all $r > 0$ there exists a constant $C_2$ which depends only on $r$ such that
\[
\|B(y) - B(z)\|_{\mathcal{L}(L^2)} \leq C_2 \|y - z\|_{H^1}
\]
for all $y, z \in B_r(0) \subseteq H^1$.

Proof. First we give an alternative representation of $B(y)$. Let us therefore introduce the multiplication operators $M_{a_y}$ and $M_{b_y}$ defined on $L^2$ given by
\[
M_{a_y}w := a_yw = \left(-1 - (Q^{-2}y) + (Q^{-2}y)^2\right)w \\
M_{b_y}w := b_yw = \left((Q^{-2}y)_x + (Q^{-2}y)(Q^{-2}y)_x\right)w
\]
for $y \in H^1$. Note that all these operators are bounded in $L^2$ and uniformly bounded on bounded sets in $H^1$. The same holds true for their restrictions on $H^1$. 
We observe that \[ \frac{\partial}{\partial x} \] holds for all \( H \)
and let \( \partial_{\xi} \) be an element of the Schwartz space \( S \) and let \( F \) denote the Fourier transform with its inverse \( F^{-1} \). Then \( \partial_{\xi}g = F^{-1}(i\xi F(g)) \) and \( Qg = F^{-1}(\sqrt{1 + \xi^2}F(g)) \), thus \( (Q\partial_{\xi})(g) = F^{-1}(\sqrt{1 + \xi^2}F(F^{-1}(i\xi F(g)))) = F^{-1}(i\xi F(F^{-1}(\sqrt{1 + \xi^2}F(g)))) = (\partial_{\xi}Q)(g) \).

We observe that \( \partial_{\xi}Q^{-1} \) is bounded in \( L^2 \), \([Q, M_{b_y}]Q^{-1} \in \mathcal{L}(L^2) \) for every \( y \in H^1 \), and families of the form \( \{[Q, M_{b_y}]Q^{-1}\}_{y \in H^1} \) have a uniform bound. In view of the representation \( (3.27) \), it remains to show that the commutator \( [Q, M_{b_y}] \) has these properties in order to prove part (i) of the lemma. Therefore we apply a remarkable result from harmonic analysis, which states that it is sufficient for a commutator \( [P, M_{a_y}] \) to have a bounded extension in \( L^2 \) satisfying if \( P \) is a standard first order pseudo differential operator and \( h \) is Lipschitz continuous; see section VII.3.5.1 in [27]. In this case it holds that
\[
\| [P, M_{b_y}] \|_{\mathcal{L}(L^2)} \leq C_P \| h_x \|_{L^\infty}.
\] (3.28)
These assumptions are fulfilled in view of the embedding \( H^3 \subseteq W^{1,\infty} \) (recall that \( a_y + 1 \in H^3 \) for any \( y \in H^1 \)).

In order to see that (ii) holds, we exploit the identity \( (3.27) \), once more, use the commutator estimate \( (3.28) \), the fact that \( H^k \) is closed under multiplication and utilize the boundedness of \( Q^{-2} : H^2 \to L^2 \) in order to obtain the following inequality:
\[
\| B(y) - B(z) \|_{\mathcal{L}(L^2)}
\begin{align*}
&\leq \| [Q, M_{a_y-a_z}]\partial_x Q^{-1} \|_{\mathcal{L}(L^2)} + \| QM_{b_y-b_z}Q^{-1} \|_{\mathcal{L}(L^2)} + \| M_{b_y-b_z} \|_{\mathcal{L}(L^2)} \\
&\leq K \left( \| [Q, M_{a_y-a_z}] \|_{\mathcal{L}(L^2)} + \| M_{b_y-b_z} \|_{\mathcal{L}(H^2)} \right) \\
&\leq K \left( \| \partial_x (a_y - a_z) \|_{L^\infty} + \| b_y - b_z \|_{H^1} \right) \\
&\leq K \left( \| (a_y - a_z) \|_{H^2} + \| b_y - b_z \|_{H^1} \right) \\
&\leq K \left( \| Q^{-2}(z - y) + Q^{-2}(y + z)Q^{-2}(y - z) \|_{H^2} \\
&\quad + \| \partial_x (Q^{-2}(y - z) + \frac{1}{2} Q^{-2}(y + z)Q^{-2}(y - z)) \|_{H^1} \right) \\
&\leq K \left( \| y - z \|_{L^2} (1 + \| y \|_{L^2} + \| z \|_{L^2}) \right),
\end{align*}
\]
which holds for all \( y, z \in H^1 \); recall that \( K \) stands for a generic constant. Therefore it is clear that one can find for any given \( r > 0 \) some constant \( C_2 > 0 \) such that
\[
\| B(y) - B(z) \|_{\mathcal{L}(L^2)} \leq C_2 \| y - z \|_{L^2}
\]
for all \( y, z \in B_r(0) \subseteq H^1 \). Note that we actually proved a stronger assertion than the one in part (ii) of the lemma. \( \square \)
Finally we show that assumption (iv) of Theorem 3.2 i.e. the required Lipschitz assumptions on \( f \), is indeed satisfied:

**Lemma 3.7.** The function \( f : H^1 \to H^1 \) has the following properties:

(i) for every \( r > 0 \) there exists some constant \( C_2 > 0 \), which depends only upon \( r \), such that

\[
\|f(y) - f(z)\|_{L^2} \leq C_2 \|y - z\|_{L^2} \quad \text{for all} \quad y, z \in B_r(0) \subseteq H^1,
\]

(ii) and for every \( r > 0 \) there exists some constant \( C_3 > 0 \), which depends only upon \( r \), such that

\[
\|f(y) - f(z)\|_{H^1} \leq C_3 \|y - z\|_{H^1} \quad \text{for all} \quad y, z \in B_r(0) \subseteq H^1.
\]

**Proof.** We prove assertion (i) first. Let \( y, z \in H^1 \), and recall that \( C > 0 \) denotes a generic constant, then

\[
\|f(y) - f(z)\|_{L^2} = \|\frac{2}{3} \partial_y \left[ Q^{-2}(y-z) \right] \left( (Q^{-2}y)^2 + (Q^{-2}y)(Q^{-2}z) + (Q^{-2}z)^2 \right) + \partial_z \left[ Q^{-2}(y-z) \right] \left( (Q^{-2}y)_z^2 + (Q^{-2}y)_z(Q^{-2}z)_z + (Q^{-2}z)_z^2 \right) \|_{L^2} 
\]

\[
\leq \left\| \frac{2}{3} Q^{-2}(y-z) \left( (Q^{-2}y)^2 + (Q^{-2}y)(Q^{-2}z) + (Q^{-2}z)^2 \right) + Q^{-2}(y-z) \left( (Q^{-2}y)_z^2 + (Q^{-2}y)(Q^{-2}z) + (Q^{-2}z)_z^2 \right) \right\|_{H^2} 
\]

\[
\leq K \|Q^{-2}(y-z)\|_{H^2} \left( \|Q^{-2}y\|_{H^2}^2 + \|Q^{-2}y\|_{H^2} \|Q^{-2}z\|_{H^2} + \|Q^{-2}z\|_{H^2}^2 \right) 
\]

\[
\leq K \|y - z\|_{L^2} \left( \|y\|_{L^2}^2 + \|y\|_{L^2} \|z\|_{L^2} + \|z\|_{L^2}^2 \right).
\]

We exploited once again, that all the involved Sobolev spaces are closed under multiplication and the continuity of \( Q^{-2} : H^2 \to L^2 \). This shows, that for given \( r > 0 \) we can always find a constant \( C_3 > 0 \) such that

\[
\|f(y) - f(z)\|_{L^2} \leq C_3 \|y - z\|_{L^2}
\]

holds for all \( y, z \in H^1 \) with \( \|y\|_{H^1} < r \) and \( \|z\|_{H^1} < r \).

In order to prove assertion (ii), we can make a very similar estimation:

\[
\|f(y) - f(z)\|_{H^1} \leq \left\| \frac{2}{3} Q^{-2}(y-z) \left( (Q^{-2}y)^2 + (Q^{-2}y)(Q^{-2}z) + (Q^{-2}z)^2 \right) + Q^{-2}(y-z) \left( (Q^{-2}y)_z^2 + (Q^{-2}y)(Q^{-2}z) + (Q^{-2}z)_z^2 \right) \right\|_{H^3} 
\]

\[
\leq K \|Q^{-2}(y-z)\|_{H^3} \left( \|Q^{-2}y\|_{H^3}^2 + \|Q^{-2}y\|_{H^3} \|Q^{-2}z\|_{H^3} + \|Q^{-2}z\|_{H^3}^2 \right) 
\]

\[
\leq K \|y - z\|_{H^1} \left( \|y\|_{H^1}^2 + \|y\|_{H^1} \|z\|_{H^1} + \|z\|_{H^1}^2 \right),
\]

which yields the claim. \( \square \)
We just verified all assumptions of Theorem 3.2 and therefore obtain Theorem 3.3, which in turn implies Theorem 3.1 (see the discussion on the choice of coefficients above). Let us finally note, that we can actually deduce the analogous well-posedness results for the families (2.19) and (2.20) from Theorem 3.3; cf. Remark 4.7.

4 Higher regularity

The main result of this section is Proposition 4.6, which states that the regularity of the solution of the Cauchy problem for equation (1.1) in terms of the Sobolev exponent is inherited by the regularity of the initial data. This result relies - like Kato’s theorem does - on the theory of linear evolution systems. The idea of the proof, see [23], is to fix the solution \( y \) of the quasi-linear first order problem that corresponds (1.1) and to consider the following time dependent linear problem

\[
\begin{cases}
    \dot{v} = A(t)v + f(t) & \text{for } 0 < t \leq T' \\
    v(0) = v_0
\end{cases}
\]  

(4.29)

with the operator \( A(t) = A(y(t)) \) and inhomogeneity \( f(t) = f(y(t)) \). If we assume that the initial data lies in a higher order Sobolev space, it turns out that the solution of the linear problem, which coincides with the solution for the original quasilinear problem, does so too.

Below we state a theorem that applies to linear inhomogeneous non-autonomous evolution equations, which covers the situation at hand; see e.g. chapter 5.2 in [?].

**Theorem 4.1** ([26]). Let \( X \) and \( Y \) be Hilbert spaces, such that \( Y \) is densely embedded into \( X \), let \( S : Y \to X \) be an isomorphism and assume that the following hold.

- \((H_1)\) \( \{A(t)\}_{t \in [0, T']} \) is a stable family in \( X \).
- \((H_2)\) There exists a strongly continuous family of bounded linear operators \( \{B(t)\}_{t \in [0, T']} \) in \( X \) such that for all \( t \in [0, T'] \),
  \[
  SA(t)S^{-1} = A(t) + B(t).
  \]
- \((H_3)\) For \( t \in [0, T'] \), \( D(A(t)) \supseteq Y \), is a bounded linear operator from \( Y \) into \( X \) and \( t \to A(t) \) is continuous in the norm \( \| \cdot \|_{L(Y, X)} \).
- \((f)\) \( f \in C([0, T']); Y \). 

Then the initial value problem (4.29) possesses a unique \( Y \)-valued solution \( v \) for every \( v_0 \in Y \), i.e. \( v \) solves (4.29) in \( X \) and

\[
v \in C([0, T']; Y) \cap C^1((0, T'); X).
\]

In order to obtain the property \((H_2)\) not only in \( L^2 \), but in \( H^k \), we will prove the boundedness of the operators \( B(y) \) in these higher order Sobolev spaces for sufficiently regular \( y \), see Lemma 4.3. Therefore we need - as in the proof of Lemma 3.6 - suitable continuity properties of commutators \([Q, M_h]\):
**Lemma 4.2.** For $k = 0, 1, 2, 3, \ldots$ let $h \in W^{k+1, \infty}$ and denote by $M_h$ the corresponding multiplication operator. Then the commutator $[Q, M_h]$ has a bounded extension in $\mathcal{L}(H^k)$, which satisfies
\[
\| [Q, M_h] \|_{\mathcal{L}(H^k)} \leq C_k \left( \sum_{i=1}^{k+1} \| h^{(i)} \|_{L^\infty} \right)^{\frac{1}{2}}
\]
for some constant $C_k > 0$ depending only upon $k$.

**Proof.** The case $k = 0$ holds true by a corollary in section VII.3.5.1 in [27].

Let us assume that the assertion holds for $k \geq 1$ and consider the linear operator
\[
\partial_x [Q, M_h] Q^{-1} = ([Q, M_h'] + [Q, M_h] \partial_x) Q^{-1}.
\]
This operator belongs to $\mathcal{L}(H^k)$, hence we infer that $\partial_x [Q, M_h] \in \mathcal{L}(H^{k+1}, H^k)$. It follows that $[Q, M_h] \in \mathcal{L}(H^{k+1})$. In order to see this, let $v \in H^{k+1}$ be arbitrary. Then
\[
\| [Q, M_h] v \|_{H^{k+1}}^2 = \| [Q, M_h] v \|_{H^k}^2 + \int \left( \partial_x^{k+1} ([Q, M_h] v) \right)^2 \, dx.
\] (4.30)

Since we assumed that $\| [Q, M_h] v \|_{H^k}^2 \leq C_k \left( \sum_{i=0}^{k+1} \| h^{(i)} \|_{L^\infty} \right) \| v \|_{H^k}^2$, it remains to estimate the integral on the right-hand side in (4.30):
\[
\| \partial_x^{k+1} ([Q, M_h] v) \|_{L^2}^2 = \sum_{i=0}^{k+1} \binom{k+1}{i} \| [Q, M_h^{(k+1-i)}] \partial_x^i v \|_{L^2}^2 \leq \sum_{i=0}^{k+1} \binom{k+1}{i} \left( \| [Q, M_h^{(k+1-i)}] \partial_x^i v \|_{L^2} \right)^2 \leq C_{k+1} \sum_{j=0}^{k+1} \| [Q, M_h^{(j)}] \|_{L^2}^2 \| \partial_x^j v \|_{L^2}^2 \leq C_{k+1} \sum_{j=0}^{k+1} \| h^{(j)} \|_{L^\infty}^2 \| v \|_{H^{k+1}}^2.
\]

Hence we can find a universal constant $C_{k+1}$, such that
\[
\| [Q, M_h] v \|_{H^{k+1}}^2 \leq C_{k+1} \left( \sum_{i=1}^{k+2} \| h^{(i)} \|_{L^\infty} \right)^{\frac{1}{2}} \| v \|_{H^{k+1}}^2
\]
and the assertion of the lemma follows by induction. \hfill \Box

As a consequence of the above lemma, we obtain Lemma 4.3, which is a generalization of Lemma 3.6 to higher order Sobolev spaces. Lemma 4.2 permits us to make similar estimates as in the proof of Lemma 3.6.

**Lemma 4.3.** For $k = 0, 1, 2, 3, \ldots$ let $y \in H^k$. Then $B(y)$ is bounded in $H^k$ and uniformly bounded on bounded sets in $H^k$. Moreover for every $r > 0$ there exists a constant $C_{r,k} > 0$ such that
\[
\| B(y) - B(z) \|_{\mathcal{L}(H^k)} \leq C_{r,k} \| y - z \|_{H^k}
\]
for all $y, z \in B_r(0) \subseteq H^k$. 
Proof. Let \( y \in H^k \), then \( a_y + 1 \in H^{k+2} \) and therefore \( a_y \in W^{k+1, \infty} \) due to Sobolev’s embedding theorem. Recall that \( B(y) \) can be represented as

\[
B(y) = \{ Q, M_{a_y} \} \partial_x Q^{-1} + QM_{b_y}Q^{-1} - M_{b_y}
\]

and that \( b_y \in H^{k+1} \) by assumption. Therefore Lemma 4.2 implies that \( B(y) \in \mathcal{L}(H^k) \) and also the uniform boundedness on bounded sets in \( H^k \). The second claim follows from the estimation below:

\[
\| B(y) - B(z) \|_{2(\mathcal{L}(H^k))}^2 \leq K \left( \| [Q, M_{a_y-a_z}] \|_{2(\mathcal{L}(H^k))}^2 + \| QM_{b_y-b_z}Q^{-1} \|_{2(\mathcal{L}(H^k))}^2 + \| M_{b_y-b_z} \|_{2(\mathcal{L}(H^{k+1}))}^2 \right)
\]

\[
\leq K \left( \sum_{i=1}^{k+1} \| \partial_x^k (a_y - a_z) \|_{L^\infty}^2 + \| b_y - b_z \|_{H^{k+1}}^2 \right)
\]

\[
\leq K \left( \sum_{i=1}^{k+1} \| a_y - a_z \|_{H^{k+1}}^2 + \| b_y - b_z \|_{H^{k+1}}^2 \right)
\]

\[
\leq K \left( \| a_y - a_z \|_{H^{k+2}}^2 + \| b_y - b_z \|_{H^{k+1}}^2 \right)
\]

\[
\leq K \left( \| Q^{-2}(z-y) \|_{L^\infty}^2 \left( 1 + \| Q^{-2}(y+z) \|_{L^\infty}^2 \right) \right)
\]

\[
\leq K \left( \| y-z \|_{H^k}^2 \left( 1 + \| y \|_{H^k}^2 + \| z \|_{H^k}^2 \right) \right),
\]

where we applied Lemma 4.2 again, exploited that the involved Sobolev spaces are algebras of functions, and used that \( Q^{-2} : H^{k+2} \rightarrow H^k \) is an isometric isomorphism, thus particularly bounded.

The next lemma will allow us to deduce assumption \((H_1)\) of Theorem 4.1 for Sobolev spaces with higher exponent under the assumption of \((H^+_2)\). This is crucial in order to perform the induction steps in the proofs of Lemma 4.3 and Proposition 4.6.

Lemma 4.4 ([23]). The properties \((H_1)\) and \((H^+_2)\) in Theorem 4.1 imply \((H_2)\) \( \{ \tilde{A}(t) \}_{t \in [0, T^*]} \), the family of parts \( \tilde{A}(t) \) of \( A(t) \) in \( Y \), is stable in \( Y \).

We are now ready to prove the stability of operator families \( A(t) \) in Sobolev spaces of higher order:

Lemma 4.5. Let \( y \in C([0, T^*]; H^k) \), \( k = 1, 2, 3 \ldots \), \( T^* > 0 \), and set \( A_k(t) := A(y(t)) \). The family of operators \( \{ A_k(t) \}_{t \in [0, T^*]} \) with domain \( D(A_k(t)) = H^{k+1} \) is a stable in \( H^k \).

Proof. Let \( y \in C([0, T^*]; H^1) \). Due to Lemma 3.4 that there exists a constant \( \beta > 0 \) such that \( A_0(t) := A(y(t)) \in C^1(1, \beta, L^2) \) for all \( t \in [0, T^*] \); we could for example take

\[
\beta := \sup_{t \in [0, T^*]} \left\{ \frac{1}{2} \sup_{x \in \mathbb{R}} \left| a_y(t, x) \right| + \sup_{x \in \mathbb{R}} \left| b_y(t, x) \right| \right\} < \infty.
\]

Therefore, \( \{ A_0(t) \}_{t \in [0, T^*]} \) is stable in \( L^2 \) with domain \( D(A_0(t)) = H^1 \) for all \( t \in [0, T^*] \). Lemma 3.6 tells us that the collection \( B_0(t) := QA_0(t)Q^{-1} - A_0(t), t \in [0, T^*] \) is a
strongly continuous family of bounded linear operators in \(L^2\), since we assumed that \(y \in C([0,T];H^1)\), and thus

\[
\|B_0(t) - B_0(t^*)\|_{L^2} \leq \|B_0(t) - B_0(t^*)\|_{L^2} \leq C_r \|y(t) - y(t^*)\|_{H^k} \rightarrow 0 \quad (t \to t^*)
\]

for all \(t^* \in [0,T]\). Now, Lemma 4.4 yields that \(\{\tilde{A}_0(t)\}_{t \in [0,T]}\), where \(\tilde{A}_0(t)\) denotes the part of \(A_0(t)\) in \(H^1\), is a stable family in \(H^1\). By setting \(A_1(t) := \tilde{A}_0(t)\) for all \(t \in [0,T]\) and noting that \(D(A_1(t)) = H^2\), the claim follows for the case \(k = 1\).

Assume that the assertion holds true for \(k \geq 2\). Due to Lemma 4.3 we infer that \(B_k(t) := QA_k(t)Q^{-1} - A_k(t), 0 \leq t \leq T'\), is a strongly continuous family of bounded linear operators in \(H^k\). Since \(Q: H^{k+1} \rightarrow H^k\) is an isomorphism, Lemma 4.4 implies that \(A_{k+1}(t) := A_k(t), t \in [0,T']\), \(D(A_{k+1}(t)) = H^{k+2}\) is stable in \(H^{k+1}\). The assertion of the lemma follows by induction.

**Proposition 4.6.** For \(k = 1, 2, 3, \ldots\) let \(y_0 \in H^k\). Then the unique solution \(y\) of (3.25), according to Theorem 3.3, lies in the space \(C([0,T];H^k) \cap C^1((0,T);H^{k-1})\), where \(T\) denotes the maximal time of existence, which is independent of \(k\).

**Proof.** The assertion is proved by induction. The case \(k = 1\) holds true due to Theorem 3.1. For the induction step we take \(y_0 \in H^{k+1} \subseteq H^k\). By assumption there is a unique solution \(y\) of (3.25), lying in \(C([0,T];H^k) \cap C^1((0,T);H^{k-1})\). Let \(0 < T' < T\) be arbitrary. For \(t \in [0,T]\) we set \(A_k(t) := A(y(t))\). In the following we check the assumptions of Theorem 4.1 for \(X := H^k\) and \(Y := H^{k+1}\).

Assumption \((H_1)\) is satisfied due to Lemma 4.5, i.e. \(\{A_k(t)\}_{t \in [0,T']}\) with domain \(D(A_k(t)) = H^{k+1}\) is a stable in \(H^k\).

Furthermore, Lemma 4.3 yields \((H^k_{2.2})\): the family of bounded linear operators \(B_k(t) := QA_k(t)Q^{-1}, t \in [0,T']\), is strongly continuous.

Next, we verify \((H_3)\). We already mentioned that \(D(A_k(t)) = H^{k+1}\) for \(t \in [0,T']\) and it is clear that \(A_t(t) \in L(H^{k+1},H^k)\) for all \(t \in [0,T']\). We infer the continuity of the map \(t \mapsto A_k(t)\) in the norm \(\| \cdot \|_{L(H^{k+1},H^k)}\) from the following estimate:

\[
\|A_k(t) - A_k(t^*)\|_{L(H^{k+1},H^k)} \leq \sup_{|w|_{H^{k+1}} \leq 1} \left( \|(a_{y(t)} - a_{y(t^*)})w\|_{H^k} + \|(b_{y(t)} - b_{y(t^*)})w\|_{H^k} \right) 
\]

\[
\|a_{y(t)} - a_{y(t^*)}\|_{H^{k+2}} + \|b_{y(t)} - b_{y(t^*)}\|_{H^{k+1}} 
\]

Both norms on the right hand side in the above inequality tend to zero as \(t \to t^*\):

\[
\|a_{y(t)} - a_{y(t^*)}\|_{H^{k+2}} \leq \|Q^{-2}(y(t^*) - y(t))\|_{H^{k+2}} + \|Q^{-2}(y(t^*) + y(t))\|_{H^{k+2}} 
\]

\[
\|y(t^*) - y(t)\|_{H^k} + \|y(t^*) + y(t)\|_{H^k} 
\]

\[
K \|y(t^*) - y(t)\|_{H^k} \rightarrow 0 \quad \text{for} \quad t \to t^* 
\]

where \(K\) can be chosen independently of \(t\). Respectively we find that

\[
\|b_{y(t)} - b_{y(t^*)}\|_{H^{k+1}} \leq K \|Q^{-2}(y(t^*) - y(t))\|_{H^{k+2}} + \|Q^{-2}(y(t^*) + y(t))\|_{H^{k+2}} 
\]

\[
K \|y(t^*) - y(t)\|_{H^k} \rightarrow 0 \quad \text{for} \quad t \to t^*. 
\]
These limits hold true for every $t^* \in [0, T']$, because $y \in C([0, T'); H^k)$ by assumption.

Finally we observe that $f_k(t) := f(y(t))$ is contained in $C([0, T'); H^{k+1})$, since all its components share this property; e.g. $\partial_x(Q^{-2}y) \in C([0, T'); H^{k+1})$, because we assumed that $y \in C([0, T); H^k)$.

Now we can apply Theorem 4.1, which yields a unique solution $y_k$ of

$$
\begin{cases}
v_t = A_k(t)v + f_k(t) & \text{for } 0 < t \leq T' \\
v(0) = y_0,
\end{cases}
$$

in $H^k$, satisfying $y_k \in C([0, T'); H^{k+1}) \cap C^1((0, T'); H^k)$. On the other hand, $y_k$ is the unique solution of the initial value problem

$$
\begin{cases}
v_t = A_1(t)v + f_1(t) & \text{for } 0 < t \leq T' \\
v(0) = y_0,
\end{cases}
$$

which by construction coincides with (3.24) considered on the time interval $[0, T']$. Therefore we find that $y_k = y$, which proves that $y$ lies in the space $C([0, T'); H^{k+1}) \cap C^1((0, T'); H^k)$. Since $T' \in (0, T)$ was arbitrary, we conclude that $y \in C([0, T); H^{k+1}) \cap C^1((0, T); H^k)$.

\[\square\]

Remark 4.7. We note that both the local well-posedness result, Theorem 3.1, as well as Proposition 4.6 on the higher regularity of solutions hold true for the entire two-parameter family (2.19) (with $\mu > 0$) and also the one-parameter family (2.20) (with $\mu < 0$). This relies on the fact, that these results do not depend on the particular values of coefficients of the corresponding equations; cf. the discussion in Section 3. The additional terms in (2.20) may be included in the semi-linear part $f$ in the abstract formulation (3.23) and it can be easily checked, that all the assumed continuity properties hold true for such $f$.

5 Towards blow-up

In this section we prove a blow-up criterion for solutions $u$ of (1.1). In particular this result tells us, that the $H^3$-norm of $u$ can be controlled, if one manages to find a bound for the $L^\infty$-norm of the slope $u_x$. In this case we would additionally obtain an $L^\infty$-bound for $u$ - a meaningful property from a physical point of view.

Theorem 5.1. According to Theorem 3.1 let $u \in C([0, T); H^3) \cap C^1((0, T); H^2)$ be the solution of the Cauchy problem for (1.1) with initial condition $u(0, x) = u_0(x)$, $u_0 \in H^3$, where $T > 0$ denotes the maximal time of existence of the solution. Then $\|u\|_{H^3}$ is bounded on $[0, T)$ if and only if $\|u_x\|_{L^\infty}$ is bounded on $[0, T)$.

Proof. Due to Sobolev’s embedding $H^1 \subseteq L^\infty$ it is clear that

$$\limsup_{t \nearrow T} \|u_x\|_{L^\infty} = \infty \quad \text{implies} \quad \limsup_{t \nearrow T} \|u\|_{H^3} = \infty.$$ 

The proof of the converse implication is divided into two steps. First we show that the $H^1$-norm stays bounded on $[0, T)$, if we assume that $\|u_x\|_{\infty} \leq M_1$ for all $t \in [0, T)$ for
some $M_1 > 0$. In order to see this, we multiply the equation (1.1) by $u$, then integrate over $\mathbb{R}$ in $x$-direction and apply integration by parts to get
\[
\int (u^2)_t \, dx + \frac{7\delta^2}{18} \int (u^2)_t \, dx = \frac{109\varepsilon^2 \delta^2}{48} \int uu_x \, dx.
\] (5.31)
Now a corollary of Lebesgue’s dominated convergence theorem enables us to pull the time derivatives in (5.31) out of the integrals and Cauchy-Schwarz’s inequality and the inequality of arithmetic and geometric means yield
\[
\frac{d}{dt} \left( \int u^2 + \frac{7\delta^2}{18} u_x^2 \, dx \right) \leq CM_1^2 \int |u| u_x \, dx \leq K_1 \int u^2 + \frac{7\delta^2}{18} u_x^2 \, dx;
\] (5.32)
here we have set
\[
C := \frac{109\varepsilon^2 \delta^2}{48} \quad \text{and} \quad K_1 := \frac{3\sqrt{14}}{14\delta} CM_1^2.
\]
Now Gronwall’s lemma yields that the function
\[
E : [0, T) \rightarrow [0, \infty), \quad E_1(t) := \int u^2 + \frac{7\delta^2}{18} u_x^2 \, dx
\]
satisfies $E_1(t) \leq E_1(0)e^{K_1 t}$ for $0 \leq t < T$. This shows that $\|u\|_{H^1}$ remains bounded on $[0, T)$. Moreover Sobolev’s embedding theorem yields that $\|u\|_{L^\infty}$ remains bounded on $[0, T)$, say by $M_2$.

We show now that $\|u\|_{H^1}$ remains bounded on $[0, T)$. Let us for the moment assume that the initial data $u_0$ belongs to $H^6$. Then Proposition 4.6 assures that $u \in C^4((0, T); H^5)$ and we may differentiate (1.1) twice with respect to $x$, multiply the outcome by $u_{xx}$ and integrate over $\mathbb{R}$ in $x$-direction to obtain
\[
\frac{d}{dt} \left( \int u_{xx}^2 + \frac{7\delta^2}{18} u_{xxx}^2 \, dx \right) = 15\varepsilon \int uu_{xx}u_{xxxx} \, dx - \frac{5\varepsilon \delta}{6} \int u_x u_{xx}^2 \, dx + \frac{1489\varepsilon^2 \delta^2}{72} \int u_x^2 u_{xx}^2 \, dx + \frac{443\varepsilon^2 \delta^2}{48} \int uu_x u_{xxx}^2 \, dx.
\] (5.33)
The integrals on the right-hand-side in (5.33) can be estimated as follows:
\[
\int uu_{xx}u_{xxxx} \, dx \leq M_2 \int |u_{xx} u_{xxxx}| \, dx \leq \frac{3\sqrt{14} M_2}{14\delta} \int u_{xx}^2 + \frac{7\delta^2}{18} u_{xxx}^2 \, dx
\]
\[
\int u_x u_{xx}^2 \, dx \leq \frac{18M_1}{7\delta^2} \int \frac{7\delta^2}{18} u_{xxx}^2 \, dx
\]
\[
\int u_x^2 u_{xx} u_{xxx} \, dx \leq M_1^2 \int |u_{xx} u_{xxx}| \, dx \leq \frac{3\sqrt{14} M_1^2}{14\delta} \int u_{xx}^2 + \frac{7\delta^2}{18} u_{xxx}^2 \, dx
\]
\[
\int uu_x u_{xxx}^2 \, dx \leq \frac{18M_1 M_2}{7\delta^2} \int \frac{7\delta^2}{18} u_{xxx}^2 \, dx
\]
By setting $E_2(t) := E_1(t) + \int u_{xx}^2 + \frac{7\delta^2}{18} u_{xxx}^2 \, dx$, the inequalities from above, together with (5.32) and (5.33) imply that
\[
\frac{d}{dt} E_2(t) \leq K_2 E_2(t) \quad \text{for} \quad t \in [0, T),
\]
where one can choose
\[ K_2 := \left( \frac{3305\sqrt{14}\varepsilon^2\delta}{672} M_1^2 + \frac{45\sqrt{14}\varepsilon}{14\delta} M_2 + \frac{15\varepsilon}{7} M_1 + \frac{1329\varepsilon^2}{56} M_1 M_2 \right). \]

Therefore, Gronwall’s lemma yields a bound for \( \|u\|_{H^3} \) on \([0, T)\). If generally \( u_0 \) lies in \( H^3 \), we may approximate it by \( H^6 \)-functions and the needed estimate still holds true for the limit of the corresponding sequence of solutions, due to the continuous dependence on the initial data; cf. Theorem 3.1.

We conclude our discussion by providing an integral of motion.

**Lemma 5.2.** According to Theorem 3.1 let \( u \in C([0, T); H^3) \cap C^1((0, T); H^2) \) be the unique solution of the Cauchy problem for (1.1) with initial condition \( u(0, x) = u_0(x) \), \( u_0 \in H^3 \). For all \( t \in [0, T) \) it holds that
\[
\int u(t, x) \, dx = \int u_0(x) \, dx. \tag{5.34}
\]

**Proof.** Integrating (1.1) over \( \mathbb{R} \) in the spatial coordinate \( x \) and interchanging differentiation with respect to \( t \) and integration yields
\[
\frac{d}{dt} \left( \int u \, dx \right) = -\int u_x \, dx + \frac{d}{dt} \left( \int u_{xx} \, dx \right) + \int u_{xxx} \, dx \tag{5.35}
\]

Observe that the last term in (5.35) is zero:
\[
\int (398uu_xu_{xx} + 45u^2u_{xxx} + 154u_x^3) \, dx
\]

\[ = \lim_{R \to \infty} \int \chi_{[-R, R]} (398uu_xu_{xx} + 45u^2u_{xxx} + 154u_x^3) \, dx \]
\[ = \lim_{R \to \infty} \left[ 45u^2u_{xx} + 154uu_x^2 \right]_{x=-R}^{x=R} = 0; \]

here we applied Lebesgue’s dominated convergence theorem to obtain the first equality. Likewise all the other integrals on the right hand side of (5.35) vanish, and therefore we obtain (5.34). \( \square \)

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**References**


Traveling wave solutions of a highly nonlinear shallow water equation

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Abstract
Motivated by the question whether higher-order nonlinear model equations, which go beyond the Camassa-Holm regime of moderate amplitude waves, could point us to new types of waves profiles, we study the traveling wave solutions of a quasilinear evolution equation which models the propagation of shallow water waves of large amplitude. The aim of this paper is a complete classification of its traveling wave solutions. Apart from symmetric smooth, peaked and cusped solitary and periodic traveling waves, whose existence is well-known for moderate amplitude equations like Camassa-Holm, we obtain entirely new types of singular traveling waves: periodic waves which exhibit singularities on both crests and troughs simultaneously, waves with asymmetric peaks, as well as multi-crested smooth and multi-peaked waves with decay or waves with asymmetric peakons. Our approach uses qualitative tools for dynamical systems and in particular, methods for integrable planar systems.

1 Introduction

Recent research literature shows strong interest in the study of singular traveling waves for model equations in hydrodynamics. On the one hand, the governing equations for water waves admit the celebrated Stokes waves of greatest height, see the discussions in \[3\] [29] [32]. Moreover, cusped traveling waves are also known to occur as solutions to the governing equations for water waves \[4\] [5] [19]. These types of solutions are real-analytic except at their peaked or cusped crests. On the other hand, such singular traveling wave solutions are encountered in the study of shallow water approximations. While weakly nonlinear model equations for small amplitude waves, like the Korteweg-de-Vries equation (KdV), do not capture these phenomena, peaked or cusped solutions do arise from model equations for waves of moderate amplitude, like the Camassa-Holm (CH) and Degasperis-Procesi (DP) equations \[2\] [20] [27]. This raises the question whether higher-order nonlinear model equations, which go beyond the regime of moderate amplitude waves, could point us to other, new types of singular traveling waves. In this paper we give an affirmative answer to this question. A natural candidate for such a new type of wave is one which exhibits singularities on both the crest and the trough of the wave simultaneously. Model equations for moderate amplitude waves do not possess this type of singular solutions, since their nonlinearities in the higher order terms are at most quadratic. In this paper we study the traveling wave solutions of the following new model equation, which encompasses stronger nonlinearities to allow for new types of singular solutions:

\[ u_t + u_x + 3 \frac{3}{2} u u_x - 4 \frac{4}{18} u_{xxx} - 7 \frac{7}{18} u_{xxt} = \frac{1}{12} (u_x^2 + 2 uu_{xx})_x - \frac{1}{96} (45 u_x^2 u_{xx} + 154 uu_x^2)_x. \]  (1.1)

Here, the dependent variable \( u = u(x, t) \) is a function of one spacial variable \( x \in \mathbb{R} \) and the time variable \( t > 0 \). This equation was derived in \[30\] as a model equation for large
amplitude gravity water waves in the shallow water regime. More precisely, its solutions describe the evolution of the horizontal velocity component of a flow field at a certain fixed depth beneath the free surface of a water wave propagating unidirectionally over a flat bed. This model is derived from the incompressible Euler equations for gravity water waves, with suitable boundary conditions neglecting surface tension, see [30]. Equation (1.1) holds in a regime which allows for the description of large amplitude waves whose strong nonlinear effects are captured by the terms on the right hand side.

It is well-known that weakly nonlinear models for shallow water waves of small amplitude, such as the KdV equation [22], admit smooth solitary and periodic traveling waves. Shallow water models for waves of moderate amplitude, such as the CH equation [2], the corresponding equation for free surface waves [6, 7, 21] as well as the DP equation [11], capture stronger nonlinear effects and admit also non-smooth solutions containing so-called peaks and cusps, see for instance [17, 26, 27]. For the present equation we discover entirely new kinds of traveling wave solutions, which are not governed by equations for moderate amplitude waves. In Fig. 1 we sketch the shapes of some of these waves in order to give the reader a first impression of the tremendously rich collection of traveling wave solutions of (1.1). Novel types of solutions include periodic waves with peaks both at the crests and the troughs, as well as multi-crested smooth and peaked solitary waves. Another interesting feature of equation (1.1) is that it allows for peaked solutions with different slopes on either side of the crest and trough, that is, we obtain non-symmetric peakons. In comparison, peaks are always symmetric in CH type equations, cf. [26]. Moreover, equation (1.1) admits peaked and cusped solutions with compact support, which was shown to be impossible for CH type equations in [17]. As we will see, the existence of such solutions requires the presence of third order terms exhibiting nonlinearities of at least cubic order in the evolution equation.

The aim of this paper is to give a complete classification of all traveling wave solutions of (1.1) in $H^1_{\text{loc}}$, where a suitable weak formulation of the evolution equation is available. Our approach relies on methods from the qualitative theory of dynamical systems, in particular on tools for integrable planar systems. Working with a suitable weak formulation we will describe precisely in which sense such non-smooth traveling waves are solutions of equation (1.1).

The paper is structured as follows. In Section 2 we provide the definition of traveling wave solutions based on a weak formulation of (1.1). Section 3 discusses the integrable structure of the planar dynamical system associated to (1.1). In Section 4 we prove a proposition which characterizes the traveling wave solutions as certain piecewise smooth $H^1_{\text{loc}}$-functions solving the aforementioned system almost everywhere in the classical sense. This opens the way to a full classification of all traveling wave solutions of (1.1) by means of a systematic phase plane analysis of a bi-parametric family of underlying dynamical systems. The construction of all possible traveling waves is finally realized in Section 5. We provide a summary of the results of our analysis in Theorems 6.1, 6.2 and 6.3 in Section 6 and conclude with a short discussion and outlook in Section 7.
Figure 1: A selection of some traveling wave solutions of (1.1). The waves on the left side from top to bottom are of the following types: smooth periodic, peaked periodic, cusped periodic, periodic with peaked crests and cusped troughs, periodic with peaked crests and troughs, composite, composite with plateaus. Right side top to bottom: smooth solitary, peaked solitary, cusped solitary, wavefront, compactly supported anticusp, multi-crest with decay, multi-peak with decay.

for a fixed $c \in \mathbb{R}$ being referred to as the wave speed. We denote by $s := x - ct$ the corresponding independent moving frame variable. In a first step, we rewrite (1.1) in the moving frame variables (2.2) and integrate with respect to the moving frame variable $s$ to obtain

$$u''(A_c + Bu + Cu^2) = K + (c - 1)u + Eu^2 + (u')^2(Gu - 1/2 B),$$

(2.3)

where the prime symbol denotes differentiation with respect to $s$ and

$$A_c = \frac{7c - 4}{18}, \quad B = -\frac{1}{6}, \quad C = \frac{45}{96}, \quad E = -\frac{3}{4}, \quad G = -\frac{154}{96},$$

and $K \in \mathbb{R}$ is a constant of integration. To facilitate the mathematical treatment of this equation, let us introduce the real polynomials

$$g(u) := (A_c + Bu + Cu^2),$$

$$f(u, v) := K + (c - 1)u + Eu^2 + v^2(Gu - 1/2 B)$$

(2.4)

and write equation (2.3) as

$$u''g(u) = f(u, u')$$

(2.5)

in a more compact form. An equivalent formulation which turns out to be convenient when working with $H^1_{loc}$-functions is

$$f(u, u') + (u')^2g'(u) = [g(u)u']',$$

(2.6)
where $g'(u) = \frac{dg}{du}(u)$, see Remark 2.2. The weak formulation of (1.1) suitable for functions $u$ of the form (2.2) is then obtained by multiplying (2.6) with a smooth and compactly supported test function $\phi$ satisfying $\phi(t,x) = \phi(x - ct) = \phi(s)$ and by a subsequent integration over $\mathbb{R}$ with respect to the moving frame variable $s$.

**Definition 2.1.** Fix $c \in \mathbb{R}$. A bounded function $u: \mathbb{R} \to \mathbb{R}$ is called a traveling wave solution, or shorter, a traveling wave of (1.1) with wave speed $c$, if $u = u(s)$ lies in $H^1_{\text{loc}}(\mathbb{R})$ and satisfies equation (2.6) in the sense of distributions, i.e. it satisfies

$$\int_{\mathbb{R}} g(u)u'\phi' + [f(u,u') + (u')^2g'(u)]\phi \delta s = 0$$

(2.7)

for all test functions $\phi$ in $\mathcal{D}(\mathbb{R}) = C^\infty_0(\mathbb{R})$, the space of compactly supported smooth real-valued functions on $\mathbb{R}$.

**Remark 2.2.** We point out that, by abuse of notation, we write $g'(u)$ to mean $\frac{dg}{du}(u)$. Moreover, when speaking of an element $u \in H^1_{\text{loc}}(\mathbb{R})$, we always refer to the absolutely continuous representative of this class of functions. Hence a traveling wave of (1.1) is absolutely continuous and bounded with a locally square integrable derivative. Definition 2.1 excludes unbounded waves, which would not be relevant from a physical point of view.

**Remark 2.3.** Note that the weak formulation of (1.1) as given in Definition 2.1 applies exclusively to functions of the form (2.2), since the test functions $\phi$ that we consider are also of that form, and hence this weak formulation is not suitable for a general formulation of the Cauchy problem that corresponds to (1.1). Note that, unlike moderate amplitude equations like CH, equation (1.1) does not permit a reformulation as a conservation law of the form $u_t + [\Phi(u)]_x = 0$. In particular, this equation does not have a Hamiltonian structure.

### 3 The associated integrable planar system

Our aim is to completely characterize all traveling wave solutions of (1.1). To this end, we study the phase portrait of the related planar differential system

$$\begin{cases}
    u' = v \\
    v' = \frac{f(u,v)}{g(u)}
\end{cases}$$

(3.8)

for all parameter pairs $(c,K) \in \mathbb{R}^2$. In Section 4 we will prove that this is sufficient since every traveling wave of (1.1) is a composition of solution curves of (3.8), cf. Proposition 4.1.

Let us first introduce some useful notation. We denote by $N_g$ the set of real zeros of the polynomial $g(u)$, that is,

$$N_g := \{u \in \mathbb{R}: g(u) = 0\},$$

(3.9)

and we denote by $U$ the domain of system (3.8), i.e.

$$U := \mathbb{R}^2 \setminus (N_g \times \mathbb{R}).$$

(3.10)
Our analysis relies heavily on the fact, that (3.8) is integrable, i.e. there exists a function $H : U \to \mathbb{R}$, called first integral, which is constant along solution curves of (3.8) within the open subset $U \subseteq \mathbb{R}^2$. In order to find a first integral, we reparametrize system (3.8) by introducing the new independent variable $\tau$ via $\frac{\delta s}{\delta \tau} = g(u)$ to obtain

$$
\begin{align*}
\dot{u} &= v g(u) \\
\dot{v} &= f(u, v).
\end{align*}
$$

(3.11)

Note that (3.11) is defined on all of $\mathbb{R}^2$. The dots in (3.11) refer to differentiation with respect to $\tau$. System (3.11) is topologically equivalent to system (3.8) on $U$: the solution curves coincide, but the orientation is reversed within the region $\{(u, v) \in U : g(u) < 0\}$ and preserved in $\{(u, v) \in U : g(u) > 0\}$, cf. [12, 18]. The set $N_g \times \mathbb{R} \subseteq \mathbb{R}^2$ is either empty, or consists of up to two vertical invariant lines.

System (3.11) has an integrating factor $\varphi : \mathbb{R} \setminus N_g \to \mathbb{R}$, i.e.

$$
\text{div} (vg(u)\varphi(u), f(u, v)\varphi(u)) = 0 \quad \text{in } U.
$$

(3.12)

It is not difficult to see that if $\varphi$ satisfies the differential equation

$$
\varphi'(u) = -2(C + G) \frac{u}{g(u)} \varphi(u),
$$

(3.13)

then $\varphi$ is an integrating factor on $\mathbb{R} \setminus N_g$. Equation (3.13) can be solved explicitly and the form of the solution $\varphi$ depends on the number of roots of the polynomial $g$; see Section 5.2 for the details. Hence (3.11) is integrable on $U$ and the first integral $H$ associated to the integrating factor $\varphi$ is given by

$$
H(u, v) := \frac{v^2}{2} \varphi(u) g(u) + \psi(u),
$$

(3.14)

where

$$
\psi(u) = -\int f_0(u) \varphi(u) \, du,
$$

(3.15)

with $f_0(u) := f(u, 0)$. Note that $\dot{u} = \frac{H_u}{\varphi(u)}$, $\dot{v} = -\frac{H_v}{\varphi(u)}$ in $U$. The solution curves, or orbits, of system (3.11) correspond to the level sets of $H$, which we denote by

$$
L_h(H) := \{(u, v) \in U : H(u, v) = h\}.
$$

(3.16)

In view of the symmetry of $H$ about the $u$-axis, these curves are composed of the two symmetric branches $(u, \psi_{\pm}(u))$, where

$$
\psi_{\pm}(u) = \pm \sqrt{\frac{2h - \psi(u)}{g(u)\varphi(u)}}.
$$

(3.17)

Let us finally note that

$$
\frac{\delta u}{\delta s} = v_{\pm}^h \quad \text{along solution curves of (3.8) in } U \cap (\mathbb{R} \times \mathbb{R}^\pm).
$$

(3.18)
A characterization of traveling wave solutions

We will construct traveling wave solutions of (1.1) by associating the solutions of (2.5) with orbits of the planar systems (3.8) and (3.11). These orbits correspond to level sets of the first integral (3.14). The following proposition ensures that we can indeed obtain all traveling wave solutions of (1.1) with this approach.

Recall that \( N_g \) denotes the zero set of the quadratic polynomial \( g \) defined in (2.4). Moreover, let \( \lambda(X) \) denote the Lebesgue measure of a measurable set \( X \subseteq \mathbb{R} \).

**Proposition 4.1.** Fix \( c \in \mathbb{R} \). A bounded continuous function \( u : \mathbb{R} \to \mathbb{R} \) is a traveling wave of (1.1) with wave speed \( c \) if and only if the following holds:

(TW1) The open set \( \mathbb{R} \setminus u^{-1}(N_g) \) is a countable disjoint union \( \bigcup_{j} I_j \) of open intervals \( I_j \). It holds that \( u|_{I_j} \in C^\infty(I_j) \) for all \( j \), \( u(s) \notin N_g \) for \( s \in \bigcup_{j} I_j \) and \( u(s) \in N_g \) for \( s \in u^{-1}(N_g) \).

(TW2) There is a \( K \in \mathbb{R} \) such that

(a) for each \( j \) there exists some \( h_j \in \mathbb{R} \) so that

\[
(u')^2 = 2 \frac{h_j - \psi(u)}{\varphi(u)g(u)} \text{ on } I_j
\]

\[u \to \alpha_i \text{ at finite endpoints of } I_j, \text{ with } \alpha_i \in N_g.\]

(b) If \( \lambda(u^{-1}(N_g)) > 0 \), then \( N_g \cap N_{f_0} \neq \emptyset \), i.e. \( K = K_{\alpha_i}(c), \ i \in \{1, 2\} \).

(TW3) \( u' \) exists a.e. and \( u' \in L^2_{loc}(\mathbb{R}) \).

**Remark 4.2.** In particular, Proposition 4.1 implies that all traveling wave solutions of (1.1) can be obtained via a systematic phase plane analysis of (3.8) for all parameter pairings \((c, K)\).

We give the proof of Proposition 4.1 at the end of this section after stating some auxiliary results.

**Lemma 4.3.** Let \( u \) be a traveling wave solution of (1.1) and let \( I \subseteq \mathbb{R} \) be an open interval. If the restriction of \( u \) on \( I \) is \( C^2 \), then \( u \) solves (2.5) pointwise on \( I \).

**Proof.** The restriction \( u|_I \) satisfies (2.5) in \( \mathcal{D}'(I) \), i.e.

\[
\int_I [u''g(u) - f(u, u')]\phi ds = 0 \quad \text{for all } \phi \in \mathcal{D}(I).
\]

Now \( \rho := u''g(u) - f(u, u') \) is continuous in \( I \) by assumption. It follows that \( \rho \) is identically zero in \( I \), proving that (2.5) is satisfied pointwise in \( I \). Indeed, otherwise there would be some \( s_0 \in I \) with \( \rho(s_0) \neq 0 \), say \( \rho(s_0) > 0 \). By continuity, \( \rho > 0 \) on a small subinterval \( I_\varepsilon \) containing \( s_0 \). Choosing a nonnegative bump-function \( \phi_0 \in \mathcal{D}(I) \) with \( \text{supp}(\phi_0) \subseteq I_\varepsilon \) would imply a strictly positive integral in (4.20) — a contradiction. \( \square \)

\[1\] See (5.37) for the definition of \( K_{\alpha_i} \).
Remark 4.4. Lemma 4.3 tells us in particular that our definition of traveling wave solutions, which is based on a weak formulation of (2.5), is a consistent generalization of the concept of classical solutions.

Lemma 4.5. Let \( u \) be a traveling wave solution of (1.1). Then \( g^k(u) := (g(u))^k \in C^2(\mathbb{R}) \) for \( k \geq 5 \).

Proof. Throughout this proof we consider derivatives \( (\cdot)' \), \( (\cdot)'' \), etc. as distributional derivatives, and terms which contain such derivatives as elements in \( \mathcal{D}' \). Once we realize that such distributions actually lie in better spaces, e.g. \( W^{1,1}_{\text{loc}} \), the symbol \( (\cdot)' \) may be interpreted as a classical (pointwise or pointwise a.e.) derivative.

Note first, that \( u''g(u) \in L^1_{\text{loc}} \) because \( u''g(u) = f(u, u') \) in \( \mathcal{D}' \) by (2.5) and \( f(u, u') \) is a regular distribution (i.e. an element of \( L^1_{\text{loc}} \)) since \( u \in H^1_{\text{loc}} \) by assumption. Therefore, we obtain that \( u'g(u) \in W^{1,1}_{\text{loc}} \) since \( u'g(u) \in L^1_{\text{loc}} \) and

\[
[u'g(u)]' = u''g(u) + (u')^2g'(u) \in L^1_{\text{loc}}. \tag{4.21}
\]

Similarly, we get that \( (u'g(u))^2 \in W^{1,1}_{\text{loc}} \), since \( (u')^2g^2(u) \in L^1_{\text{loc}} \) and

\[
[(u'g(u))^2]' = 2u'g(u)[u''g(u) + (u')^2g'(u)] \in L^1_{\text{loc}}. \tag{4.22}
\]

As a consequence we also have that \( u''g^3(u) \in W^{1,1}_{\text{loc}} \), since

\[
u''g^3(u) = f(u, u')g^2(u) = f(u, 0)g^2(u) + (u')^2g^2(u)(Gu - B/2), \tag{4.23}
\]

and \( (u')^2g^2(u) \in W^{1,1}_{\text{loc}} \). For \( k \geq 4 \) we calculate

\[
g^k(u)' = kg^{k-1}(u)g'(u)u' = kg^{k-4}(u)g'(u)u'g^3(u). \tag{4.24}
\]

Hence, we may write \( [g^k(u)]'' \) for \( k \geq 5 \) as

\[
k(k - 4)g^{k-5}(u)g(u)(u')^2g^3(u) + kg^{k-4}(u)[g'(u)(u')^2g^3(u) + g'(u)\left(u''g^3(u) + 3(u')^2g^2(u)g(u)\right)], \tag{4.25}
\]

which lies in \( W^{1,1}_{\text{loc}} \) by our previous considerations. Thus \( [g^k(u)]'' \) is absolutely continuous and therefore \( g^k(u) \in C^2(\mathbb{R}) \) for \( k \geq 5 \).

Let us denote by \( \partial X \) the set of all boundary points of a subset \( X \subseteq \mathbb{R} \), and recall that \( N_g = g^{-1}(0) \) defined in (3.9) is the preimage of 0 under \( g \).

Lemma 4.6. Let \( u \) be a traveling wave solution of (1.1). Then \( u \in C^\infty(\mathbb{R} \setminus \partial(u^{-1}(N_g))) \).

Proof. The set \( N_g \) is either empty, a singleton, or it contains two elements. Assuming that \( s_0 \) is an interior point of \( u^{-1}(N_g) \), there exists an open interval \( I_\varepsilon = (s_0 - \varepsilon, s_0 + \varepsilon) \subseteq u^{-1}(N_g) \). Since \( u \) is continuous, \( u(I_\varepsilon) \) is a singleton by the mean value theorem. Therefore, the restriction \( u|_{I_\varepsilon} \) is a constant function, in other words \( u|_{I_\varepsilon} \in C^\infty(I_\varepsilon) \).

For the remaining part of the proof let \( O := \mathbb{R} \setminus N_g \). The preimage of the closed set \( N_g \) under \( u \) is a closed set since \( u \) is continuous, and hence \( O \) is an open subset. Assuming that \( s_0 \in O \), we will find a small open interval \( I \) containing \( s_0 \), such that the restriction \( u|_I \) lies in \( C^\infty(I) \), which establishes the claim. For this purpose, let us first denote by \( I_\varepsilon := (s_0 - \varepsilon, s_0 + \varepsilon) \subseteq O \) a suitable \( \varepsilon \)-neighborhood of \( s_0 \). The key
observation is that \( u'_I \in C^2(I) \). Indeed, by Lemma 4.5 we find that \((g(u))^k \) is \( C^2(\mathbb{R})\) for any \( k \geq 5 \), and hence \( g(u) \) is \( C^2(\mathbb{R} \setminus u^{-1}(N_g)) \). Therefore the restriction of \( u \) on \( I \) is twice continuously differentiable, since \( u(I) \cap N_g = \emptyset \) by construction. In particular, \( u \) has a classical derivative in \( s_0 \), say \( u'(s_0) =: v_0 \); furthermore we set \( u_0 := u(s_0) \). Let us consider the following initial value problem on \( I \):\[
\begin{cases}
u' = v \\
u'(u,v) = \frac{f(u,v)}{g(u)}, \quad (u(s_0), v(s_0)) = (u_0, v_0).
\end{cases}
\tag{4.26}
\]
The classical Picard-Lindelöf theory provides a unique smooth solution \((\bar{u}, \bar{v})\) of \(4.26\), at least on a small open subinterval \( I \subseteq I \) with \( s_0 \in I \), since the right hand side of \(4.26\) is smooth on \( I \). Since \( \bar{u} = u_I \) we conclude that \( u_I \in C^\infty(I) \), due to our construction and Lemma 4.9 \( \square \)

Remark 4.7. Lemma 4.6 implies in particular that a traveling wave of \(1.1\) with wave speed \( c > \bar{c} \) is smooth. An alternative straightforward (but tedious) way to prove Lemma 4.6 is to show that for any given \( k \in \mathbb{N} \) one can find some \( n \in \mathbb{N} \) such that \( g^n(u) \in C^k(\mathbb{R}) \), similarly as in \[26\].

Lemma 4.8. Let \( w : \mathbb{R} \to \mathbb{R} \) be an absolutely continuous function and let \( A \subseteq \mathbb{R} \) be a finite subset. Then the classical derivative \( w' \) exists a.e. on \( \mathbb{R} \) and \( w' = 0 \) a.e. on the preimage \( w^{-1}(A) \).

Proof. Let us first prove the special case where \( A \) contains only one element, say \( A = \{\alpha\} \), for \( \alpha \in \mathbb{R} \). Since \( w \) is absolutely continuous, \( w' \) exists almost everywhere in \( \mathbb{R} \). By continuity, \( R := w^{-1}(\alpha) \) is a closed subset of \( \mathbb{R} \). As a closed set, \( R \) is the disjoint union of a perfect set \( P \) (i.e. closed without isolated points) and a countable set \( S \), due to the Cantor-Bendixson theorem (see for example \[20\]). Let \( p \in P \) be a point, such that \( w'(p) \) exists. We choose a sequence \((p_i)_i\) of points \( p_i \in P \) with \( p_i \to p \) for \( i \to \infty \) in order to see that\[
w'(p) = \lim_{i \to \infty} \frac{w(p) - w(p_i)}{p - p_i} = \lim_{i \to \infty} \frac{\alpha - \alpha}{p - p_i} = 0. \tag{4.27}
\]
Since \( S \) is countable, its Lebesgue measure is zero and hence \( w' = 0 \) a.e. on \( R \).

For the case of a general finite subset \( A \), we apply the same line of arguments as before. We see that the sequence \((w(p_i))_i\) might not be constant but take different values of the finite set \( A \). By the continuity of \( w \) however, we infer that \( w(p_i) \to w(p) \) as \( p_i \to p \). Therefore, the sequence \((w(p_i))_i\) takes the constant value \( w(p) \) for almost all \( n \in \mathbb{N} \), which shows that the limit in \(4.27\) is zero also in the general case \( \square \).

Proof of Proposition 4.1. Let us first assume that \( u \) is a traveling wave of \(1.1 \). Thus, \( u \in H^1_{\text{loc}} \), hence \((\text{TW3})\) is satisfied. Property \((\text{TW1})\) follows from Lemma 4.6 and its proof, and the fact that every open subset of \( \mathbb{R} \) can be represented as a countable union of open intervals. Property \((\text{TW2})\) follows from the fact that \( u \) is smooth on \( I_j \) and solves the planar differential system \((3.8)\) on \( I_j \). In Section 3 we proved that this system is integrable and \((3.17)\) implies the first relation in \(4.19\) for some \( h_j \in \mathbb{R} \). The continuity of \( u \) and \((\text{TW1})\) yield the second assertion in \(4.19\). Suppose that \( \lambda(u^{-1}(N_g)) > 0 \). Since both \( u \) and \( u'g(u) \) are absolutely continuous in view of the proof of Lemma 4.5 we deduce from Lemma 4.8 that both\[
u' = 0 \quad \text{and} \quad [u'g(u)]' = 0 \quad \text{a.e. on} \ u^{-1}(N_g), \tag{4.28}\]
since \( u^{-1}(N_g) \subseteq [u'g(u)]^{-1}\{0\} \). Thus in particular, \( u''g(u) = 0 \) a.e. on \( u^{-1}(N_g) \). In summary this implies, using (2.5), that \( f(u,0) = 0 \) on \( u^{-1}(N_g) \). Therefore, \( K = K_\alpha \), with \( \alpha \in N_g \) as defined in (3.37).

Let us now assume that a bounded continuous function \( u: \mathbb{R} \to \mathbb{R} \) satisfies (TW1)–(TW3).

If \( \lambda(u^{-1}(N_g)) = 0 \), then \( u \) satisfies (2.5) pointwise a.e. in \( \mathbb{R} \) by (TW1) and (TW2). To see this, one has to differentiate the equation in (4.19) with respect to \( s \), and use the formula (3.13) for the integrating factor and the fact that \( u' \) is nonzero a.e.

Due to (TW3) and the boundedness of \( u \), we obtain that \( f(u,u') \) is locally integrable. Therefore \( u''g(u) \) lies in \( L^1_{\text{loc}}(\mathbb{R}) \) as well, and since they agree almost everywhere, we obtain:

\[
\int_{\mathbb{R}} u''g(u) \phi \, ds = \int_{\mathbb{R}} f(u,u') \phi \, ds \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}),
\]

which means that \( u \) is a traveling wave of (1.1), since we can rewrite this equation in the form (2.7). For the case \( \lambda(u^{-1}(N_g)) > 0 \), let us first observe, that \( u''g(u) \in W^{1,1}_{\text{loc}}(\mathbb{R}) \) and is therefore absolutely continuous. To see this, note that similar to the proof of Lemma 4.3 we obtain

\[
u''(u) \in L^1_{\text{loc}}(\mathbb{R} \setminus u^{-1}(N_g))
\]

and

\[
[u''(u)]' = u''g(u) + g'(u)(u')^2 \in L^1_{\text{loc}}(\mathbb{R} \setminus u^{-1}(N_g)),
\]

since \( u \) is bounded, and both \( (u')^2 \) and \( u''g(u) \) are locally integrable. Therefore (4.28) holds true and hence \( u''g(u) = 0 \) a.e. on \( u^{-1}(N_g) \). Since \( K = K_\alpha \), with \( \alpha \in N_g \) and in view of (4.28), we find that \( f(u,u') = 0 \) a.e. on \( u^{-1}(N_g) \), and therefore equation (2.5) holds a.e. on \( u^{-1}(N_g) \). Since we already know that the equation holds on \( \mathbb{R} \setminus u^{-1}(N_g) \), and use the formula (3.13) for the integrating factor and the fact that \( u' \) is nonzero a.e. on \( \mathbb{R} \).

By (TW1) and (TW2) we conclude that (2.5) holds a.e. on \( \mathbb{R} \). By (TW3) we obtain that \( f(u,u') \in L^1_{\text{loc}}(\mathbb{R}) \) and therefore \( u \) is a traveling wave solution of (1.1) in the sense of Definition 2.1.

5 Construction of traveling wave solutions

In the current section we construct traveling waves of (1.1) by combining solution curves of (3.8) in a suitable way. According to Proposition 1.1 we can obtain all traveling waves of (1.1) by following this approach. For the study of these solution curves we exploit the fact that (3.8), and the topologically equivalent system (3.11), are integrable on \( U \) and the respective solution curves are the level sets \( L_h(H) \) of the first integral \( H \), which can be expressed in the form (3.17). In particular, we observe that the functions \( f_0 \), \( g \) and \( \varphi \) fully determine the phase portrait of the systems (3.8) and (3.11) for a given pair \((c, K) \in \mathbb{R}^2\).

Before studying the qualitative behavior of the solution curves in detail, we give a rough overview of the phase portraits of (3.11) by discussing the fixed points for all parameter combinations \((c, K)\). After that, we provide explicit formulae for the integrating factors \( \varphi \) for all wave speeds \( c \) and summarize their basic properties.

5.1 Fixed points of system (3.11).

The fixed points of system (3.11) are of the form \((u,0)\) and \((\alpha_i, v)\), where \( \alpha_i \in N_g \), \( i \in \{1,2\} \), denote the real zeros of \( g \), cf. (3.9). The number and type of fixed points are
determined by the zeros of the polynomials \( g \) and \( f_0 \), which vary with the parameters \( c \) and \( K \).

Fixed points of the form \((u, 0)\) on the horizontal axis are determined by the roots of \( f_0(u) = f(u, 0) \). For any wave speed \( c \in \mathbb{R} \), we denote by \( K_0 = K_0(c) \) the zero of the discriminant \( \Delta_{f_0} := (c - 1)^2 - 4EK \) of the quadratic polynomial \( f_0 \), that is,

\[
K_0 := \frac{(c - 1)^2}{4E}.
\] (5.32)

Then \( f_0 \) has a double root \( \bar{u} \) when \( K = K_0 \), it has two zeros \( u_1 < \bar{u} < u_2 \) whenever \( K > K_0 \), where

\[
u_i = \frac{1 - c \pm \sqrt{\Delta_{f_0}}}{2E}, \quad i = 1, 2,
\]

and \( f_0 < 0 \) if \( K < K_0 \). Provided that \( N_g \) is nonempty, there exist invariant vertical lines \( \{u = \alpha_i\} \), where \( \alpha_i \in N_g, i = 1, 2 \). These invariant sets exist for wave speeds \( c \leq \bar{c} \), where

\[
\bar{c} := \frac{64}{105}
\] (5.33)

is the zero of the discriminant \( \Delta_g = B^2 - 4AcC \) of the quadratic polynomial \( g \). Then \( g \) is strictly positive if and only if \( c > \bar{c} \) and \( g(u) = C(u - \alpha_1)(u - \alpha_2) \) if \( c \leq \bar{c} \), with

\[
\alpha_i = \frac{-B \pm \sqrt{\Delta_g}}{2Ac}, \quad i = 1, 2.
\] (5.34)

Note that the two zeros of \( g \) coincide precisely when \( c = \bar{c} \) in which case we denote the double root of \( g(u) \) by \( \alpha \). Observe that \( \bar{u}(c) < \alpha_1(c) < \alpha_2(c) \) for all \( c \in (-\infty, \bar{c}) \) and \( \bar{u}(\bar{c}) < \alpha \) as displayed in Fig. 2a. The second component of the fixed point \((\alpha_i, v)\) is determined by the relation \( f(\alpha_i, v) = 0 \), which we will analyze below.

![Figure 2](image)

**Figure 2:** (a) the graphs of the functions \( \alpha_2 \) [bold], \( \alpha_1 \) [plain] and \( \bar{u} \) [dashed]; (b) the graphs of \( K_{\alpha_2} \) [bold], \( K_{\alpha_1} \) [plain] and \( K_0 \) [dashed], cf. (5.32) and (5.37).

Next we determine the type of fixed points \((u, 0)\). The Jacobian of a point \((u, v) \in \mathbb{R}^2 \) of (3.11) is given by

\[
J(u, v) = \begin{pmatrix} \frac{vg'(u)}{\partial_u f(u, v)} & g(u) \\ \frac{\partial_v f(u, v)}{\partial_u f(u, v)} & \frac{\partial_v f(u, v)}{\partial_v f(u, v)} \end{pmatrix}.
\] (5.35)
We recall that system \((3.11)\) is integrable on \(U = \mathbb{R}^2 \setminus (N \times \mathbb{R})\). Therefore, the type of any fixed point in \(U\) is determined by the sign of the determinant of the Jacobian at the fixed point: a negative determinant implies a saddle, a positive determinant implies a center and a vanishing determinant implies a cusp. The determinant of \(J\) at the fixed points \((u_i, 0) \in U\) is given by

\[
det[J(u_i, 0)] = \mp \sqrt{\Delta_0} g(u_i), \quad i = 1, 2. \tag{5.36}
\]

Therefore, its sign depends on the positions of the fixed points \((u_i, 0)\) relative to each other and the invariant lines \(u = \alpha_i, i = 1, 2\). Whenever these invariant lines exist, that is, for \(c \leq \bar{c}\), we define the values \(K_{\alpha_i}(c)\) as follows:

\[
K_{\alpha_i}(c) \text{ denotes the unique zero of } u_2(c, K) - \alpha_i, \quad i = 1, 2. \tag{5.37}
\]

That is, for each \(c \leq \bar{c}\) the root \(\alpha_i\) of \(g(u)\) coincides with the root \(u_2\) of \(f_0(u)\) precisely when \(K = K_{\alpha_i}(c)\). The relative positions of \(u_1(c, K), u_2(c, K), \alpha_1(c)\) and \(\alpha_2(c)\), and hence the sign of \((5.36)\), are determined by the relative position of \(K\) with respect to \(K_0(c), K_{\alpha_1}(c)\) and \(K_{\alpha_2}(c)\), cf. Fig. 2. Since \(K_0(c) < K_{\alpha_1}(c) < K_{\alpha_2}(c)\) for all \(c \in (-\infty, \bar{c})\), and \(K_0(\bar{c}) < K_\alpha\), there are precisely seven different scenarios which classify the types of fixed points lying on the \(u\)-axis. We summarize them in Table 1.

<table>
<thead>
<tr>
<th>scenario</th>
<th>parameter</th>
<th>order relation</th>
<th>fixed points and type</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(K &lt; K_0(c))</td>
<td>(u &lt; \alpha_1 &lt; \alpha_2)</td>
<td>((u, 0)) s, ((u_2, 0)) c</td>
</tr>
<tr>
<td>II</td>
<td>(K = K_0(c))</td>
<td>(u &lt; \alpha_1 &lt; \alpha_2)</td>
<td>((u_1, 0)) s, ((u_2, 0)) c</td>
</tr>
<tr>
<td>III</td>
<td>(K_0(c) &lt; K &lt; K_{\alpha_1}(c))</td>
<td>(u_1 &lt; u_2 &lt; \alpha_1 &lt; \alpha_2)</td>
<td>((u_1, 0)) s, ((u_2, 0)) c</td>
</tr>
<tr>
<td>IV</td>
<td>(K = K_{\alpha_1}(c))</td>
<td>(u_1 &lt; \alpha_1 &lt; u_2 &lt; \alpha_2)</td>
<td>((u_1, 0)) s, ((\alpha_1, 0)) n</td>
</tr>
<tr>
<td>V</td>
<td>(K_{\alpha_1}(c) &lt; K &lt; K_{\alpha_2}(c))</td>
<td>(u_1 &lt; \alpha_1 &lt; u_2 &lt; \alpha_2)</td>
<td>((u_1, 0)) s, ((u_2, 0)) s</td>
</tr>
<tr>
<td>VI</td>
<td>(K = K_{\alpha_2}(c))</td>
<td>(u_1 &lt; \alpha_1 &lt; u_2 = \alpha_2)</td>
<td>((u_1, 0)) s, ((\alpha_2, 0)) n</td>
</tr>
<tr>
<td>VII</td>
<td>(K &gt; K_{\alpha_2}(c))</td>
<td>(u_1 &lt; \alpha_1 &lt; \alpha_2 &lt; u_2)</td>
<td>((u_1, 0)) s, ((u_1, 0)) c</td>
</tr>
</tbody>
</table>

Table 1: A list of all possible scenarios for the ordering of fixed points on the horizontal axis. Here \(s\) stands for saddle, \(c\) for center and \(n\) means that the Jacobi matrix at the fixed point is nilpotent.

The local behavior near the nilpotent fixed points will be determined in the phase plane analysis in Section \(5.4\). Let us point out, that in the situation \(c = \bar{c}\), i.e. when \(g(u)\) has the double root \(\alpha\), we distinguish between the five scenarios I–IV and VII in Table 1 since \(K_\alpha(\bar{c}) = K_{\alpha_1}(\bar{c}) = K_{\alpha_2}(\bar{c})\). In case that \(c > \bar{c}\), i.e. when \(N_g\) is empty, we distinguish only between the first three scenarios, where \(K\) has no upper bound in scenario III.

To determine the type of the fixed points \((\alpha_i, v), i \in \{1, 2\}\), on the invariant lines, recall that their second component is determined by the relation \(f(\alpha_i, v) = 0\), which holds if and only if \(v = \pm v_{\alpha_i}\), where

\[
v_{\alpha_i} := \sqrt{-\frac{f_0(\alpha_i)}{-\frac{1}{2} B + G\alpha_i}}
\]
We have that
\[ g > 0, \] for all \( c \in (-\infty, \bar{c}), \] whenever this expression is real and finite. We observe that
\[ -\frac{1}{2} B + G\alpha_2(c) < 0 \] for all \( c \in (-\infty, \bar{c}), \) thus the fixed points \((\alpha_2, \pm \nu_{\alpha_2})\) exist whenever \( f_0(\alpha_2) > 0,\) that is, for \( K > K_{\alpha_2}(c)\) defined in \((5.37)\). These fixed points are saddles, since the local linearization of \((3.11)\), which is a lower triangular matrix in view of \((5.35)\), has two nonzero eigenvalues of opposite sign. The existence of the fixed points \((\alpha_1, \pm \nu_{\alpha_1})\) does not depend solely on \( K,\) but also on the parameter \( c,\) since \(-\frac{1}{2} B + G\alpha_1(c)\) changes its sign at the particular wave speed
\[ c_1 := \frac{24505}{41503} < \bar{c}. \] We find that \( \pm \nu_{\alpha_1} \) are real numbers – and hence the points \((\alpha_1, \pm \nu_{\alpha_1})\) are fixed points of \((3.11)\) – provided that either \( c < c_1 \) and \( K < K_{\alpha_1}(c) \) or \( c \in (c_1, \bar{c}) \) and \( K > K_{\alpha_1}(c) \). In the first case, \((\alpha_1, \pm \nu_{\alpha_1})\) are saddles since the Jacobian \( J \) is a lower triangular matrix with eigenvalues of opposite sign. In the second case, these fixed points are stable or unstable nodes, i.e. both eigenvalues of their lower triangular matrix \( J \) are nonzero real numbers of the same sign. In the case that \( K = K_{\alpha_1}(c) \) and \( c \in (-\infty, \bar{c}) \setminus \{c_1\},\) we have \( \nu_{\alpha_1} = 0,\) hence the corresponding fixed point lies on the horizontal axis and the situation is as described in scenario IV in Table 1. Similarly, the case \( K = K_{\alpha_2}(c),\) where \( \nu_{\alpha_2} = 0 \) for all \( c \in (-\infty, \bar{c}),\) corresponds to scenario VI. It remains to discuss the case \( c = c_1 \) and \( K = K_{\alpha_1}(c_1) \) for which the function \( f \) can be written as
\[ f(u, v) = E(u - u_1)(u - \alpha_1(c_1)) - G(u - \alpha_1(c_1))v^2. \] This implies that every point on the invariant line \( \{u = \alpha_1(c_1)\}\) is a fixed point of system \((3.11).\)

If \( c = \bar{c},\) the function \( g \) has a unique double root at \( u = \alpha \) and for \( K > K_{\alpha},\) we have that the fixed points \((\alpha, \pm \nu_{\alpha})\) are non-hyperbolic.

### 5.2 Integrating factor.

In this subsection, we give explicit formulae for the integrating factor \( \varphi \) of system \((3.11)\) for various wave speeds. Recall that \( \varphi \) solves the differential equation \((3.13).\) Obviously, its explicit form depends on the number of real roots of the polynomial \( g,\) and thus on the wave speed \( c.\) Therefore, we treat the three cases \( c > \bar{c}, \) \( c = \bar{c} \) and \( c < \bar{c} \) separately.

#### 5.2.1 Case \( c > \bar{c}.\)

We have that \( g > 0,\) hence we may define \( \gamma := \sqrt{-\Delta_g} \in \mathbb{R}^+.\) We find that the positive real analytic function
\[ \varphi(u) = (g(u))^\rho \exp \left( -\frac{2 \rho B}{\gamma} \arctan \left( \frac{g'(u)}{\gamma} \right) \right) \] solves \((3.13)\) in \( \mathbb{R},\) where we have set \( \rho := -\left(1 + \frac{\rho}{c} \right) > 0,\) cf. Fig. 3a. The first integral \( H \) associated to \( \varphi \) is analytic in \( \mathbb{R}^2.\)
5.2.2 Case $c = \bar{c}$.

In this situation, the polynomial $g$ has a double root in $\alpha$, cf. (5.34), so that $g(u) = C(u - \alpha)^2$, and

$$\varphi(u) = \left(\frac{g(u)}{C}\right)^{\rho} \exp\left(-2\rho \frac{\alpha}{u - \alpha}\right) > 0$$  \hfill (5.42)

solves (3.13) in $\mathbb{R} \setminus \{\alpha\}$, where again $\rho = -\left(1 + \frac{G}{C}\right) > 0$. We note that $\varphi$ is real analytic and strictly positive in its domain $\mathbb{R} \setminus \{\alpha\}$, cf. Fig. 3b. Therefore, the first integral $H$ associated to $\varphi$ is analytic in $U = \mathbb{R} \setminus \{\alpha\}$.

![Graphs](image)

Figures (a)-(c): $c \in (-\infty, 4/7)$ and $\varphi$ is continuous with two zeros at $\alpha_{1,2}$.

(d) $c = 4/7$: $\varphi$ has a zero at $\alpha_2$. (e) $c \in (4/7, \bar{c}_1)$: $\varphi \in H^1_{\text{loc}}(\mathbb{R})$. (f) $c \in [\bar{c}_1, \bar{c})$: $\varphi \notin H^1_{\text{loc}}(\mathbb{R})$.

Figure 4: The graph of $\varphi$ for increasing values of $c \in (-\infty, \bar{c})$.

5.2.3 Case $c < \bar{c}$.

We have that $g(u) = C(u - \alpha_2)(u - \alpha_1)$ and

$$\varphi(u) = \left(|u - \alpha_2|^\alpha_2 |u - \alpha_1|^{-\alpha_1}\right)^{\theta_c}$$  \hfill (5.43)
solves \[ (3.13) \] in \( \mathbb{R} \setminus \{\alpha_1, \alpha_2\} \), where
\[
\theta_c := -\frac{2(1 + \frac{c}{2})}{\alpha_2(c) - \alpha_1(c)} > 0.
\]

Note that \( \varphi \) is real analytic and positive in \( U = \mathbb{R} \setminus \{\alpha_1, \alpha_2\} \) and \( \varphi \in C(\mathbb{R}) \) as long as \( \alpha_1 \leq 0 \), cf. Fig. 4. For all \( c < \bar{c} \) we have that \( \lim_{|u| \to \infty} \varphi(u) = \infty \) and that \( \varphi \) is continuous in \( \alpha_2 \) with \( \varphi(\alpha_2) = 0 \), while
\[
\lim_{u \to \alpha_1} \varphi(u) = \begin{cases} 0 & \text{if} \ \alpha_1 < 0 \\ (\alpha_2 - \alpha_1)^{\alpha_2} & \text{if} \ \alpha_1 = 0 \\ \infty & \text{if} \ \alpha_1 > 0. \end{cases}
\]

Formula (5.43) tells us that \( \varphi \) has a certain regularity in \( \alpha_2 \), and also in \( \alpha_1 \) provided that \( \alpha_1 \) is small enough. Furthermore we deduce from (5.43) that \( \varphi \in L^1_{\text{loc}}(\mathbb{R}) \) if and only if \( \alpha_1(c) \theta_c < 1 \), which holds true if and only if \( c < c_1 \). Recall that \( c_1 \), defined in (5.39), is the bifurcation value for the existence of fixed points of the form \( (\alpha_1, \pm \nu_{\alpha_1}) \). Moreover, we see that the function \( \psi = -\int f(u) \varphi \psi \alpha \) is continuous at \( \alpha_1 \) if \( c < c_1 \). Observe that \( \psi \) is continuous even for wave speeds slightly larger than \( c_1 \) if \( \alpha_1 \) is a root of \( f_0 \). More precisely, for \( c \in (-\infty, \bar{c}) \) we consider \( f_0 = f_0(c, K) \) and set \( K = K_{\alpha_1(c)} \) so that \( f_0 \) vanishes at \( u_2(c, K_{\alpha_1}) = \alpha_1(c) \), cf. (5.37). We define
\[
c_2 := \sup\{c \in (-\infty, \bar{c}) : f_0 \varphi \in L^1_{\text{loc}}(\mathbb{R})\},
\]
and find that \( c_2 = \sup\{c \in (-\infty, \bar{c}) : \alpha_1(c) \theta_c < 2\} = \frac{165796}{277207} \). Then \( \psi \in C(\mathbb{R}) \) if and only if \( c < c_2 \). We clearly have that
\[
c_1 < c_2 < \bar{c}.
\]

This concludes our discussion of the fixed points and integrating factor of system (3.11). The remainder of this section deals with the systematic construction of traveling wave solutions of (1.1).

### 5.3 Traveling waves – the ”smooth” case.

We claim that for any \( c > \bar{c} \) and \( K > K_0(c) \) there exist smooth solitary and smooth periodic traveling waves. Indeed, recall that when \( c > \bar{c} \) we have \( g(u) > 0 \), and the first integral is of the form (5.41). Therefore \( H \) is analytic on \( \mathbb{R}^2 \) with \( \nabla H(u, v) = 0 \) if and only if \( v = 0 \) and \( \psi''(u) = 0 \). In view of the fact that \( \int \frac{v}{2} \varphi(u) g(u) \geq 0 \), we find that \( (u_2, 0) \) is a local minimum of \( H \) whereas \( (u_1, 0) \) is a saddle point since
\[
\psi''(u) = -(\varphi(u) f_0(u))' = \varphi(u) \left( f_0(u) \frac{2(C + G)u}{g(u)} - f_0'(u) \right),
\]
implying that
\[
\psi''(u_1) = -\varphi(u_1) f_0'(u_1) = -\varphi(u_1) \sqrt{\Delta f_0} < 0, \quad \psi''(u_2) = \varphi(u_2) \sqrt{\Delta f_0} > 0,
\]
where \( \Delta f_0 \) is the discriminant of \( f_0 \) defined in Section 5.1. Note that \( H(u, 0) = \psi(u) \) with \( \lim_{u \to \pm \infty} \psi(u) = \pm \infty \), and \( \lim_{|u| \to \infty} H(u, v) = \infty \), for any fixed \( u \in \mathbb{R} \). Since \( \psi \) decreases strictly in the interval \( (u_1, u_2) \) and increases strictly in \( (u_2, \infty) \), there exists precisely one \( u_r \in (u_2, \infty) \) with \( \psi(u_1) = \psi(u_r) \). Thus, the level-set \( L_{\theta_1}(H) \) contains
the two branches \( \{(u, v_{h_1}^\pm(u)) : u \in [u_1, u_r]\} \) for \( h_1 := \psi(u_1) \) and \( v_{h_1}^\pm(u) \) given in (3.17), which form a homoclinic orbit of system (3.8) representing a smooth solitary traveling wave of (1.1). This solution is symmetric with respect to its unique maximum in view of the symmetry of \( H \) in the second variable, cf. (3.14). Moreover, the solitary wave decays exponentially to the constant value \( u_1 \) on either side of the maximum at infinity, since the vector field is locally \( C^1 \)-conjugate to its linearization at the hyperbolic saddle \((u_1, 0)\) by the Hartman-Grobman Theorem, cf. [31].

The level-sets \( L_h(H) \) with \( h \in (\psi(u_2), \psi(u_1)) \) correspond to periodic orbits around the center \((u_1, 0)\) of system (3.8), that is, closed loops contained in the region bounded by the homoclinic orbit corresponding to \( h = h_1 \). These periodic orbits represent smooth periodic traveling wave solutions of (1.1) which are symmetric with respect to their local extrema and have a unique maximum and minimum per period.

These are all non-constant solutions of (3.8) for \( K > K_0 \) which are bounded in the \( u \)-component. In Fig. 5 we indicate the unbounded solutions by grey lines. There are no non-trivial bounded solutions for \( K \leq K_0 \), cf. Table 1. Indeed, for \( K < K_0 \) the system has no critical points, while for \( K = K_0 \) it has a nilpotent fixed point (a cusp), hence there are no non-constant orbits in the phase plane, which are bounded in the \( u \)-component.

![Figure 5](image)

**Figure 5:** In (5a) we sketch a phase portrait representing scenario III in Table 1, which yields a smooth solitary and smooth periodic traveling waves as illustrated in (5b).

### 5.4 Traveling waves – the ”singular” case.

We recall that the zero set \( N_g \) of the quadratic polynomial \( g \) is nonempty for wave speeds \( c \leq \bar{c} \). This yields the existence of one invariant vertical line \{\( u = \alpha \)} in the phase plane of (3.11) if \( c = \bar{c} \) and \( N_g = \{\alpha\} \), or two such lines \{\( u = \alpha_1 \)\} and \{\( u = \alpha_2 \)\} if \( c < \bar{c} \) and \( N_g = \{\alpha_1, \alpha_2\} \). These lines form the complement of the domain \( U \subseteq \mathbb{R}^2 \) of system (3.8). However, it turns out that for certain parameter combinations \((c, K) \in (-\infty, \bar{c}) \times \mathbb{R}\) solutions of (3.8) can have a continuous extension to a fixed point of (3.11) of the form \((\alpha_i, \pm v_{\alpha_i})\), \( i \in \{1, 2\} \), on \( N_g \times \mathbb{R} \), and possible even beyond that point. It may also happen that a solution of (3.8) becomes unbounded in its \( v \)-component as its \( u \)-component approaches an element of \( N_g \). In view of Proposition 4.1 we will combine such solutions of system (3.8), which are defined not globally, but on subintervals of \( \mathbb{R} \), in a suitable way to construct non-smooth traveling waves of (1.1). In Example 5.3 we explain a prototypical construction in full detail.
We discuss the cases $c < \bar{c}$ in Section 5.4.1 and the case $c = \bar{c}$ in Section 5.4.2. For each scenario we provide sketches of the corresponding phase portraits. Let us point out that the orientation of the orbits—indicated by arrows—reflects the parametrization of system (3.8). For convenience, also fixed points of the form $(\alpha_i, v)$, $i = 1, 2$, of the reparametrized system (3.11) are included in the sketches, even though they are not contained in the domain $U$ of (3.8).

5.4.1 Case $-\infty < c < \bar{c}$.

For this parameter range our qualitative analysis distinguishes between the scenarios I–VII of Table 1. Moreover, we divide each scenario into the subcases $c < c_1$, $c = c_1$ and $c_1 < c < \bar{c}$, where $c_1$ defined in (5.39) is the bifurcation value for fixed points on the invariant line $\{u = \alpha_1\}$. It is convenient to consider the following $c$-dependent subregions of the $(u, v)$-plane

\[
L := \{(u, v) \in \mathbb{R}^2 : -\infty < u < \alpha_1\} \\
M := \{(u, v) \in \mathbb{R}^2 : \alpha_1 < u < \alpha_2\} \\
R := \{(u, v) \in \mathbb{R}^2 : \alpha_2 < u < \infty\}.
\]

Furthermore we denote by $L^+$, $M^+$, and $R^+$ the intersection of $L$, $M$ and $R$ respectively with the upper half-plane $\mathbb{R} \times \mathbb{R}^+$. We define the lower half-regions $L^-$, $M^-$ and $R^-$ accordingly. In the following we will refer to the restrictions of level sets $L_h(H)$ to these regions as segments. We will often analyze the regions $L$, $M$ and $R$ separately. Let us emphasize, however, that orbits may cross the invariant lines through fixed points.

In the constructions below we will frequently discover solution curves of (3.8) that give rise to global piecewise defined continuous functions $\hat{u} : \mathbb{R} \rightarrow \mathbb{R}$ satisfying properties (TW1) and (TW2) of Proposition 4.1. If all involved orbits are not only bounded in the $u$-component but also in the $v$-component, then property (TW3) of Proposition 4.1, which ensures that $\hat{u} \in H^1_{\text{Loc}}(\mathbb{R})$, is trivially satisfied as well and hence $\hat{u}$ turns out to be a traveling wave solution of (1.1). The following result clarifies under which conditions property (TW3) is still satisfied in the case that certain involved orbits become unbounded in the $v$-component.

Lemma 5.1. Let $(c, K) \in (-\infty, \bar{c}) \times \mathbb{R}$, let $h \in \mathbb{R}$ and suppose that $\alpha_i$, $i \in \{1, 2\}$, is an adherent point of the $u$-component of the level set $L_h(H) \subseteq U$. Let $\omega = (\omega_1, \omega_2) : I \rightarrow U$ be a maximal solution of (3.8), whose orbit is contained in $L_h(H) = \{(u, v_h(u))\}$.

1. If $\lim_{u \rightarrow \alpha_2} v_h^+(u) = \infty$, then $\omega$ is not suitable for the construction of a traveling wave solution of (1.1).

2. Suppose that assumption (1) is not satisfied. If $\lim_{u \rightarrow \alpha_1} v_h^+(u) = \infty$, then $\omega$ is suitable for the construction of a traveling wave solution of (1.1), if and only if $\omega_1$ is bounded and $c \in (c_0, c_1)$, where

\[c_0 := 511/1024.\]  

(5.50)

Proof. First we observe the following. If $v_h^+$ blows up as $u \rightarrow \alpha_i$, $i \in \{1, 2\}$, then the blow up of $\omega_2$ happens on a finite subinterval of $I$. To this end, we assume without loss of generality that $\omega$ runs through the set $\{(u, v^+(u)) : \alpha_i - \epsilon \leq u < \alpha_i\}$, $i \in \{1, 2\}$,
where \( \varepsilon > 0 \) is sufficiently small, and \( \lim_{u \to \alpha_i} v_h^+(u) = \infty \). In view of (3.18) we see that \( \omega \) passes this set within a finite interval of length
\[
\delta(\varepsilon) = \int_{\alpha_1 - \varepsilon}^{\alpha_1} \frac{\delta u}{v_h^+(u)} < \infty.
\]
(5.51)

More precisely we have that \( \delta(\varepsilon) = |s_1 - s_\varepsilon| \), where \( s_\varepsilon \) is defined via \( \omega(s_\varepsilon) = (\alpha_i - \varepsilon, v_h^+(\alpha_i - \varepsilon)) \), and \( s_1 \in \mathbb{R} \) is the boundary point of the interval \( I \) where the blow up of \( \omega \) occurs, that is,
\[
\lim_{s \to s_1} \omega_1(s) = \alpha_1 \quad \text{and} \quad \lim_{s \to s_1} \omega_2(s) = \infty.
\]

Ad [1]. We show that \( \omega_2 \not\in L^2_{\text{loc}}(I) \). Let us assume that \( \{(u, v_h^+(u)) : \alpha_2 - \varepsilon \leq u < \alpha_2 \} = \omega([s_\varepsilon, s_1]) \) for some \( \varepsilon > 0 \) sufficiently small with \( \lim_{u \to \alpha_2} v_h^+(u) = \infty \) such that \( \omega(s_\varepsilon) = (\alpha_2 - \varepsilon, v_h^+(\alpha_2 - \varepsilon)) \) with \( \lim_{s \to s_1} \omega_1(s) = \alpha_2 \) and \( \lim_{s \to s_1} \omega_2(s) = \infty \). Observe that \( h \neq \psi(\alpha_2) \), since otherwise this limit would be zero in the case \( K = K_{\alpha_2} \) and equal to \( v_{\alpha_2} \) in case that \( K > K_{\alpha_2} \), as an application of de l’Hôpital’s rule shows:
\[
\lim_{u \to \alpha_2} (v_h^+(u))^2 = \lim_{u \to \alpha_2} 2\frac{h - \psi(u)}{\varphi(u)g(u)} = \lim_{u \to \alpha_2} \frac{-2\psi'(u)}{2f_0(u) \varphi(u) + g(u)\varphi'(u)} = \frac{f_0(\alpha_2)}{2\varphi(\alpha_2)}
\]
(5.52)
where \( \frac{1}{2}B - Go_2 > 0 \) for all \( c \in (-\infty, \varepsilon) \), as discussed in (5.38). Since \( \psi \) is continuous on \( (\alpha_1, \infty) \), we infer that \( h - \psi \) is bounded on \( [\alpha_2 - \varepsilon, \alpha_2] \) and for sufficiently small \( \varepsilon > 0 \) we may assume that \( |h - \psi| > \delta \) on \( [\alpha_2 - \varepsilon, \alpha_2] \) for some \( \delta > 0 \). Therefore we obtain that
\[
v_h^+(u) = \sqrt{\frac{2(h - \psi(u))}{g(u)\varphi(u)}} \in \Theta(|u - \alpha_2|^{-(1+\theta_\alpha_2)/2}) \quad \text{for} \quad u \not\to \alpha_2,
\]
(5.53)
where
\[
1 + \frac{\theta_\alpha_2}{2} \geq 1 \quad \text{for all} \quad c \in (-\infty, \varepsilon).
\]
(5.54)

From (3.18) we obtain that
\[
\int_{s_\varepsilon}^{s_1} [\omega_2(s)]^2 \, ds = \int_{\alpha_2 - \varepsilon}^{\alpha_2} v_h^+(u) \, \delta u,
\]
(5.55)
hence \( \omega_2 \) is not locally square integrable by (5.54).

Ad [2]. For \( c < c_1 \) we use the analogous notation and simplifying assumptions as in the proof of part (i). In this case we obtain by similar arguments as in the proof of part (i) that
\[
v_h^+(u) \in \Theta(|u - \alpha_1|^{-(1-\theta_\alpha_1)/2}) \quad \text{for} \quad u \not\to \alpha_1.
\]
(5.56)
Therefore, the corresponding integral in (5.55) is convergent, and hence \( \omega_2 \in L^2_{\text{loc}}(I) \), if and only if
\[
\frac{1 - \theta_\alpha_1}{2} < 1,
\]
(5.57)
which is equivalent to requiring that \( c > c_0 \). For the case \( c = c_1 \), we recall that \( \psi \) develops a singularity in \( \alpha_1 \), provided that \( K \neq K_{\alpha_1}(c_1) \) (if \( K = K_{\alpha_1}(c_1) \) we use the same reasoning as above). However, since \( \varphi \in \Theta(1) \) for \( u \not\to \alpha_1 \) and \( \sqrt{|h - \psi|} \in L^2_{\text{loc}}(\mathbb{R}) \), we infer that the corresponding integral (5.55) is finite also in this case. \( \square \)
Remark 5.2. We point out that Lemma 5.1 does not tell us whether there exist solutions as stated in the assumptions. We will see that there exist no solutions of (3.11) which become unbounded in the second component at \( \alpha_1 \) if \( c > c_1 \). Note that \( c_0 < c_1 \).

In order to get a first impression of the construction of non-smooth traveling waves of (1.1), we provide a very detailed description of the construction of one particular wave in the following example.

Example 5.3 (A cusped solitary wave). Let \( K \in (K_{\alpha_1}(c), K_{\alpha_2}(c)) \), i.e. we find ourselves in scenario V of Table 1, and let \( c \in (-\infty, c_1) \). The corresponding phase portrait is sketched in Fig. 12a. The function \( \psi : \mathbb{R} \to \mathbb{R} \) is continuous, and it is smooth on \( \mathbb{R} \setminus \{\alpha_1, \alpha_2\} \). We restrict our attention to the region \( L \) and observe that \( \psi \) increases strictly on \( (-\infty, u_1) \), takes a local maximum in \( u_1 \) — recall that \( (u_1, 0) \) is a saddle — and decreases strictly on \( (u_1, \alpha_1) \). Let \( h := \psi(u_1) \) and consider the two branches \( \{ (u, v_h^\pm(u)) : u_1 < u < \alpha_1 \} \). The corresponding orbits are indicated by red lines emerging from the saddle point in Fig. 12a. \( v_h^\pm \) increases strictly on \( (u_1, \alpha_1) \) and becomes unbounded as \( u \nearrow \alpha_1 \), as an application of de L'Hôpital shows. We use these two branches to construct a solitary wave. To this end, recall that \( s \) is the moving frame variable corresponding to the wave speed \( c \), that is, \( s \) is the independent variable of system (3.8). We choose \( s_0 \in (-\infty, 0) \), let \( (u_0, v_0) \in \{ (u,v_h^\pm(u)) : u_1 < u < \alpha_1 \} \) be a point on the upper branch, and consider the Cauchy problem of (3.8) with initial condition \( (u(s_0), v(s_0)) = (u_0, v_0) \). The solution, denoted by \( \omega^-(s) = (\omega_1^-, \omega_2^-) \), runs through the upper branch by construction. It is clear that the maximal interval of existence \( I^- \) is of the form \( (-\infty, s^*) \) with \( s^* \in (s_0, \infty) \). Indeed, \( \omega^-(s) \) approaches the saddle \( (u_1, 0) \) for \( s \to -\infty \), and the upper bound is obtained from (3.18), since

\[
s_1 - s_0 = \int_{u_0}^{u_1} \frac{\delta u}{v_h^+(u)} < \infty, \tag{5.58}
\]

for a unique finite \( s_1 \in \mathbb{R} \). Let us for simplicity assume that \( s^* = 0 \), i.e. \( I^- = (-\infty, 0) \), then

\[
\lim_{s \to -\infty} (\omega_1^-(s), \omega_2^-(s)) = (u_1, 0), \quad \lim_{s \nearrow 0} \omega_1^-(s) = \alpha_1, \quad \lim_{s \nearrow 0} \omega_2^-(s) = \infty. \tag{5.59}
\]

Similarly we obtain a solution \( \omega^+ \) of (3.8) satisfying \( (\omega_1^+, \omega_2^+) = (u_0, -v_0) \), which is defined on \( I^+ = (0, \infty) \) and runs through the lower branch with

\[
\lim_{s \to \infty} (\omega_1^+(s), \omega_2^+(s)) = (u_1, 0), \quad \lim_{s \nearrow 0} \omega_1^+(s) = \alpha_1, \quad \lim_{s \nearrow 0} \omega_2^+(s) = -\infty. \tag{5.60}
\]

We are now ready to construct a composition of these wave segments by defining the bounded continuous function \( \hat{u} : \mathbb{R} \to \mathbb{R} \) as

\[
\hat{u}(s) := \begin{cases} 
\omega_1^- & \text{for } s \in I^- \\
\alpha_1 & \text{for } s = 0 \\
\omega_1^+ & \text{for } s \in I^+,
\end{cases} \tag{5.61}
\]

which is smooth on \( \mathbb{R} \setminus \{0\} \) and decreases exponentially to \( u_1 \) for \( |s| \to \infty \), cf. the upper sketch in Fig. 9b. Due to our construction, \( \hat{u} \) clearly satisfies properties (TW1) and (TW2) of Proposition 4.1, independent of the value \( c \in (-\infty, c_1) \). It remains to confirm (TW3) i.e. to show that the weak derivative \( \hat{u}' \) lies in \( L^p_{\text{loc}}(\mathbb{R}) \), in order to infer that \( \hat{u} \) is
indeed a traveling wave of (1.1). In view of Lemma 6.1 we find that \( \hat{u}' \in L^2_{\text{loc}}(\mathbb{R}) \) if and only if \( c > c_0 \), with \( c_0 \) defined in (5.50).

A similar construction yields cusped periodic traveling waves, cf. the lower sketch in Fig. 9b. Indeed, for \( c \in (c_0, c_1) \) and \( K \in (K_{\alpha_1}(c), K_{\alpha_1}(c)) \), each \( h \in (\psi(\alpha_1), \psi(u_1)) \) corresponds to an orbit similar to the one indicated by the wine red line in Fig. 12a. We identify such an orbit with a smooth solution of (3.8) on some bounded open interval, which may be continued periodically and continuously on the whole real line. Since \( c > c_0 \), the weak derivative of this function is locally square integrable, and hence this periodic extension clearly satisfies all properties of Proposition 4.1.

In the sequel we will omit the details of such "gluing-processes" in our constructions and just identify suitable combinations of orbits in the phase plane of (3.8) with traveling waves of (1.1).

Remark 5.4. In the sketches of the phase portraits for scenarios I–IV we display orbits which are bounded in the \( u \)-component and satisfy Lemma 4.1 (i) with dashed lines to indicate that they are not suitable to construct traveling waves of (1.1).

![Figure 6: Sketches of phase portraits for scenario I, i.e. \( K < K_0(c) \).](image)

**Scenario I [Fig. 6].** We begin with the case \( c < c_1 \), cf. Fig. 6a, where system (3.11) has two saddles \( (\alpha_1, \pm v_{\alpha_1}) \) and no other fixed points. The continuous function \( \psi: \mathbb{R} \to \mathbb{R} \) is strictly increasing.

The orbits in \( L^+ \) can be grouped into the following three categories: orbits corresponding to the segment \( \{(u, v^+_h(u))\}: -\infty < u < \alpha_1 \) for the level \( h = \psi(\alpha_1) \), which reach the fixed point \( (\alpha_1, v_{\alpha_1}) \) and separate \( L^+ \) into an upper region with orbits of the form \( \{(u, v^+_h(u))\}: -\infty < u < \alpha_1 \) for the levels \( h > \psi(\alpha_1) \), and a lower region with orbits corresponding to the level sets \( \{(u, v^+_h(u))\}: -\infty < u < r \) for \( h < \psi(\alpha_1), r < \alpha_1 \), with \( \lim_{u \to -\infty} v^+_h = -\infty \) in each case. By the symmetry of the system we obtain the analogous picture for \( L^- \). We sketch these orbits with grey lines in Fig. 6a since they are unbounded in the first component and therefore do not give rise to traveling wave solutions.

The level sets of the first integral in \( M \) can be divided into three groups as well: \( h < \psi(\alpha_1), h = \psi(\alpha_1) \) and \( \psi(\alpha_1) < h < \psi(\alpha_2) \), cf. the green, red and dark blue dashed...
lines in Fig. 6a. The corresponding (maximal) solutions of (3.8), whose first component runs from $\alpha_1$ to $\alpha_2$, are defined on bounded intervals of length
\[
\int_{\alpha_1}^{\alpha_2} \frac{\delta u}{v_h^\alpha(u)} = \int_{\alpha_1}^{\alpha_2} \sqrt{\varphi(u)g(u)} \frac{\delta u}{2(h - \psi(u))} < \infty,
\]
for corresponding $h \in \mathbb{R}$. However these solutions do not yield traveling waves due to Lemma 5.1, since $\lim_{u \uparrow \alpha_2} v_h^\alpha(u) = \infty$ for all $h$.

All orbits in $R$ have a similar shape and correspond to a level set segment of $L_h(H)$ with $h > \psi(\alpha_2)$, cf. the light blue dashed line in Fig. 6a. Once again, Lemma 5.1 implies that they are not suitable to construct traveling waves.

If $c \geq c_1$, cf. Fig. 6a, the phase portrait of (3.8) changes qualitatively in $L$ and in $M$ due to the absence of the fixed points $(\alpha_1, \pm \psi(\alpha_1))$. All orbits in $L$ correspond to segments of $L_h(H)$ with $h \in \mathbb{R}$ and are unbounded in the first component. The orbits in $M$ correspond to segments of $L_h(H)$ with $h < \psi(\alpha_2)$. They are all of the same type and are not suitable for the construction of traveling waves due to Lemma 5.1, cf. Fig. 6b.

**Scenario II** [Fig. 7]. In comparison with scenario I, the phase portraits (for both cases $c < c_1$ and $c_1 < c < \bar{c}$) change only within $L$, where one fixed point $(\bar{u}, 0)$, a cusp, is present, cf. Fig. 7. However, we see that all (non-constant) orbits in $L$ are unbounded in the first component, so none of them can be used to construct a traveling wave. The trivial solution $u \equiv \bar{u}$ of (2.5) is the only traveling wave solution of (1.1).

![Figure 7: Sketches of phase portraits for scenario II, i.e. $K = K_0(c)$](image)

**Scenario III** [Fig. 8]. The picture in $M$ and $R$ is unchanged, but we will see that the orbits in $L$ give rise to both smooth and non-smooth traveling waves.

We begin with the case $c < c_1$, cf. Fig. 8a,b,c. Note that $\psi$ has the following monotonicity properties: $\psi$ increases strictly on $(-\infty, u_1)$ and attains a local maximum at $u_1$, decreases strictly on $(u_1, u_2)$, takes a local minimum at $u_2$ and increases strictly on $(u_2, \infty)$. Recall that $\psi$ is continuous at $\alpha_1$ since $\varphi \in L_1^{\text{loc}}$ if $c < c_1$. The sign of $\psi(u_1) - \psi(\alpha_1)$, which depends on the choice of the parameter $K$, determines the qualitative behavior of the phase portrait of Fig. 8a,b,c. Let us suppose for the moment that $c$ is fixed such that $\alpha_1 \leq 0$, that is, $c \leq 4/7$. We define the differentiable function
\[
F(K) := \psi(u_1) - \psi(\alpha_1) = \int_{u_1(K)}^{\alpha_1} f_0(K, u) \varphi(u) \delta u, \quad F: [K_0, K_{\alpha_1}] \to \mathbb{R},
\]
for
where we write \( f_0(K, u) \) to emphasize the \( K \)-dependence of \( f_0 \). By Leibniz’ integral rule, the derivative of \( F \) with respect to \( K \) is given by

\[
F'(K) = \int_{u_1(K)}^{\alpha_1} f_0(K, u) \varphi(u) \delta u - u_1'(K) \int_{K_0}^{f_0(K, u_1(K))} \varphi(u_1(K)) > 0. \tag{5.64}
\]

The positive sign follows from the fact that \( u_1 \) is by definition a zero of \( f_0 \), and moreover we have that \( f_0(K, u) = \partial_K f_0(K, u) = 1 \) for all \( K, u \in \mathbb{R} \). Thus \( F \) is strictly increasing. Furthermore, \( F(K_0) < 0 \) and \( F(K_{\alpha_1}) > 0 \), therefore \( F \) has a unique zero which we denote by \( K_1 \in (K_0, K_{\alpha_1}) \), and \( F < 0 \) in \([K_0, K_1] \), \( F > 0 \) in \((K_1, K_{\alpha_1}] \). If \( 4/7 < c < c_1 \) then \( \alpha_1(c) > 0 \), and we can not apply Leibniz’ rule on \( F \) since \( \psi \) is not continuous at \( \alpha_1 \). Note however, that \( F \) is still differentiable on \([K_0, K_{\alpha_1}] \) and continuous at \( \alpha_1 \), with \( F(K_0) < 0 \) and \( F(K_{\alpha_1}) > 0 \). Let \( F_\varepsilon(K) = \psi(u_1) - \psi(\alpha_1 - \varepsilon) \) for some sufficiently small \( \varepsilon > 0 \), which is defined on a subinterval \([K_0, K_{\varepsilon}] \subseteq [K_0, K_{\alpha_1}] \). By continuity we can choose \( \varepsilon \) small enough such that \( F_\varepsilon(K) = 0 \). Then \( F_\varepsilon' > 0 \), which in turn shows that \( F \) is strictly increasing on \([K_0, K_{\alpha_1}] \) for all \( c \in (-\infty, c_1) \), since \( \varepsilon \) can be chosen arbitrarily small.

If \( K \in (K_0, K_1) \), cf. Fig. 8a, we find periodic orbits around the center \((u_2, 0)\) which are surrounded by a homoclinic orbit starting at the saddle \((u_1, 0)\). The periodic orbits, which correspond to energies \( \psi(u_2) < h < \psi(u_1) \), yield smooth periodic traveling waves and the energy \( h = \psi(u_1) \) corresponds to a homoclinic orbit, which gives rise to a smooth solitary wave, cf. Fig. 5b.

If \( K = K_1 \), cf. Fig. 8b, then \( F(K) = 0 \), which implies the existence of a heteroclinic orbit in \( L^\pm \) connecting \((u_1, 0)\) with \((\alpha_1, \pm v_{\alpha_1})\); the corresponding energy is given by \( h = \psi(u_1) = \psi(\alpha_1) \). The region in \( L \) inside these heteroclinic orbits and the invariant line at \( \alpha_1 \) is filled with periodic orbits encircling \((u_2, 0)\), which correspond to level set segments of \( L_0(H) \) with \( \psi(u_2) < h < \psi(u_1) \). The two heteroclinic branches form a peaked solitary wave, cf. Fig. 9a. The periodic orbits yield smooth periodic waves cf. Fig. 5b.

If \( K < K < K_{\alpha_1} \), cf. Fig. 8c, there exists a heteroclinic orbit linking \((\alpha_1, v_{\alpha_1})\) with \((\alpha_1, -v_{\alpha_1})\) corresponding to the energy \( h = \psi(\alpha_1) < \psi(u_1) \). This heteroclinic orbit bounds a region in \( L \), which is filled with periodic orbits encircling \((u_2, 0)\) with energies \( \psi(u_2) < h < \psi(\alpha_1) \). The heteroclinic orbit yields peaked periodic waves, the periodic orbits yield smooth periodic waves. The energy \( h = \psi(u_1) \) corresponds to an orbit in \( L^+ \) which arises from \((u_1, 0)\) and becomes unbounded in the \( v \)-component as \( u \searrow \alpha_1 \). This orbit combined with its counterpart in \( L^- \) yields a cusped solitary wave, cf. Fig. 8d, provided that the additional condition \( c \in (c_0, c_1) \) is satisfied in view of Lemma 5.1. For energies \( \psi(\alpha_1) < h < \psi(u_1) \) we obtain periodic cusped traveling wave solutions if \( c \in (c_0, c_1) \), again due to Lemma 5.1.

If \( c_1 \leq c < \hat{c} \), cf. Fig. 8d, \( \psi \) has a pole at \( \alpha_1 \) and we obtain a homoclinic orbit for energy \( h = \psi(u_1) \) and periodic orbits for energies \( \psi(u_2) < h < \psi(u_1) \). Hence we obtain a smooth solitary wave and smooth periodic traveling waves, cf. Fig. 5b. Energies other than these correspond to orbits which are unbounded in the \( u \)-component.

Remark 5.5. So far we have constructed non-smooth waves by combining orbits that correspond to one particular energy level \( h \), see for instance Fig. 3 and Fig. 9. More precisely, these waves satisfy the following special version of property [TW2] in
Figure 8: Sketches of the phase portraits of scenario III: (8a) and (8d) yield smooth periodic waves, (8a) and (8d) yield smooth solitary waves, (8b) yields peaked solitary waves, (8c) yields peaked periodic and – provided that \( c \in (c_0, c_1) \) – both periodic and solitary cusped traveling waves.

Figure 9: Sketches of peaked (9a) and cusped (9b) traveling waves.

Proposition 4.19

It holds that \( \lambda(u^{-1}(N_g)) = 0 \) and there exist \( K, h \in \mathbb{R} \) such that

\[
(u')^2 = 2 \frac{h - \psi(u)}{\varphi(u) g(u)} \quad \text{on all intervals } I_j
\]

\[
u \rightarrow \alpha_i \quad \text{at finite endpoints of } I_j, \text{with } \alpha_i \in N_g.
\]

(TW2')

(5.65)
In particular all smooth traveling waves satisfy this property. Note however, that Proposition 4.1 also permits the combination of orbits which correspond to different energy levels. Scenarios IV and VII for instance yield rich collections of such solutions of system (3.8), see Fig. 11 and Fig 17c. To distinguish between these two types of traveling waves, we make the following definition:

Definition 5.6. A traveling wave solution of (1.1) is called an elementary wave, if \((TW2')\) is satisfied. Otherwise we speak of a composite wave.

![Phase portrait sketches](image-url)

Figure 10: Sketches of phase portraits in scenario IV, i.e. \(K = K_{\alpha_1}(c)\).

Scenario IV [Fig. 10]. We refer to the previous scenarios for the discussion of solution curves within the region \(R\). Note that \(\psi \in \mathbb{C}(\mathbb{R})\) if and only if \(c < c_2\), cf. (5.46), due to the fact that \(f_0\) vanishes in \(\alpha_1\).

Let us consider the region \(M\) first. For wave speeds \(c \in (-\infty, c_1) \setminus \{c_1\} \), cf. Fig. 10a and Fig. 10c we distinguish between three different types of level sets \(L_h(H)\), similar as in the previous scenarios. The only qualitative difference is, that the upper branch of the \(M\)-segment of \(L_h(H)\) for energy \(h = \psi(\alpha_1)\) connects to a point on the \(u\)-axis, namely \((\alpha_1, 0)\). Indeed, if \(c \neq c_1\) we find that

\[
\lim_{u \searrow \alpha_1} (u^\pm_h(u))^2 = \lim_{u \searrow \alpha_1} 2 \frac{\psi(\alpha_1) - \psi(u)}{\varphi(u) g(u)} = \lim_{u \searrow \alpha_1} \frac{f_0(u)}{B - 2Gu} = 0, \quad (5.66)
\]

where we have used L'Hôpital’s rule in the second equality. Furthermore we have that

\[
\lim_{u \nearrow \alpha_2} (u^\pm_h(u))^2 = \lim_{u \nearrow \alpha_2} 2 \frac{\psi(\alpha_1) - \psi(u)}{\varphi(u) g(u)} = \infty. \quad (5.67)
\]
For $c \in [c_2, \hat{c})$, cf. Fig. 10d, each orbit in $M$ corresponds to the $M$-segment of a level set $L_h(H)$ with $h < \psi(\alpha_2)$. These segments cross the $u$-axis and satisfy $\lim_{u \to \alpha_2} v_h^+(u) = \pm \infty$. None of the orbits we considered so far are suitable for the construction of traveling waves by Lemma 5.1. We will analyse the case $c = c_1$ separately below.

Next we analyze the phase portraits within $L$. For the wave speeds $c < c_1$, cf. Fig. 10a, the function $\psi$ is continuous on $\mathbb{R}$, increases strictly in the interval $(-\infty, u_1)$, takes a local maximum at $u = u_1$, decreases strictly on $(u_1, \alpha_1)$, has a local minimum at $\alpha_1$ and increases on $(\alpha_1, \infty)$ with $\psi'(\alpha_1) = 0$. This yields, for the energy $h = \psi(u_1)$, a solution branch $\{(u, v_h^+(u)) : u_1 < u < \alpha_2\}$ which connects to the saddle $(u_1, 0)$ with $\lim_{u \to \alpha_1} v_h^+(u) = \infty$, cf. the upper red orbit in Fig. 10a. We can identify this orbit together with its counterpart in $L^-$ with a cusped solitary wave, provided that $c \in (c_0, c_1)$, cf. Lemma 5.1. Energies $\psi(\alpha_1) < h < \psi(u_1)$ yield smooth orbits in $L$ as indicated by the wine red line in Fig. 10a. The elementary traveling waves that correspond to these orbits are periodic ones with cusps. There are no other solution curves possessing a bounded first component.

For $c \in (c_1, \hat{c})$, cf. Fig. 10b and Fig. 10d, the level $h = \psi(u_1)$ yields a heteroclinic orbit connecting $(u_1, 0)$ with $(\alpha_1, 0)$. This is obvious for $c \in (c_1, c_2)$, since in this case $(g\varphi)(u) \to \infty$ as $u \nrightarrow \alpha_1$, whereas $\psi(u_1) - \psi(u)$ stays bounded. For $c \in [c_2, \hat{c})$ we have that $\psi(u_1) - \psi(u) \to \infty$ as $u \nrightarrow \alpha_1$ and by applying de l'Hôpital's rule we obtain

$$\lim_{u \to \alpha_1} \frac{2(\psi(u_1) - \psi(u))}{\varphi(u)g(u)} = \frac{f_0(\alpha_1)}{2 - \theta \alpha_1} = 0.$$  \hspace{1cm} (5.68)

The point $(\alpha_1, 0)$ is reached by a solution of (3.8) at some finite value of the moving frame variable $s$. Again this is obvious for $c \in (c_1, c_2)$ where $\psi$ is continuous at $\alpha_1$, because then $\varphi g$ has a finite improper integral and in particular

$$\int_{\alpha_1}^{\alpha_1 - \varepsilon} \frac{\delta u}{\sqrt{v_h^+(u)}} = \int_{\alpha_1 - \varepsilon}^{\alpha_1} \frac{\varphi(u)g(u)}{2(h - \psi(u))} \delta u < \infty \hspace{1cm} (5.69)$$

for sufficiently small $\varepsilon > 0$ such that $u_1 < \alpha_1 - \varepsilon$. If $c \in (c_2, \hat{c})$, we use that

$$\lim_{u \to \alpha_1} \frac{\varphi(u)g(u)(\alpha_1 - u)}{2(h - \psi(u))} = \frac{1}{2} \frac{2 - \theta \alpha_1 - C(\alpha_2 - \alpha_1)\alpha_1}{E(u_1 - \alpha_1)} \in (0, \infty),$$  \hspace{1cm} (5.70)

which implies that the improper integral in (5.69) is again finite. Energies $h \in (\psi(\alpha_1), \psi(u_1))$ in case that $c \in (c_1, c_2)$, and energies $h \in (-\infty, \psi(u_1))$ in case that $c \in [c_2, \hat{c})$, yield homoclinic orbits of (3.8), cf. the wine red loop in Fig. 10b and Fig. 10d. The corresponding solutions of (3.8) are defined on intervals of finite length.

Finally we analyze the case $c = c_1$, cf. Fig. 10b. We have that every point on the invariant line $\{u = \alpha_1\}$ of system (3.11) is a fixed point. Recall that the function $f$ can be written as

$$f(u, v) = E(u - u_1)(u - \alpha_1) - G(u - \alpha_1)v^2 \hspace{1cm} (5.71)$$

in this case, and that $\psi$ is continuous on $\mathbb{R}$ in view of (5.46). It satisfies the same monotonicity properties as in the case $c < c_1$, but its graph is not smooth at the local minimum in $\alpha_1$: the corresponding one-sided derivatives exist, but do not coincide. The function $\varphi g$ has a (finite) jump at $\alpha_1$, since $1 - \theta \alpha_1 = 0$ if $c = c_1$ and therefore

$$(\varphi g)(u) = -C \text{ sgn}(u - \alpha_1)(u - \alpha_2)^{\theta \alpha_2 + 1}, \quad u \in \mathbb{R} \setminus \{\alpha_1\},$$  \hspace{1cm} (5.72)
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for some constant $C$. Note, however, that the limit $v_h^+(\alpha_1^-) := \lim_{u \searrow \alpha_1^-} v_h^+(u)$ is still defined. For $h = \psi(u_1)$ we obtain a heteroclinic orbit connecting $(u_1, 0)$ with the fixed point $(\alpha_1, v_h^+(\alpha_1^-))$. We obtain a peaked solitary wave by combining this orbit with its counterpart in $L^-$. Energies $h \in (\psi(\alpha_1), \psi(u_1))$ yield heteroclinic orbits in $L$ which connect the fixed points $(\alpha_1, v_h^-(\alpha_1^-))$ and $(\alpha_1, v_h^+(\alpha_1^-))$ and cross the $u$-axis in some point $(u_h, 0)$. Due to the strict monotonicity of $\psi$ and $\varphi g$ on $(u_1, \alpha_1)$ it is obvious, that $0 < v_{h_1}^-(\alpha_1^-) < v_{h_2}^-(\alpha_1^-)$ for $\psi(\alpha_1) < h_2 < h_1 \leq \psi(u_1)$. Each one of these solution curves gives rise to a peaked periodic traveling wave of elementary type. By combining orbits of different energies, we obtain a rich collection of (not necessarily periodic) peaked waves. The orbits that correspond to energies $h > \psi(u_1)$ and $h < \psi(\alpha_2)$ are unbounded in $u$. The orbits in $M$ correspond to energy levels $h < \psi(\alpha_2)$. All these orbits become unbounded in the second coordinate as $u$ approaches $\alpha_2$ from the left side. Therefore they are not useful for the construction of traveling waves in view of Lemma 5.1. For the sake of completeness we analyze their behavior as $u \searrow \alpha_1$. For $h = \psi(\alpha_1)$ we have that $\lim_{u \searrow \alpha_1} v_h^+(u) = 0$ – this orbit (and its reflection about the $u$-axis) is indicated by the light blue dashed line in Fig. 10b. Energies $h < \psi(\alpha_1)$ imply $\lim_{u \searrow \alpha_1} v_h^+(u) > 0$, and energies $\psi(\alpha_1) < h < \psi(\alpha_2)$ yield orbits indicated by a dark blue dashed line in Fig. 10b.

To conclude the discussion of scenario IV we observe that for any wave speed $-\infty < c < \bar{c}$ the constant solution $u \equiv \alpha_1$ is a classical solution of (2.5), since in this case $u' \equiv u'' \equiv 0$, $g(\alpha_1) = 0$, and $f_0(\alpha_1(c), K_{\alpha_1(c)}) = 0$. This enables the construction of composite waves, which are piecewise constant equal to $\alpha_1$. We thereby obtain smooth and peaked traveling waves with plateaus and so-called compactons, cf. Fig. 11.

Figure 11: Some examples of composite waves constructed from orbits corresponding to scenario IV: smooth (11a) and peaked (11b) traveling wave solutions with plateaus at height $\alpha_1$, smooth (11c) and peaked (11d) multi-crested solutions with decay, and smooth (11e) and non-smooth (11f) compactons.
Compactons are solitary waves with compact support in the sense that they take a constant value outside an interval of finite length; in other words a solitary wave of finite length.

\begin{align}
\text{(a) } c &\in (-\infty, c_1] \\
\text{(b) } c &\in (c_1, \bar{c})
\end{align}

Figure 12: Sketches of phase portraits in scenario V, i.e. \( K \in (K_{a_1}(c), K_{a_2}(c)) \).

**Scenario V [Fig. 12].** We refer to the previous scenarios for the phase portraits within \( \mathbb{R} \). Recall that the set of fixed points of (3.11) consists of the two saddles \((u_1, 0) \in L\) and \((u_2, 0) \in M\) if \( c \leq c_1 \). For larger wave speeds, i.e. \( c \in (c_1, \bar{c}) \), system (3.11) has the additional fixed points \((\alpha_1, \pm v_{\alpha_1})\).

First we consider the case \( c \leq c_1 \), cf. Fig. 12a. We begin with the description of the orbits in \( L \). If \( c \neq c_1 \), the function \( \psi \in C(\mathbb{R}) \) decreases strictly on \((u_1, \alpha_1)\). We obtain a phase portrait similar as in scenario IV with \( c \in (-\infty, c_1) \), cf. Fig. 10a. There are two kinds of relevant orbits: the two orbits with corresponding energy \( h = \psi(\alpha_1) \), indicated by the red lines in Fig. 12a and the orbits corresponding to energies \( h \in (\psi(\alpha_1), \psi(u_1)) \), which are indicated by wine red lines in Fig. 12a. These orbits are suitable for the construction of traveling waves if \( c \in (c_0, c_1) \). The situation is similar in the case \( c = c_1 \). The only difference is, that the orbits of the latter type (wine red) correspond to energies \( h < \psi(u_1) \).

Next we describe the orbits in \( M \). Let us again first assume that \( c \neq c_1 \). Observe that \( \psi \in C(\mathbb{R}) \) decreases strictly on \((\alpha_1, u_2)\), takes a local minimum in \( u_2 \) and increases strictly on \((u_2, \alpha_2)\). There are two kinds of relevant orbits in \( M \), whose second components become unbounded at \( \alpha_1 \): the two orbits corresponding to \( h = \psi(u_2) \), indicated by the dark blue lines in Fig. 12a and the orbits corresponding to \( \psi(u_2) < h < \psi(\alpha_1) \), indicated by the light blue line. The corresponding elementary traveling waves are solitary and periodic anti-cusps, cf. Fig 14a. The situation is similar in the case \( c = c_1 \). The only difference is, that the orbits of the latter type (light blue) correspond to energies \( h > \psi(u_2) \). We may combine orbits in \( L \) and \( M \) to obtain composite waves, such as steep wavefronts, see Fig. 13b.

If \( c_1 < c < \bar{c} \), cf. Fig. 12b, then (3.11) has the additional fixed points \((\alpha_1, \pm v_{\alpha_1})\) which are stable and unstable nodes. The suitable \( M^+ \)-segments of the level-sets \( L_h(H) \) with \( h \geq \psi(u_2) \) in \( M \) and the \( L^+ \)-segments of \( L_h(H) \) with \( h \leq \psi(u_1) \) in \( L \) reach the point \((\alpha_1, v_{\alpha_1})\) as \( u \to \alpha_1 \) and the corresponding solutions of (3.8) reach this point as the moving frame variable approaches some finite value \( s_0 \). Therefore we obtain elementary
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Figure 13: Traveling waves constructed from orbits corresponding to the phase portrait in Scenario V, Fig. 12a: Fig. 13a shows solitary and periodic anti-cusped waves. Fig. 13b shows composite waves: a steep wave front and a periodic composition.

Figure 14: Traveling waves constructed from orbits corresponding to the phase portrait in Scenario V, Fig. 12b: Fig. 14a shows solitary and periodic anti-peaked waves. Fig. 14b shows composite waves: a wave front and a periodic composition.

waves with peaks, cf. Fig. 14a. Moreover, we can construct a large variety of composite waves, e.g. wave fronts, see Fig. 14b.

Scenario VI [Fig. 15]. We refer to the previous scenario for the description of the regions \( L \) and \( R \), and proceed with the discussion for the region \( M \).

Let \( c \leq c_1 \), cf. Fig. 15a, and assume for the moment that \( c \neq c_1 \). Observe that \( \psi \in C(\mathbb{R}) \) decreases strictly on \((\alpha_1, \alpha_2)\). There are two types of suitable orbits in \( M \). The two orbits corresponding to \( h = \psi(\alpha_2) \) reaching the nilpotent fixed point \((\alpha_2, 0)\) of (3.11), which are indicated by the dark blue lines, and the orbits corresponding to \( h \in (\psi(\alpha_2), \psi(\alpha_1))\); one of them is indicated by a light blue line. All these orbits are suitable for the construction of traveling waves, provided that \( c > c_0 \) is satisfied in view of Lemma 5.1. We observe that (maximal) solutions of (3.8) in \( M \), which correspond to the energy \( h = \psi(\alpha_2) \), are defined on a bounded interval \( I \) of length \( \delta \). Similar as in
(5.70) it holds that
\[
\lim_{u \to \alpha_2} \frac{u g(u)(\alpha_2 - u)}{2(h - \psi(u))} = \frac{1}{2} B - G \alpha_2 \in (0, \infty),
\]

hence \(\delta\) is given by
\[
\delta = \int_{\alpha_2}^{\alpha_1} \frac{\delta u}{|v^+(u)|} = \int_{\alpha_2}^{\alpha_1} \sqrt{\frac{\varphi(u)g(u)}{2(h - \psi(u))}} \delta u < \infty.
\]

The situation is similar in the case \(c = c_1\), the difference being that the orbits of the light blue type correspond to energies \(h > \psi(\alpha_2)\). Let now \(c \in (c_1, \bar{c})\), cf. Fig. [15]. Similar as in the previous case we obtain that all orbits in \(M^+\) are connected to \((\alpha_2, v_{\alpha_2})\).

We observe that the constant function \(u \equiv \alpha_2\) is a classical solution of (2.5) for all wave speeds \(c < \bar{c}\). We may therefore construct traveling waves of (1.1), which are piecewise constant. For instance, there exist \(\text{anti-cuspecompactons}\) for \(c_0 < c \leq c_1\), and \(\text{anti-peakompactons}\) for \(c_1 < c < \bar{c}\). These solitary waves have the finite length \(2\delta\) with a cusp or peak respectively at their trough, see Fig. [17a].

Figure 15: Sketches of phase portraits in scenario VI, i.e. \(K = K_{\alpha_2}(c)\).

**Scenario VII [Fig. 16]**. We refer to the previous scenario for the discussion of \(L\).

The restriction of the phase portrait to \(M\) is similar as in the previous scenario for all \(-\infty < c < \bar{c}\). The main difference is that the orbits corresponding to \(h = \psi(\alpha_2)\) reach the points \((\alpha_2, \pm v_{\alpha_2})\), which are saddles of (3.11). We may use these orbits for the construction of periodic traveling waves with peaked crests and cusped troughs, if \(c_0 < c \leq c_1\), or waves where both crests and troughs are peaked, if \(c_1 < c < \bar{c}\), cf. Fig [17b].

The situation in \(R\) differs from all the other scenarios. Energy \(h = \psi(\alpha_2)\) yields a heteroclinic orbit connecting \((\alpha_2, v_{\alpha_2})\) with \((\alpha_2, -v_{\alpha_2})\). It crosses the \(u\)-axis at some point to the right of the center \((u_2, 0)\). The region bounded by this orbit and the vertical line \(\{u = \alpha_2\}\) is filled with periodic orbits corresponding to energies \(\psi(u_2) < h < \psi(\alpha_2)\) encircling the center \((u_2, 0)\). Energies \(h > \psi(\alpha)\) yield orbits, which are not suitable for the construction of traveling waves.

Scenario VII enables the construction of composite waves, where orbits from all three regions \(L\), \(M\) and \(R\) with different energy levels are combined. We sketch an example of such a composite wave in Fig. [17c].
5.4.2 Case $c = \bar{c}$.

The quadratic polynomial $g$ has one double root $\alpha$ and $\varphi$ given by (5.41) is an integrating factor for (3.11), which satisfies

$$\lim_{u \to -\infty} \varphi(u) = \lim_{u \to \infty} \varphi(u) = \infty, \quad \lim_{u \to \alpha} \varphi(u) = 0.$$  \hspace{1cm} (5.75)

The corresponding first integral $H$ is defined on $\mathbb{R}^2 \setminus \{\alpha\} \times \mathbb{R}$. The vertical line $\{u = \alpha\}$ is an invariant set of system (3.11), which separates the $(u,v)$-plane into the two regions

$L := \{(u,v): -\infty < u < \alpha\}$ and $R := \{(u,v): \alpha < u < \infty\}$.  \hspace{1cm} (5.76)

We observe that $\varphi$ vanishes faster than any polynomial as $u \searrow \alpha$. This implies that orbits in $R$, which become unbounded in the $v$-component as $u \searrow \alpha$ are not suitable for the construction of traveling waves. Indeed, similarly as in Lemma 5.1, we see that the existence interval of these orbits is finite, but

$$\int_{\alpha}^{\alpha+\varepsilon} v_h^+(u) \delta u = \infty,$$  \hspace{1cm} (5.77)

for any $\varepsilon > 0$ and all $h > 0$. Therefore the second component of the corresponding solution curves in $R$ are not locally square integrable in view of (3.18).

In the following we study the solutions of (3.8) for increasing values of $K$. We distinguish between five scenarios, cf. I-V in Table 1.

Scenarios I and II [Fig. 18]. These scenarios do not yield traveling waves apart from the trivial constant wave $u \equiv \bar{u}$ in scenario II. The $L$-segment of any (nonempty) level-set $L_h(H)$ is unbounded in $u$. The orbits in $R$, which correspond to $R$-segments of $L_h(H)$ for $h > 0$, are bounded in $u$ but the second components of these solution curves are not locally square integrable and therefore not suitable for the construction of traveling waves.

Scenario III [Fig. 19a]. We refer to scenarios I and II for the phase portrait in $R$. The phase portrait in $L$ looks similar as in scenario III in Section 5.4 for wave speeds $c \in [c_1, \bar{c})$. We find a homoclinic orbit of (3.8) corresponding to the $L_h(H)$-segment.
Figure 17: In Fig. 17a we see sketches of an anti-cusped and an anti-peaked solitary wave taking the constant value $\alpha_2$ outside a bounded interval - they correspond to the dark blue lines in Fig. 15a and 15b, respectively. The first image in Fig. 17b shows a periodic wave with peaked crests and cusped troughs. In the second sketch in Fig. 17b we see a periodic wave with peaked crests and troughs. These waves correspond to the dark blue lines in Fig. 16a and 16b, respectively. In Fig. 17c we see an example of a composite wave constructed from orbits of different energy levels in Fig. 16b.

with $h = \psi(u_1)$, which yields a smooth solitary wave. Energies $h \in (\psi(u_2), \psi(u_1))$ yield periodic orbits encircling the center $u_2$, which correspond to smooth periodic waves, cf. Fig. 5b.

**Scenario IV** [Fig. 19b]. The situation in $R$ is similar as in the previous scenarios. The restriction of phase portrait to $L$ as well as the corresponding traveling wave solutions are similar as in scenario IV in Section 5.4 with $c \in [v_2, c)$, cf. Fig. 10d and Fig. 11b. The heteroclinic orbits indicated by red lines correspond to the energy $h = \psi(u_1)$, whereas the homoclinic orbits of system (3.11), indicated by a wine-red line, correspond to energies $h < \psi(u_1)$.

**Scenario V** [Fig. 20]. For this choice of the parameter $K$, systems (3.8) and (3.11) have a saddle in $(u_1, 0)$ and a center in $(u_2, 0)$. Additionally, system (3.11) the two non-hyperbolic fixed points $(\alpha, \pm v_\alpha)$.

We describe the phase portrait in $L$. Only the energies $h \leq \psi(u_1)$ yield orbits that are bounded in the first coordinate. Let $h = \psi(u_1)$. The branches $\{(u, v_\alpha^\pm(u)): u_1 < u < \alpha\}$ link $(u_1, 0)$ with $(\alpha, \pm v_\alpha)$. The corresponding orbits of (3.11) are indicated by the red lines in the left half-plane in Fig. 20. The energies $h < \psi(u_1)$ correspond to heteroclinic orbits from $(\alpha, -v_\alpha)$ to $(\alpha, v_\alpha)$, cf. the wine red orbit in the left half-plane in Fig. 20. The following types of elementary waves can be obtained: peaked solitary and peaked periodic traveling waves.
Next we discuss the phase portrait in $\mathbb{R}$. The energy $h = 0$ yields a heteroclinic orbit from $(\alpha, -v_\alpha)$ to $(\alpha, v_\alpha)$, which is indicated by the red line in the right half-plane in Fig. 20. Energies $h \in (\psi(u_2), 0)$ correspond to periodic orbits around the center in $(u_2, 0)$. Thus we obtain smooth periodic waves and an anti-peaked periodic wave. The orbits that correspond to energies $h > 0$ are not suitable for the construction of traveling waves.

**Remark 5.7 (Cantor waves).** Let us point out that it is possible to obtain composite waves of fractal type, for instance Cantor waves as indicated in Fig. 21, using constructions based on iterative schemes. Such fractal functions appear also as traveling wave solutions of the CH equation, cf. [26]. In order to construct the aforementioned Cantor wave, we use a suitable collection of either cusp or peak elements with the property that for each value $\delta \in (0, \delta_0)$, with $\delta_0 > 0$, there exists an element such that the corresponding solutions of (3.8) passing through this element are defined on a maximal interval of existence of length $\delta$. In the following we demonstrate that such collections do indeed exist e.g. in scenarios V – VII of Table 1. For $c \in (c_0, c_1)$ and $K > K_{\alpha_1}(c)$ we can construct cusped as well as anti-cusped Cantor waves; for every $c \in (c_1, \bar{c})$ and $K > K_{\alpha_1}(c)$ we obtain peaked as well as anti-peaked Cantor waves. We carry out the details only for cusped waves, the other cases being similar.

Fix $h_0 \in (\psi(\alpha_1), \psi(u_1))$ and consider the cusp component in $L$ of the level set $L_{h_0}(H)$,
e.g. the orbit indicated by the wine red line in Fig. 16a. We have already shown that solutions of (3.8) passing through this curve are defined on an interval of finite length $2\delta_0 > 0$ with

$$
\delta_0 = \int_{u_0(h_0)}^{\alpha_1} \frac{du}{v_{h_0}(u)},
$$

(5.78)

where $u_0(h_0)$ denotes the unique point in $(u_1, \alpha_1)$, at which $\psi$ takes the value $h_0$. The collection of cusp components corresponding to energies $\psi(\alpha_1) < h < h_0$ fills the entire region between the $h_0$-curve and the vertical line $\{u = \alpha_1\}$. Let

$$
\delta: (\psi(\alpha_1), h_0) \to \mathbb{R}, \quad \delta(h) := \int_{u_0(h)}^{\alpha_1} \frac{du}{v_{h}(u)},
$$

(5.79)

where $u_0(h)$ denotes the unique point in $[u_0(h_0), \alpha_1)$, at which $\psi$ takes the value $h$. Then $\delta(h_0) = \delta_0$ and it is clear that $\delta$ is positive. We will show that

$$
\delta(h) \to 0 \quad \text{as} \quad h \to \psi(\alpha_1),
$$

(5.80)

that is, the maximal existence interval of a cusp element can be arbitrarily small. It follows from the continuity of the functions $g$, $\varphi$ and $\psi$ on $[u_0(h_0), \alpha_1)$, that $\delta: (\psi(\alpha_1), h_0) \to (0, \delta_0]$ is onto. This guarantees that the collection of cusp elements with energies $h \in (\psi(\alpha_1), h_0]$ is suitable for the construction of a composite wave as illustrated in Fig. 21b, where the preimage of $\{\alpha_1\}$ under this wave is a Cantor set. In order to show (5.80), we fix $h \in (\psi(\alpha_1), h_0]$ and split the integral $\delta(h)$ into two parts:

$$
\delta(h) = \delta_1(h) + \delta_2(h) = \int_{u_0(h)}^{\psi(h)} \frac{du}{v_{h}(u)} + \int_{\psi(h)}^{\alpha_1} \frac{du}{v_{h}(u)},
$$

(5.81)

where $u^*(h) := u_0(h) + (\alpha_1 - u_0(h))/2$. Then

$$
\delta_2(h) \leq \frac{\alpha_1 - u^*(h)}{v_{h}(u^*(h))} \leq \frac{\alpha_1 - u^*(h)}{v_{h_0}(u^*(h_0))} \to 0 \quad \text{as} \quad h \to \psi(\alpha_1),
$$

(5.82)

since clearly $u^*(h) \to \alpha_1$ for $h \to \psi(\alpha_1)$. In order to see that also $\delta_1(h) \to 0$ as $h \to \psi(\alpha_1)$, we observe that $1/v_{h}(u) \in \Theta(1/\sqrt{u - u_0(h)})$ for $u \searrow u_0(h)$. Therefore, there exists a constant $C > 0$ such that

$$
\delta_1(h) \leq C \int_{u_0(h)}^{u^*(h)} \frac{du}{\sqrt{u - u_0(h)}} = 2C \sqrt{u^*(h) - u_0(h)} \to 0 \quad \text{as} \quad h \to \psi(\alpha_1),
$$

(5.83)
since \( u_0(h) \to \alpha_1 \) in this limit. Thus we have shown that the maximal existence intervals of cusp elements (and similarly for peak elements) can be arbitrarily small. More precisely, the length of such intervals can take any value in \((0, 2\delta_0]\). Note that this implies in particular that equation \((1.1)\) admits peaked and cusped periodic traveling waves of arbitrarily small wave length.

![Figure 21: Sketches of Cantor waves with peak elements (a) and cusp elements (b).](image)

### 6 Main results

In this section we summarize our main results, based on the comprehensive phase plane analysis of the integrable system \((3.8)\) in Section 5. To this end, let us briefly recall that solution curves of \((3.8)\) correspond to the level sets \(L_1(H)\) of the first integral \(H\). We denote the energy level of each solution curve by \(h \in \mathbb{R}\). Note that \(H(u, 0) = \psi(u)\) is defined in \((3.15)\), while \(u_{1,2}\) and \(\alpha_{1,2}\) denote the real zeros of the quadratic polynomials \(g\) and \(f_0\), see Section 5.1. We review the following "critical" values of the parameters \(c\) and \(K\): \(\bar{c}\) is the bifurcation point of the double root \(\alpha\) of \(g(u)\), cf. \((5.33)\); for every \(c\), the value \(K_0(c)\) is the bifurcation point of the double root \(\bar{u}\) of \(f_0(u)\), cf. \((5.32)\); for \(c \leq \bar{c}\), the value \(K_1(c)\) is defined in Section 5.4.1 Scenario III, and the values \(\bar{c} = K_\alpha(c), i \in \{1, 2\}\), are the ones at which the root \(\alpha_i(c)\) of \(g\) coincides with the root \(u_2(c), K\) of \(f_0\), cf. \((5.37)\); \(c_1\) denotes the wave speed at which fixed points on the invariant line \(\{u = \alpha_1\}\) of system \((3.11)\) bifurcate, cf. \((5.39)\); it is the supremum over all \(c \leq \bar{c}\) with \(K \neq K_\alpha\) such that \(\psi\) is still continuous; \(c_2\) is the supremum over all \(c \leq \bar{c}\) such that \(\psi\) is still continuous in the case that \(K = K_\alpha\), cf. \((5.46)\); \(c_0\) is the infimum over all \(c \leq \bar{c}\) such that solution candidates, which satisfy \([TW1], [TW2]\) and contain cusped-type singularities at the value \(u = \alpha_1\), do also satisfy \([TW3]\) cf. \((5.50)\); it holds that \(c_0 < c_1 < c_2 < \bar{c}\), and for fixed \(c < \bar{c}\) we have that \(K_0 < K_1 < K_\alpha_1 < K_{\alpha_2}\).

Our first theorem classifies all elementary traveling waves of \((1.1)\), that is, traveling waves constructed from orbits corresponding to a single energy level \(h\), cf. Definition 5.6

**Theorem 6.1** (Elementary waves). Every elementary traveling wave \(u\) of \((1.1)\) belongs to one of the following types:

1. Smooth periodic. They appear if and only if the parameters \(c, K\) and \(h\) satisfy one of the following relations:
   
   (a) \(c > \bar{c}\), and \(K > K_0(c), \psi(u_2) < h < \psi(u_1)\) [Fig. 5]
   
   (b) \(c = \bar{c}\), and \(K_0(c) < K < K_\alpha(c), \psi(u_2) < h < \psi(u_1)\) [Fig. 19a]
   
   (c) \(c < \bar{c}\), and \(K > K_\alpha(c), \psi(u_2) < h < 0\) [Fig. 20]
   
   (d) \(c < \bar{c}\), and \(K_0(c) < K < K_{\alpha_1}(c), \psi(u_2) < h < \psi(u_1)\) [Fig. 8]
2. Smooth solitary. They appear if and only if the parameters $c$, $K$, and $h$ satisfy one of the following relations.

(a) $c > c$, and $K > K_0(c)$, $h = \psi(u_1)$ [Fig. 5]
(b) $c_1 \leq c \leq c$, and $K_0(c) < K < K_1(c)$, $h = \psi(u_1)$ [Fig. 8c, 19a]
(c) $c < c_1$, and $K_0(c) < K < K_1(c)$, $h = \psi(u_1)$ [Fig. 8a]
(d) $c_1 < c \leq c$, and $K = K_1(c)$, for $h = \psi(1)$ [Fig. 10c, 10d]

3. Peaked solitary. They appear if and only if the parameters $c$, $K$, and $h$ satisfy one of the following relations.

(a) $c = c$, and $K > K_1$, $h = \psi(u_1)$ [Fig. 20]
(b) $c_1 < c < c$, and $K > K_1(c)$, $h = \psi(u_1)$ [Fig. 12b, 15b, 16b]
(c) $c = c_1$, $K = K_1(c)$ and $h = \psi(u_1)$ [Fig. 10a]
(d) $c < c_1$, and $K = K_1(c)$, $h = \psi(u_1)$ [Fig. 8a]

4. Anti-peaked solitary. They appear if and only if the parameters $c$, $K$, and $h$ satisfy $c_1 < c < c$, and $K_1(c) < K < K_2(c)$, $h = \psi(u_2)$ [Fig. 12b].

5. Peaked periodic. They appear if and only if the parameters $c$, $K$, and $h$ satisfy one of the following relations.

(a) $c = c$, and $K > K_1$, $h < \psi(u_1)$ [Fig. 20]
(b) $c_1 < c < c$, and $K > K_1(c)$, $h < \psi(u_1)$ [Fig. 12b, 15b, 16b]
(c) $c = c_1$, and $K = K_1(c)$, $\psi(1) < h < \psi(u_1)$ [Fig. 10b]
(d) $c < c_1$, and $K = K_1(c)$, $h = \psi(u_1)$ [Fig. 8c]

6. Anti-peaked periodic. They appear if and only if the parameters $c$, $K$, and $h$ satisfy one of the following relations.

(a) $c_1 < c < c$, and $K_1(c) < K < K_2(c)$, $h > \psi(u_2)$ [Fig. 12b]
(b) $c_1 < c < c$, and $K \geq K_2(c)$, $h > \psi(2)$ [Fig. 15b, 16b]
(c) $c < c$, and $K > K_2(c)$, $h = \psi(2)$ [Fig. 16b]

7. Cusped solitary. They appear if and only if the parameters $c$, $K$, and $h$ satisfy one of the following relations.

(a) $c_0 < c < c_1$, and $K_1(c) < K \leq K_1(c)$, $h = \psi(u_1)$
(b) $c_0 < c \leq c_1$, and $K = K_1(c)$, $h = \psi(u_1)$

8. Cusped periodic. They appear if and only if the parameters $c$, $K$, and $h$ satisfy one of the following relations.

(a) $c_0 < c < c_1$, and $K_1(c) < K \leq K_1(c)$, $\psi(1) < h < \psi(u_1)$ [Fig. 8c, 10a]

---

2These traveling waves are smooth in the sense that they belong to $C^1(\mathbb{R})$. 
The wave profiles of periodic elementary waves have exactly one maximum and one minimum per period, while the wave profiles of solitary elementary waves have a unique maximum or minimum. The smooth, peaked and cusped solitary waves decay exponentially to the constant \( u = u_1 \) at infinity, while the anti-peaked and anti-cusped solitary waves tend exponentially to the constant \( u = u_2 \). The cusps always have the value \( u = \alpha_1 \) while the peaks take the value \( u = \alpha_2 \). Anti-peaks take either the value \( u = \alpha_1 \) or \( \alpha_2 \).

We now emphasize certain types of composite waves of special interest. The next theorem deals with wavefronts, which are composite waves consisting of two components: one half of a solitary peak and one half of a solitary anti-peak, or vice versa. Steep wavefronts consist of a solitary cusp component and a solitary anti-cusp component, cf. Fig. 12. Moreover, we consider "fast" wavefronts, which attain their maximum at a finite value of the moving frame variable, by combining a solitary cusp or peak component with a solitary anti-cusp or anti-peak component, cf. Fig. 15.

**Theorem 6.2 (Wavefronts).** The appearance of wavefronts is characterized as follows:

1. Wavefronts occur if and only if \( c_1 < c < \bar{c} \) and \( K_{\alpha_1}(c) < K < K_{\alpha_2}(c) \) for \( h_i = \psi(u_i), \ i = 1, 2 \). [Fig. 12b].

2. Steep wavefronts occur if and only if \( c_0 < c \leq c_1 \) and \( K_{\alpha_1}(c) < K < K_{\alpha_2}(c) \) for \( h_i = \psi(u_i), \ i = 1, 2 \). [Fig. 12a].

3. Fast wavefronts occur if and only if \( c_1 < c < \bar{c} \) and \( K = K_{\alpha_2}(c) \) for \( h_i = \psi(u_i), \ i = 1, 2 \). [Fig. 15b].

4. Fast steep wavefronts occur if and only if \( c_0 < c \leq c_1 \) and \( K = K_{\alpha_2}(c) \) for \( h_i = \psi(u_i), \ i = 1, 2 \). [Fig. 15a].
The profile of wavefronts (i) and (ii) tends to its maximal value \( u = u_2 \) and to its minimal value \( u = u_1 \) as \( s \to \pm \infty \), while fast wavefronts attain their maximal value \( u = \alpha_2 \) at a finite value of the moving frame variable \( s \). The slope of steep wavefronts becomes infinite precisely at the value \( u = \alpha_1 \), while the slope of a wavefront remains bounded everywhere.

Our last theorem classifies composite waves with constant components of finite or infinite length. In particular, there exist waves with plateaus and so-called compactons, which are solitary waves with compact support.

**Theorem 6.3** (Compactons, plateau waves, multicrests, multipeaks). Traveling waves of \((1.1)\) involving constant components can occur if \( c_0 < c \leq \bar{c} \) and either \( K = K_{\alpha_1}(c) \) or \( K = K_{\alpha_2}(c) \).

If \( K = K_{\alpha_1}(c) \), Scenario IV, Fig. 10, the following composite waves are possible.

1. Combinations of cusp-components and plateaus at height \( u = \alpha_1 \) for \( c_0 < c < c_1 \).
2. Combinations of peak-components and plateaus at height \( u = \alpha_1 \) for \( c = c_1 \).
3. Waves \( u \) with plateaus at height \( u = \alpha_1 \) and vanishing classical derivative on the preimage \( u^{-1}(\{\alpha_1\}) \) for \( c_1 < c \leq \bar{c} \).

In particular, there exist smooth and peaked multi-crested solutions with decay, cf. Fig. 11c and 11d, and waves with plateaus at height \( \alpha_1 \), cf. Fig. 11a and 11b. Moreover, there exist smooth and non-smooth anti-compactons, i.e. solitary waves of depression with compact support in the sense that they are constant equal to \( \alpha_1 \) outside a finite interval, cf. Fig. 11e.

If \( K = K_{\alpha_2}(c) \), Scenario VI, Fig. 15, the following composite waves are possible:

1. Combinations of anti-cusp-components and plateaus at height \( u = \alpha_2 \) for \( c_0 < c \leq c_1 \).
2. Combinations of anti-peak-components and plateaus at height \( u = \alpha_2 \) for \( c_1 < c < \bar{c} \).

In particular, there exist anti-cuspcapactons and anti-peakompactons, i.e. solitary cusped and peaked waves of depression with compact support, cf. Fig. 11a.

Note that the classification of elementary traveling waves in Theorem 6.1 is exhaustive, while Theorems 6.2 and 6.3 merely highlight certain composite waves of special interest.

### 7 Discussion and Outlook

In this paper we have studied traveling wave solutions of a highly nonlinear model equation for shallow water waves of large amplitude. Driven by the quest for new kinds of traveling waves which are not described by present day shallow water models, we resort to equation \((1.1)\) which is a natural extension to moderate amplitude models such as the CH equation or the corresponding equation for the free surface. The structure of the equation’s higher order nonlinearities is responsible for the fact that the corresponding ordinary differential equation \((2.5)\), which governs the traveling wave solutions of \((1.1)\), becomes singular in potentially two points. This loss of uniqueness allows one to construct singular traveling wave solutions in \( H^1_{\text{loc}}(\mathbb{R}) \) by combining various components of solution curves of the associated integrable planar system \((3.8)\) corresponding to different level sets of the first integral. These components are smooth solutions of \((3.8)\) defined on
a (possibly bounded) subinterval of the real line. By gluing them together one finds traveling wave solutions, which are bounded and globally absolutely continuous with singularities at points where values $\alpha_i$, $i \in \{1, 2\}$ – the roots of the quadratic polynomial $g(u)$ in (2.5) – are taken. The degree of this polynomial is two as a consequence of the presence of the third order cubic term in (1.1). This gives rise to entirely new types of traveling waves, whose wave profiles may exhibit singularities at two different heights. In particular, one obtains traveling waves where peaks and cusps form at both the wave crest and trough. These novel traveling wave solutions cannot be described using CH type equations [26, 27, 15] since they lack a second singularity due to the absence of higher nonlinearities in higher order order terms. Another novelty is that we can construct traveling waves involving non-symmetric peaks, whose slope differs from on either side of the crest or trough to the other. Moreover, recall that the CH solitary waves are monotone from crest to trough with a unique maximum. In contrast, here we have solitary waves whose profiles are non-monotonic as well as multi-crested peaked or smooth. Furthermore, we obtain negative smooth as well as anti-peaked and anti-cusped solitary waves with compact support, that is, the wave profile attains a constant value outside a finite interval. For comparison notice that it was shown to be impossible for moderate amplitude models to have peaks with compact support in [17]. Finally let us assert that we also recover all known smooth and singular traveling wave solutions of CH type equations.

It is known that the traveling wave solutions of models for moderate amplitude waves are symmetric with respect to their crest or trough as long as the solution conserves the energy of the corresponding planar system, see the discussions in [14, 16]. This is true for equation (1.1) as well. Whether all symmetric solutions of (1.1) are necessarily traveling waves will be studied in a subsequent paper.

Regarding stability, we remark that smooth solitary as well as periodic traveling wave solutions of several shallow water equations have been shown to be orbitally stable, i.e. they are stable under small perturbations, which makes them physically detectable, see for example [1, 9, 10, 13, 25]. Moreover, singular traveling waves involving peaks, for instance the CH and DP peakons, are also known to be orbitally stable, cf. [8, 23, 24, 28]. This naturally raises the question whether traveling waves of (1.1) with peaked crests and troughs are stable in that sense as well. We expect the investigation of these issues to be quite involved since we are not aware of a Hamiltonian formulation for (1.1), and therefore the methods put forth to prove the aforementioned stability results in the moderate amplitude regime are not applicable here, cf. Remark [2, 3].

Finally, we observe that it is possible to construct bounded continuous functions satisfying the properties (TW1) and (TW2) but not (TW3) of the characterization of traveling wave solutions given in Proposition 4.1. This can be achieved by using orbits indicated by dashed lines in the sketches of the phase portraits in Section 5. Such functions can not be interpreted as traveling wave solutions of (1.1) in the sense of Definition 2.1. In a forthcoming paper, we will investigate in which sense these functions can still be regarded as traveling solutions of (1.1) in Sobolev spaces $H^r_{loc}(\mathbb{R})$ for certain suitable indices $r < 1$.

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References


On irrotational flows beneath periodic traveling equatorial waves

Ronald Quirchmayr


Abstract

We discuss some aspects of the velocity field and particle trajectories beneath periodic traveling equatorial surface waves over a flat bed in a flow with uniform underlying currents. The system under study consists of the governing equations for equatorial ocean waves within a non-inertial frame of reference, where Euler’s equation of motion has to be suitably adjusted, in order to account for the influence of the earth’s rotation.

1 Introduction

Irrotational flows of symmetric periodic traveling water waves over a flat bed are well-studied; from a theoretical point of view [2, 10, 19], numerically [11, 1, 9] and experimentally [23].

This paper deals with some qualitative properties of certain geophysical waves, which are not governed by Euler’s equations of motion (which apply for inertial systems), but by suitable extensions that account for the influence of the earth’s rotation. More precisely we study irrotational flows and particle paths beneath symmetric periodic traveling surface waves in regions close to the equator. In the case of equatorial surface waves, which propagate practically unidirectionally in the East-West direction due to the prevailing wind pattern (known as trade winds), it is justifiable to consider the $f$-plane approximation for two-dimensional flows instead of the full geophysical governing equations in three dimensions. This reduction and the simplifying assumption of irrationality, implying that underlying currents are uniform, make it possible to analyze qualitative properties of the flow with the aid of well-known tools from complex and harmonic analysis, and conformal mapping theory. The qualitative techniques developed in [2, 10] to study Stokes waves can be adapted for the investigation of irrotational equatorial waves.

For recent studies of equatorial waves under the influence of the Coriolis effect we refer to [3, 4, 7, 13, 14, 16, 18, 22, 20] and the references therein. In particular, the influence of the Coriolis force on the dispersion relation is brought to light in the papers [3, 13] dealing with surface waves; we refer moreover to [5] for effects on internal waves, and [17, 21] for edge waves.

The outline of the paper is the following. First we introduce the governing equations for equatorial waves in Section 2. After transforming this system to moving frame coordinates, we provide two alternative reformulations of the problem: a stream function formulation and its transformation on a strip via a conformal hodograph mapping.

\footnote{We refer to the discussion in [7, 12].}

\footnote{While non-uniform currents are prevalent in the equatorial ocean, these are typically subsurface currents. We treat gravity waves that are relevant for mean-surface flows.}
Section 3 is devoted to the study of the velocity field beneath a surface wave, which provides the basis for the qualitative description of particle trajectories in Section 4.

2 The governing equations for equatorial waves

The governing equations for gravity water waves in the equatorial $f$-plane are given by

\[
\begin{align*}
U_T + UU_X + VU_Y + 2\Omega V &= -\frac{1}{\rho} P_X, \\
V_T + UV_X + VV_Y - 2\Omega U &= -\frac{1}{\rho} P_Y - g, \\
U_X + V_Y &= 0.
\end{align*}
\] (2.1)

They hold in the domain $D_H(T) := \{(X,Y) \in \mathbb{R}^2: -d < Y < H(X,T)\}$, where $H = H(X,T)$ parametrizes the free surface, see Fig. 1. The velocity field is denoted by $(U,V) = (U(X,Y,T),V(X,Y,T))$, $P = P(X,Y,T)$ denotes the pressure, $\rho$ is the constant fluid density, $g \approx 9.81 \text{ m s}^{-2}$ denotes the gravitational acceleration and $\Omega \approx 7.29 \times 10^{-5} \text{ rad s}^{-1}$ denotes the rotational speed of the earth. The boundary conditions associated with (2.1) on the free surface and the flat bed are given by

\[
\begin{align*}
P &= P_{\text{atm}} \\
V &= H_T + UH_X \\
\end{align*}
\] on $Y = H(X,T),

where $P_{\text{atm}}$ is the constant atmospheric pressure, and

$V = 0$ on $Y = -d.$

We restrict our considerations on flows being irrotational at the initial time $T = 0$. The corresponding vorticity equation for such two-dimensional flows implies that the curl will remain zero for all times. We may therefore assume that $(U,V)$ is curl-free throughout the fluid domain:

\[U_Y = V_X.\] (2.2)

2.1 Moving frame re-formulation for traveling waves

The goal of this paper is the analysis of flows beneath traveling periodic surface waves. Let us therefore consider the corresponding moving frame coordinates for a given wave speed $c > 0$:

\[
\begin{align*}
x &:= X - cT, \quad y := Y \\
u(x,y) &:= U(X - cT,Y), \quad v(x,y) := V(x - cT,Y), \\
\eta(x) &:= H(X - cT), \quad P(x,y) := \frac{1}{\rho} P(X - cT,Y),
\end{align*}
\]

which transform (2.1) and (2.2) into

\[
\begin{align*}
(u - c)u_x + vu_y + 2\Omega v &= -P_x \\
(u - c)v_x + vv_y - 2\Omega u &= -P_y - g
\end{align*}
\] (2.3)

\[\text{The details are provided in the appendix.}\]
IRROTATIONAL FLOWS BENEATH EQUATORIAL WAVES

Figure 1: The fluid domain $D_H(T)$ beneath a periodic traveling surface wave of wavelength $L$ in the physical frame at some fixed time $T$.

\begin{align*}
  u_x + v_y &= 0 \
  u_y &= v_x 	ag{2.4}
\end{align*}

in the fluid domain $D_\eta := \{(x,y): -d < y < \eta(x)\}$. The corresponding boundary conditions on the free surface and on the flat bed are given by

\begin{align*}
  v &= (u - c)\eta_x \quad \text{on} \quad y = \eta(x), \tag{2.6} \\
  P &= \frac{1}{\rho}P_{\text{atm}} \quad \text{on} \quad y = \eta(x), \tag{2.7}
\end{align*}

and

\begin{equation}
  v = 0 \quad \text{on} \quad y = -d. \tag{2.8}
\end{equation}

We assume that $u, v, P$ and $\eta$ are periodic in the $x$-direction. The period $L > 0$ corresponds to the wavelength of the surface wave.

### 2.2 Stream function formulation

Equation (2.4) permits the definition of a stream function satisfying

\begin{align*}
  \psi_x &= -v, \quad \psi_y = u - c. \tag{2.9}
\end{align*}

We see that $\psi$ is unique up to an additive constant and observe that $\psi$ is constant on both parts of the boundary of $D_\eta$: on $y = -d$ due to (2.8) and also on the surface, since its derivative along the free surface $\eta$ is zero by (2.6):

\begin{equation}
  \partial_x(\psi(x, \eta(x)) = -v(x, \eta(x)) + (u(x, \eta(x)) - c)\eta_x(x) = 0. \tag{2.10}
\end{equation}

With the choice

\begin{equation}
  \psi = 0 \quad \text{on} \quad y = \eta(x) \tag{2.10}
\end{equation}

we obtain

\begin{equation}
  \psi(x, y) = m + \int_{-d}^{y} (u(x, s) - c) \, ds \quad \text{for} \quad -d \leq y \leq \eta(x). \tag{2.10}
\end{equation}
This formula tells us in particular, that $\psi$ inherits the $x$-periodicity from $u$. The constant $m$ is called relative mass flux; its value is given by

$$m = \int_{-d}^{\eta(x)} (c - u(x,y)) \, dy. \quad (2.11)$$

Indeed, $m$ is an invariant of the flow; differentiating the right hand side of (2.11) with respect to the $x$-variable and employing (2.4), (2.6) and (2.8) gives zero.

Due to (2.5) and (2.9) we obtain

$$\begin{align*}
&\frac{(u-c)u_x + vv_y}{(u-c)v_x + vv_y} = \frac{(u-c)u_y + vv_y}{(u-c)u_y + vv_y} = \frac{1}{2} \nabla \psi^2 + \frac{1}{2} \nabla \psi_x^2 \\
&\quad \text{and} \\
&-2\Omega \left( \frac{-v}{u} \right) = -2\Omega \nabla (\psi + cy).
\end{align*}$$

Therefore (2.3) yields Bernoulli’s law for irrotational equatorial flows

$$\nabla \left( \frac{\psi_x^2 + \psi_y^2}{2} - 2\Omega(\psi + cy) + P + g(y + d) \right) = 0;$$

in other words, the expression

$$\frac{\psi_x^2 + \psi_y^2}{2} - 2\Omega(\psi + cy) + P + g(y + d) \quad (2.12)$$

is constant throughout the fluid. By means of (2.7) and (2.10) we infer that

$$\frac{\psi_x^2 + \psi_y^2}{2g} + \left( 1 - \frac{2c\Omega}{g} \right) y + d = Q \quad \text{on} \quad y = \eta(x)$$

for some physical constant $Q$, called head. In order to ensure positivity of the coefficient of $y$ in the above relation, we impose the following realistic upper bound for the wave speed $c$:

$$c < \frac{g}{2\Omega}. \quad (2.13)$$

In summary we obtained the following reformulation of (2.3) - (2.8):

$$\begin{align*}
\Delta \psi &= 0 \quad \text{in} \quad -d < y < \eta(x) \\
\psi &= m \quad \text{on} \quad y = -d \\
\psi &= 0 \quad \text{on} \quad y = \eta(x) \\
\frac{\psi_x^2 + \psi_y^2}{2g} + \left( 1 - \frac{2c\Omega}{g} \right) y + d &= Q \quad \text{on} \quad y = \eta(x).
\end{align*} \quad (2.14)$$

Note that the pressure $P$ does not appear explicitly in (2.14). It can be recovered by means of (2.12) and (2.7):

$$P = gQ + \frac{1}{\rho} P_{\text{atm}} - \frac{[\nabla \psi]^2}{2} + 2\Omega \psi - g \left[ \left( 1 - \frac{2c\Omega}{g} \right) y + d \right].$$

Since we are interested in solutions different from the trivial solution, that would be $\psi \equiv 0$ (with $m = 0$) throughout the strip $D_0$, it follows from the strong maximum
principle applied to the harmonic function $\psi$, that $m \neq 0$. Without loss of generality we take $m > 0$, then $\psi > 0$ throughout $D_\eta$ and

$$\psi_y = u - c < 0 \quad \text{in} \quad D_\eta,$$  
(2.15)

because $\psi$ attains a minimum at every surface point $(x, \eta(x))$ and a maximum at every point $(x, -d)$ on the flat bed (by Hopf’s lemma, this strict inequality holds everywhere on the boundary $\partial D_\eta$ of the fluid domain $D_\eta$, therefore the strong maximum principle applied on the harmonic function $\psi_y$ implies (2.15)). The irrotationality condition (2.5) and $L$-periodicity imply that

$$0 = \int_{-d}^{y_0} \int_0^L (u_y - v_x) \, dx \, dy = \int_0^L u(x, y_0) \, dx - \int_0^L u(x, -d) \, dx$$  

at all depths $y_0$ below the trough level $\eta(L/2)$ by means of Green’s theorem. In other words: the mean of $u$ at any depth below the trough level takes the constant value $\kappa < c$ referred to as mean current:

$$\kappa := \frac{1}{L} \int_0^L u(x, -d) \, dx = \frac{1}{L} \int_0^L u(x, y_0) \, dx \quad \text{for all} \quad y_0 \in [-d, \eta(L/2)].$$  
(2.16)

We distinguish between three cases: if $\kappa$ is positive, there is a uniform underlying current moving with the wave, if $\kappa$ is negative, it moves against the wave and $\kappa = 0$ indicates the absence of an underlying current. In the latter case we have that:

$$\int_0^L u(x, -d) \, dx = 0.$$  
(2.17)

An immediate consequence of (2.16) is that

$$\frac{1}{L} \int_0^L (u(x, -d) - c) \, dx = \kappa - c.$$  
(2.18)

In the remaining sections we investigate flows beneath symmetric periodic traveling ocean waves. More precisely, we study qualitative properties of smooth periodic solutions $(\eta, \psi)$ of (2.14) (for a given wave speed $c > 0$, relative mass flux $m > 0$ and mean current $\kappa$), having period $L$ in the $x$-variable and mean level $y = 0$, i.e.

$$\int_0^L \eta(x) \, dx = 0.$$  

Moreover $\eta$ and $\psi$ are supposed to be symmetric about the crest line, i.e. the vertical line from $(0, \eta(0))$ to $(0, -d)$, where $\eta$ attains its maximum. Furthermore we assume that the surface wave $\eta$ has only one crest per period. Its wave profile is supposed to be strictly increasing from the trough at $x = -L/2$, where the minimum is attained, to the crest at $x = 0$. In terms of the velocity field $(u, v)$ and the pressure $P$, the symmetry of $\psi$ means that $u$ and $P$ are symmetric, whereas $v$ is anti-symmetric about the crest line. We refer the reader to [15] for the existence of solutions of (2.14) with these properties.

Let us note that the approach developed in [6] can be adapted to our setting for equatorial waves, yielding that the free surface is actually real-analytic.
2.3 Hodograph transform

We introduce new coordinates $q$ and $p$, which will turn out to be of great use in the further analysis.

Due to (2.5), there exists a potential $\phi$ for the velocity field $(u - c, v)$:

$$\phi_x = u - c, \quad \phi_y = v \quad \text{(2.19)}$$

We set

$$\phi(x, y) := \int_0^x (u(\xi, -d) - c) d\xi + \int_0^y v(x, s) ds \quad \text{(2.20)}$$

for $-d \leq y \leq \eta(x)$. Then $\phi$ satisfies (2.19) and $\phi(0, y) = 0$ for all $y \in [-d, \eta(0)]$ since $v$ is anti-symmetric about the crest line. Moreover, $\phi(L, y) = L(\kappa - c)$ for all $y$ below the trough level by (2.16), whereas $\phi(-L, y) = -L(\kappa - c)$. The map $(x, y) \mapsto \phi(x, y) + (c - \kappa)x$ is odd and periodic in the $x$-variable with period $L$; in particular $\phi(jL, y) = -(c - \kappa)jL$ for every integer $j$. Furthermore $\phi_x < 0$ by (2.15). Observe that

$$\phi(L/2, y_0) = -\lambda/2 \quad \text{and} \quad \phi(-L/2, y_0) = \lambda/2 \quad \text{(2.21)}$$

for all $y_0 \in [-d, \eta(L/2)]$, with $\lambda := L(c - \kappa)$, because the restrictions $u|_{(-L/2, 0)}$ and $u|_{(L/2, L)}$ coincide due to $L$-periodicity, hence

$$L\kappa = \int_0^L u(x, y_0) dx = \int_0^{L/2} u(x, y_0) dx + \int_{-L/2}^0 u(x, y_0) dx.$$

The two integrals on the right hand side are equal, since $u$ is even, which gives (2.21).

The stream function $\psi$ and the potential $\phi$ are harmonic conjugates: the mapping $x + iy \mapsto \phi(x, y) + i\psi(x, y)$ is holomorphic throughout the fluid domain. Let us consider the orientation preserving conformal hodograph transform

$$\begin{cases}
q = -\phi(x, y) \\
p = -\psi(x, y)
\end{cases} \quad \text{(2.22)}$$

which transforms the free boundary value problem (2.14) into a nonlinear boundary value problem for the harmonic function

$$h(q, p) = y + d \quad \text{(2.23)}$$

in a fixed strip; cf. Fig. 2. The corresponding system for $h$ reads

$$\begin{cases}
\Delta_{q, p} h = 0 \quad \text{in the strip} \\
h = 0 \quad \text{on} \\
-p = -m
\end{cases} \quad -m < p < 0$$

$$\begin{cases}
2g\left(\hat{Q} - \left(1 - \frac{2\Omega d}{g}\right)h\right)\left(h_q^2 + h_p^2\right) = 1 \quad \text{on} \\
p = 0
\end{cases} \quad \text{(2.24)}$$

where

$$\hat{Q} = Q - \frac{2\Omega d}{g}. \quad \text{(2.25)}$$

We will frequently make use of the following identities:

$$\partial_q = h_p \partial_x + h_q \partial_y$$

$$\partial_p = -h_q \partial_x + h_p \partial_y \quad \text{(2.26)}$$
Figure 2: The conformal hodograph transform maps the domain $D_\eta$, which is an unknown part of the solution, to the closed strip between $p = 0$ and $p = -m$ by inverting the dependent and independent variables. The price is paid by a higher nonlinearity in the corresponding boundary condition.

\begin{align}
\frac{\partial x}{\partial \psi} &= (c-u)\frac{\partial q}{\partial p} + v\frac{\partial p}{\partial q} \\
\frac{\partial y}{\partial \psi} &= -v\frac{\partial q}{\partial p} + (c-u)\frac{\partial p}{\partial q} \\
\frac{\partial y}{\partial \eta} &= -\frac{v}{(u-c)^2 + v^2} \\
\frac{\partial x}{\partial \eta} &= \frac{c-u}{(u-c)^2 + v^2}.
\end{align}

Note for instance, that we obtained the boundary condition in (2.24) by multiplying the corresponding condition in (2.14) by $2g\frac{v^2 + (c-u)^2}{(c-u)^2 + v^2}$ to get

\[1 = 2g \left[ Q - \left( 1 - \frac{2c\Omega}{g} \right) y + d \right] \left( \frac{v^2 + (c-u)^2}{(c-u)^2 + v^2} \right) \text{ on } y = \eta(x).\]

This is precisely the boundary condition in (2.24) in view of (2.28), (2.23), (2.22), (2.25) and the fact, that $y = \eta(x)$ translates into $p = 0$ due to the definition of $\psi$.

### 3 Properties of the velocity field

This section is dedicated to the study of basic properties of the velocity field $(u,v)$ of solutions $(\psi,\eta)$ to (2.14) as we introduced them in the last paragraph of Section 2.2. In particular we will determine the zero level sets of $u$ and $v$, where sign changes occur. This will permit a detailed qualitative analysis of fluid particle paths in the physical frame, see Section 4.
Due to periodicity we may restrict our considerations to the particular periodicity window
\[ D := \{(x, y) : x \in (-L/2, 0), -d < y < \eta(x)\}. \]

We denote by \( D \) the closure of \( D \) in \( \mathbb{R}^2 \); its left and right boundary\(^4\) are referred to as trough lines. The crest line is the intersection of \( D \) with the vertical line \( \{x = 0\} \). In view of the symmetry of \( \psi \) it is convenient to distinguish between the right half \( D_+ \) of \( D \) and its left half \( D_- \):

\[ D_+ := \{(x, y) : x \in (0, L/2), -d < y < \eta(x)\} \]
\[ D_- := \{(x, y) : x \in (-L/2, 0), -d < y < \eta(x)\}. \]

Let us furthermore denote by
\[ S_+ := \{(x, y) : x \in (0, L/2), y = \eta(x)\} \]
\[ S_- := \{(x, y) : x \in (-L/2, 0), y = \eta(x)\} \]

and
\[ B_+ := \{(x, y) : x \in (0, L/2), y = -d\} \]
\[ B_- := \{(x, y) : x \in (-L/2, 0), y = -d\}. \]

the parts of the free surface and the flat bed, which correspond to \( D_+ \) and \( D_- \) respectively. We denote the images of \( D_- \) and \( D_+ \) under the conformal transformation of variables \((2.22)\) by \( \hat{D}_- \) and \( \hat{D}_+ \):

\[ \hat{D}_- := \{(q, p) : -\lambda/2 < q < 0, -m < p < 0\} \]
\[ \hat{D}_+ := \{(q, p) : 0 < q < \lambda/2, -m < p < 0\}. \]

We start our analysis by determining the sign of the vertical velocity component \( v \) in \( D \), cf. Fig. 3.

**Proposition 3.1.** The vertical velocity component \( v \) is strictly positive in \( D_+ \); it is strictly negative in \( D_- \) and vanishes on the crest and trough lines.

**Proof.** Recall that \( v \) is harmonic in the entire fluid domain; this is ensured by \((2.4)\) and \((2.5)\). Due to \((2.6)\) and \((2.15)\) we have that \( v(x, \eta(x)) > 0 \) for all \( x \in (0, L/2) \), since \( \eta'(x) < 0 \) on this interval by assumption, cf. the last paragraph of Section 2.2. Recall that \( v = 0 \) on the bottom. The strong maximum principle implies that \( v > 0 \) in \( D_+ \), and since \( v \) is anti-symmetric we infer that \( v < 0 \) in \( D_- \). By continuity we have that \( v = 0 \) on the crest line \( x = 0 \) and also on the trough lines \( x \pm L/2 \) in view of periodicity.

As a consequence we obtain the following corollary, which tells us that all streamlines except the flat bed \( y = -d \) replicate the shape of the free surface.

**Corollary 3.2.** All streamlines \( \{(\psi = p) \cap D \} \) with \( p \in (0, m) \), are analytic, symmetric and strictly decreasing between \( x = 0 \) and \( x = L/2 \).
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Figure 3: The vertical component $v$ of the velocity field is negative in the left half $D_-$ of the periodicity window $D$, it is positive within the right half $D_+$ and vanishes on the crest line and both trough lines.

Proof. Recall that $\psi$ has no critical points due to (2.15). Let us identify an arbitrary streamline in $D$ with the function

$$y: (-L/2, L/2) \to \mathbb{R}, \ x \mapsto y(x).$$

It is clear, that $y$ is analytic, since it is a parametrization of a level set of the harmonic map $\psi$, i.e. $\psi(x, y(x))$ is constant for all $x \in (-L/2, L/2)$. Therefore we have that

$$0 = \partial_x (\psi(x, y(x))) = \psi_x(x, y(x)) + \psi_y(x, y(x)) y_x$$

and hence

$$y_x(x) = \frac{v(x, y(x))}{u(x, y(x))}.$$  

Thus we see, in view of the sign of $v$ together with (2.15), that all streamlines within $D$ are symmetric and strictly decreasing between the wave crest and the wave trough. \Box

Remark 3.3. Let $y$ be a streamline in $D$. The continuous periodic continuation $\tilde{y}: \mathbb{R} \to \mathbb{R}$ to the entire fluid domain attains its maxima at crests and its minima at troughs, since $\tilde{y}_x$ vanishes precisely on the crest and trough lines.

The next proposition reveals the monotonicity properties of the horizontal velocity component $u$ along streamlines, the crest line and trough lines, cf. Fig. 4.

Proposition 3.4. The horizontal velocity component $u$ is strictly decreasing along streamlines in $D_+ \cup S_+ \cup B_+$ and strictly increasing along streamlines in $D_- \cup S_- \cup B_-$; in particular $u_x = 0$ on the crest and trough lines. On the crest line it holds that $u_y > 0$, whereas $u_y < 0$ along the trough lines.

Proof. Let us first observe that the pressure $P$ is superharmonic throughout the entire fluid region, in particular within $D$. By (2.3) and (2.5) we have that

$$P_{xx} = -u_x^2 - uu_{xx} + cu_{xx} - v_x u_y - vu_{yx} - 2\Omega v_x$$

in view of Corollary 3.2
\[ P_{yy} = -u_y v_x - u u_{xy} + c u v_x - v^2 y - v v_{yy} + 2 \Omega u_y \]
\[ = -u_y^2 - u u_{yy} + c u_{yy} - u_x^2 - v v_{yy} + 2 \Omega u_y, \]
which sums up to
\[ \Delta P = P_{xx} + P_{yy} \]
\[ = -(v_x^2 + u_y^2) - 2 u_x^2 - v \Delta v - u \Delta u + c \Delta u + 2 \Omega (u_y - v_x). \]
Therefore we obtain that
\[ \Delta P = -(\psi_{xx}^2 + \psi_{yy}^2) - 2 \psi_{xy}^2 \leq 0, \]
since both \( u \) and \( v \) are harmonic in the whole fluid domain by (2.4) and (2.5).

Let us now consider the function \( Q \) defined on the entire fluid domain and given by
\[ Q(x, y) := P(x, y) - 2 \Omega \psi(x, y). \]
The map \( Q \) is superharmonic as a sum of the superharmonic function \( P \) and a multiple of the harmonic function \( \psi \). Therefore we have that the minimum of \( Q \) is attained on the boundary of the fluid domain.

On the bottom we find that
\[ Q_y = P_y - 2 \Omega u = -g + 2(\Omega - \Omega)u = -g < 0, \]
because of (2.8) and the second component of (2.3). Hence \( Q \) decreases in \( y \)-direction on the bottom, therefore the minimum can only be attained on the surface, where the maximum principle tells us that the minimum of \( Q \) is attained on the boundary of the fluid domain.

On the flat bed we have that
\[ u_q(q, -m) = \frac{u_x(x, -d)}{c - u(x, -d)} = -\frac{v_y(x, -d)}{c - u(x, -d)}. \]
Hopf’s maximum principle yields that $v_y(x, -d) > 0$ for $x \in (0, L/2)$, since $v$ is minimal along the bottom $B_+$, and therefore we infer that

$$u_q(q, -m) < 0 \quad \text{for} \quad q \in (0, \lambda/2).$$

In summary we have that $u_q \leq 0$ all along the boundary of $\hat{D}_+$; the inequality being strict on the upper and lower boundary. The harmonicity of $u$ is preserved under (2.22), and so we can apply the strong maximum principle in order to find that

$$u_q < 0 \quad \text{throughout} \quad \hat{D}_+. \quad (3.30)$$

Since $u$ is even, we obtain that $u_q > 0$ in $D_-$. Thus we achieved the desired monotonicity along horizontal lines in the conformal frame, which correspond to streamlines in the moving frame. The transformation to $(x, y)$-coordinates yields the claim: we know from the proof of Corollary 3.2 and (2.29) that the derivatives of parameterizations $y = y(x)$ of streamlines satisfy

$$\frac{dy}{dx} = \frac{v}{u - c} = \frac{h_q}{h_p},$$

hence (2.28) yields

$$\frac{\partial}{\partial x} [u(x, y(x))] = u_x + \frac{dy}{dx} u_y = u_x + \frac{h_q}{h_p} u_y = 1 \frac{h_p}{h_p} u_q,$$

which determines the claimed monotonicity of $u$ along streamlines, since $h_p > 0$ due to (2.28) and (2.15).

In order to establish the claimed monotonicity along the crest and trough lines we recall first that $u_y = (c - u)u_q$ on these lines because of (2.27) and the fact that $v$ vanishes there due to Proposition 3.1. We recall that $u_p = u_q$ as a consequence of (2.4), (2.5) and (2.26). We recall from Proposition 3.1 that the harmonic function $v$ is positive in $D_+$ and vanishes on the crest line and the trough lines, hence these points are minimizers of $v$ in $\overline{D}_+$. Therefore Hopf’s lemma tells us that $v_x(0, y) > 0$ for $y \in (-d, \eta(0))$ and $v_x(L/2, y) < 0$ for $y \in (-d, \eta(L/2))$.

The claim follows since $u_p = h_p v_x$ on the images of the crest and trough lines under the conformal mapping (2.22) and due to (2.15). \qed

**Corollary 3.5.** The horizontal velocity component $u$ attains its maximum at the wave crest $(0, \eta(0))$ and it is minimal at the wave troughs $(\pm L/2, \eta(\pm L/2))$.

**Proof.** The maximum of $u$ can only be attained somewhere along the crest line and its minimum has to be found on the trough lines, since $u_x < 0$ throughout the region

$$D_+ \cup S_+ \cup B_+$$

and $u_x > 0$ in

$$D_- \cup S_- B_-.$$

Due to Proposition 3.4 we know that $u$ increases in $y$-direction on the crest line and decreases in $y$-direction on the trough lines, which proves the assertion of the corollary. \qed
Corollary 3.6. The slopes of streamlines decrease in absolute value along vertical lines from the surface to the bottom.

Proof. Recall first, that the slope of any streamline vanishes at crest and trough lines; see Remark 3.3. Let the functions $y: (-L/2, 0) \to \mathbb{R}, x \mapsto y(x)$ be the parameterizations of the restrictions of all streamlines to the left periodicity window $\mathcal{D}_-$. We want to compare their slopes $y'(x)$ at some fixed point $x \in (-L/2, 0)$.

$$
\frac{\partial}{\partial y} \left[ \frac{v(x, y)}{(u(x, y) - c)} \right] = \frac{v_y(x, y)(u(x, y) - c) - u_y(x, y)v(x, y)}{(u(x, y) - c)^2} = \frac{u_x(x, y)(c - u(x, y)) - u_y(x, y)v(x, y)}{(u(x, y) - c)^2} > 0,
$$

because $u_q(q, p) = \frac{u_x(x, y)(c - u(x, y)) - u_y(x, y)v(x, y)}{(c - u(x, y))^2 + v^2} > 0$ by Proposition 3.4. This proves the assertion for vertical lines intersecting $x \in (-L/2, 0)$, since the slopes of all streamlines are positive at such $x$, see Corollary 3.2. Likewise, $\frac{\partial}{\partial y} \left[ \frac{v(x, y)}{(u(x, y) - c)} \right] < 0$ for $x \in (0, L/2)$, where all slopes are negative.

The monotonicity properties of the horizontal velocity component $u$ allows a qualitative description of its zero-set $\{u = 0\}$. We discuss the special case without an underlying current in full detail in Proposition 3.7, cf. the first image in Fig. 5. The general cases can be deduced from this case, see Remark 3.8.

Proposition 3.7. If $\kappa = 0$, the the zero-level-set of $u$ in $\mathcal{D}$ consists of a smooth curve $C_+$ in $D_+$ and a smooth curve $C_-$ in $D_-$. Both $C_+$ and $C_-$ connect the respective parts $B_+$ and $B_-$ of the flat bed with the corresponding surface parts $S_+$ and $S_-$ and intersect...
each streamline exactly at one point. They separate $\mathcal{D}$ into the three regions where $u$ has a sign: $u$ is strictly positive in the set between $\mathcal{C}_+$ and $\mathcal{C}_-$ and strictly negative in the corresponding complement with respect to $\overline{\mathcal{D}}$.

**Proof.** Recall that (2.16) implies, cf. (2.21), that

$$
\int_0^{L/2} u(x, -d) \, dx = 0.
$$

Since $u_x(x, -d) < 0$ for all $x \in (0, L/2)$, there exists a unique $x_0 \in (0, L/2)$ such that $u(x_0, -d) = 0$. This implies the existence of a unique $q_0 \in (0, \lambda/2)$ such that $u(q_0, -m) = 0$. Strict monotonicity of $u$ on the bottom between 0 and $\lambda/2$ implies that $u(q, -m) > 0$ for $q \in [0, q_0)$ and $u(q, -m) < 0$ for $q \in (q_0, \lambda/2]$. By combining the inequality $u(0, -m) > 0 > u(\lambda/2, -m)$ with the monotonicity along the crest and trough lines, see Proposition 3.4, we obtain

$$
u(0, p) > 0 > u(\lambda/2, p) \quad \text{for} \quad p \in [-m, 0].
$$

From (3.30) and (3.31) we infer that $u$ vanishes exactly once along the intersections of horizontal lines with $D_+$. These line segments correspond to the intersections of level sets of $\psi$ with $D_+$ which represent (parts of) the streamlines of the velocity field in the moving frame. The corresponding pictures for the domain $\overline{D_-}$ and $D_-$ are obtained by reflecting $\overline{D_+}$ and $D_+$ at the vertical lines $\{q = 0\}$ and $\{x = 0\}$ respectively. Let us denote by $(\hat{\alpha}(p), p), p \in (-m, 0]$ the path of points in $\overline{D_+}$ where $u$ vanishes. Due to (3.30) we may apply the implicit function theorem at every point $(\hat{\alpha}(p), p)$ and can thereby reconstruct the map $p \mapsto \hat{\alpha}(p)$ globally as a smooth curve $\hat{\mathcal{C}}_+$. The pre-image of this curve under the coordinate transform (2.22) is a smooth curve $\hat{\mathcal{C}}_+$ in $D_+$, parametrized by $(\alpha(y), y)$, that connects the flat bed $y = -d$ with the free surface $y = \eta(x)$ by intersecting each streamline exactly once.

**Remark 3.8** (Sign of $u$ in flows with an underlying current). This case can be traced back to the case $\kappa = 0$: let $\hat{(\psi, \eta)}$ be a symmetric $L$-periodic solution to (2.14) representing the wave speed $c > 0$, relative mass flux $m > 0$ and mean current $\kappa$. We may identify $(\psi, \eta)$ with $(\psi_0, \eta_0)$ which represents an $L$-periodic solution for the wave speed $c_0 = c - \kappa > 0$ without the presence of an underlying current; namely via taking $u_0 = u - \kappa$, $v_0 = v$, $\eta_0 = \eta$ (and $\psi_0 = \psi$, since $u_0 - c_0 = u - c$). Then $u_0 < c_0$ is guaranteed. We distinguish between the cases $\kappa > 0$ and $\kappa < 0$, which are divided into three sub-cases:

1. $\kappa > 0$.

   - (a) $\kappa \leq -u_0(L/2, -d).$ This situation matches qualitatively the case $\kappa = 0$: there is exactly one sign change of $u$ somewhere on the flat bed. In the borderline case $u(L/2, -d) = 0$, the set $\{u = 0\}$ connects the point $(L/2, -d)$ with some surface point $(x_0, \eta(x_0))$, where $0 < x_0 < L/2$.

   - (b) $-u_0(L/2, -d) < \kappa < -u_0(L/2, \eta(L/2))$. Here, the curve $\{u = 0\}$, connects a point $(L/2, y_1)$ on the trough line with some surface point $(x_1, \eta(x_1))$, $0 < x_1 < L/2$.

---

6We describe the situation for $D_+$; the picture for $D_-$ is obtained by reflection about the $y$-axis. The observations rely on the strict monotonicity of $u$ along streamlines in $D_+ \cup S_+ \cup \mathcal{B}_+$ and its monotonicity in $y$-direction along the crest and trough line, cf. Proposition 3.4 and the proof of Proposition 3.7.
Figure 5: The dashed curves are the zero sets of $u$, which separate $\overline{D}$ into parts of positive and negative sign. The image on the top illustrates the case without or with a weak favorable or weak adverse current, the second picture shows the scenario of a moderate favorable current and the last image explains the situation of a moderate adverse current.
(c) $\kappa \geq -u_0(L/2, \eta(L/2))$. In this case, $u$ is nonnegative throughout $D_+$, i.e. no change of sign occurs.

2. $\kappa < 0$

(a) $-u_0(0, -d) \leq \kappa$. As in (1a), the situation matches qualitatively the case $\kappa = 0$.

(b) $-u_0(0, \eta(0)) < \kappa < -u_0(0, -d)$. The curve $\{u = 0\}$ connects a point $(0, y_2)$ on the crest line with some surface point $(x_2, \eta(x_2))$, $0 < x_2 < L/2$.

(c) $-g/2 - c < \kappa \leq -u_0(0, \eta(0))$. Similar as in (1c), no change of sign occurs: $u \leq 0$ throughout $D_+$.

4 The particle path pattern

In this section we study the trajectory $t \mapsto (X(t), Y(t))$ of a fluid particle initially located at $(X_0, Y_0)$, which satisfies the system

$$
\begin{align*}
X' &= U(X - cT, Y) \\
Y' &= V(X - cT, Y)
\end{align*}
$$

with initial condition $(X(0), Y(0)) = (X_0, Y_0)$. In the moving frame the above time dependent problem is transformed to the autonomous system

$$
\begin{align*}
x' &= u(x, y) - c \\
y' &= v(x, y)
\end{align*}
$$

with initial data $(x(0), y(0)) = (X_0, Y_0)$. We note that $\psi$ is a Hamiltonian for this system, i.e. $\psi$ satisfies

$$
\begin{align*}
x' &= \partial_y \psi \\
y' &= \partial_x \psi
\end{align*}
$$

and moreover $\psi$ is an integral of motion,

$$
\psi(x(t), y(t)) = \psi(X_0, Y_0) \quad \text{for all } t \in \mathbb{R},
$$

since it does not depend on time.

Let us assume, that the fluid particle is initially (at time $t = 0$) located at the point $(L/2, Y_0)$ and observe that this is no restriction of the general case, since any fluid particle would cross the vertical line $\{x = L/2\}$ after some finite time in view of the moving frame. Indeed, due to (2.13) and (4.33) there exists a $\delta > 0$ such that $x' = u(x, y) - c \leq -\delta < 0$ for all $(x, y)$ in the entire fluid domain, and this means that $x$ runs from $\infty$ to $-\infty$ when $t$ goes from $-\infty$ to $\infty$. Since $u$ is periodic in $x$, there exists a uniquely determined positive time (depending only on the initial height $Y_0$) it takes a fluid particle to traverse one periodicity window of length $L$:

**Definition 4.1.** The time $\tau = \tau(Y_0) > 0$ which satisfies $x(\tau) = -L/2$ is called the **elapsed time**.

\footnote{Note that we require $-\kappa < g/2 - c$ to ensure that $c_0$ satisfies (2.13).}
Remark 4.2 (Viewpoint q,p-coordinates). Let us denote by \((q(t), p(t))\) the images of points \((x(t), y(t))\) under the conformal transformation of variables \((2.22)\), and recall that 

\[
\frac{dq}{dt} = \frac{d\phi}{dt} = -\phi_x x' - \phi_y y' = -(c - u)^2 - v^2 \leq \delta^2
\]

by \((2.19)\) and \((4.33)\). So \((q(t), p(t))\) moves along straight horizontal lines from right to left in the \((q, p)\)-plane. This shows (by keeping in mind that \(q(0) = \lambda/2\)) that there exists a unique time \(\theta = \theta(p)\) such that \(q(\theta) = -\lambda/2\). By construction we have that \(\theta\) coincides with the elapsed time: \(\theta(p) = \tau(Y_0)\) and consistently we obtain

\[
y(\tau) - y(0) = \int_0^\tau v(x(t), y(t)) \, dt = \int_0^\theta p'(t) \, dt = \int_{-\lambda/2}^{\lambda/2} -h_q(q, p) \, dq = \int_{-\lambda/2}^{\lambda/2} v \left( \frac{(c - u)^2 + v^2}{2} \right) \, dq = 0,
\]

where the last equality holds true because \(v\) is odd in the \(q\)-variable.

**Proposition 4.3.** The elapsed time is given by

\[
\theta(p) = \int_{-L/2}^{L/2} \frac{dx}{c - u(x, y_{\gamma_0}(x))} = \int_{-\lambda/2}^{\lambda/2} \left[ h_p^2(q, p) + h_q^2(q, p) \right] \, dq \tag{4.35}
\]

and satisfies

\[
\theta(p) > \frac{L}{c - \kappa} \quad \text{for all} \quad p \in [-m, 0]. \tag{4.36}
\]

**Proof.** Let us first consider the case \(p \neq 0\), i.e. \(Y_0 \neq -d\). By parameterizing the particular streamline containing the point \((L/2, Y_0)\) via the map \(x \mapsto y_{\gamma_0}(x)\), \(y_{\gamma_0} : [-L/2, L/2] \to \mathbb{R}\), we may write the elapsed time \(\theta(p)\) as

\[
\theta(p) = \int_0^\theta 1 \, dt = \int_0^\theta \frac{x'(t)}{u(x(t), y(t)) - c} \, dt = \int_{-L/2}^{L/2} \frac{dx}{c - u(x, y_{\gamma_0}(x))}. \tag{4.37}
\]

In order to recognize the second representation of \(\theta\) we recall that

\[
h_p^2 + h_q^2 = \frac{1}{(c - u)^2 + v^2}
\]

by means of \((2.28)\) and calculate the derivative of \(q\) along streamlines:

\[
\frac{dq}{dx} = -\frac{d}{dx} [\phi(x, y(x))] = c - u(x) + \frac{v(x)^2}{c - u(x)} = \frac{(c - u(x))^2 + v(x)^2}{c - u(x)}.
\]

Let us now consider the divergence-free vector field \((v, c - u)\) (recall \((2.5)\)) restricted to the region \(D_{y_{\gamma_0}} \subseteq D\) beneath the streamline \(y = y_{\gamma_0}(x)\), above the flat bed \(y = -d\) and between the trough lines \(x = \pm L/2\). The divergence theorem implies that

\[
0 = \int_{\partial D_y} (v, c - u) \cdot n \, dS = \int_{-L/2}^{L/2} (u(x, -d) - c) \, dx + \int_{-L/2}^{L/2} (c - u(x, y_{\gamma_0}(x))) \sqrt{1 + (\partial_x y_{\gamma_0}(x))^2} \, dx, \tag{4.38}
\]
where we exploited the anti-symmetry of \( v \) and the fact that \( y_{Y_0} \) has the same shape as the free surface, see Corollary 3.2. By exploiting \((2.21)\) and the periodicity of \( u \) in the \( x \)-variable, we infer from \((4.38)\) that
\[
(c - \kappa)L = \int_{-L/2}^{L/2} (c - u(x, y_{Y_0}(x))) \sqrt{1 + (\partial_x y_{Y_0}(x))^2} \, dx.
\] (4.39)

Due to the Cauchy-Schwarz inequality we get
\[
L^2 = \left( \int_{-L/2}^{L/2} \frac{c - u(x, y_{Y_0}(x))}{\sqrt{c - u(x, y_{Y_0}(x))}} \, dx \right)^2
\leq \int_{-L/2}^{L/2} \frac{dx}{c - u(x, y_{Y_0}(x))} \int_{-L/2}^{L/2} (c - u(x, y_{Y_0}(x))) \, dx.
\] (4.40)

We recall that the streamline \( y_{Y_0} \) satisfies (see Corollary 3.2)
\[
\partial_x y_{Y_0}(x) \neq 0 \quad \text{for} \quad x \in (-L/2, L/2) \setminus \{0\},
\]
and hence
\[
\int_{-L/2}^{L/2} (c - u(x, y_{Y_0}(x))) \sqrt{1 + (\partial_x y_{Y_0}(x))^2} \, dx > \int_{-L/2}^{L/2} (c - u(x, y_{Y_0}(x))) \, dx.
\] (4.41)

In summary we get from \((4.37)\), \((4.40)\), \((4.41)\) and \((4.39)\) that
\[
\theta(p) = \int_{-L/2}^{L/2} \frac{dx}{c - u(x, y(x))} \geq \frac{L^2}{\int_{-L/2}^{L/2} (c - u(x, y(x))) \, dx}
> \frac{L^2}{\int_{-L/2}^{L/2} (c - u(x, y(x))) \sqrt{1 + (\partial_x y_{Y_0}(x))^2} \, dx} = \frac{L^2}{(c - \kappa)L} = \frac{L}{c - \kappa}
\]
for all \( p \in (-m, 0] \).

In order to see that \( \theta(p) > L/(c - \kappa) \) holds still true for \( p = -m \), we argue by contradiction: if we assume the contrary, namely that \( \theta(-m) = L/(c - \kappa) \), we enforce equality in the Cauchy-Schwarz inequality \((4.38)\):
\[
L^2 = \int_{-L/2}^{L/2} \frac{dx}{c - u(x, y(x))} \int_{-L/2}^{L/2} (c - u(x, y(x))) \, dx,
\]
which means that the functions \( x \mapsto \frac{1}{c - u(x, y(x))} \) and \( x \mapsto (c - u(x, -d)) \) are linearly dependent. But this would only be possible if \( u \) was constant on the flat bed, but this is not the case, since we know that \( u_x < 0 \) on \( (0, L/2) \) by Proposition 3.4. And so we conclude that \((4.36)\) is satisfied. \( \square \)

**Proposition 4.4.** The elapsed time \( \theta(p) \) decreases strictly with depth \( p \in [-m, 0] \).

**Proof.** We show that \( \theta'(-m) = 0 \) and \( \theta''(p) > 0 \) for \( p \in (-m, 0] \). Differentiation of \((4.35)\) with respect to \( p \) gives
\[
\theta' = 2 \int_{-L/2}^{L/2} [h_q h_{qp} + h_p h_{pp}] \, dq.
\]
Since $h$ is periodic the $q$-variable with period $\lambda$, we obtain
\[
0 = \int_{-\lambda/2}^{\lambda/2} (h_{qq})_q \, dq = \int_{-\lambda/2}^{\lambda/2} [h_{qp}^2 + h_{pp}^2] \, dq
\]
for all $p \in [-m, 0]$; therefore the fact that $h$ is harmonic yields that
\[
\theta' = 4 \int_{-\lambda/2}^{\lambda/2} h_{qp} \, dq = -4 \int_{-\lambda/2}^{\lambda/2} h_{pp} \, dq
\]
for all $p \in [-m, 0]$, in particular we get that
\[
\theta'(-m) = 0
\] (4.42)
, since $h_q$ is constantly zero on the flat bed by (2.8). The second derivative of $\theta$ is given by
\[
\theta'' = 4 \int_{-\lambda/2}^{\lambda/2} [h_{pp}^2 + h_{pp}h_{ppp}] \, dq = 4 \int_{-\lambda/2}^{\lambda/2} [h_{pp}^2 + h_{pp}^2] \, dq \geq 0
\] (4.43)
for all $p \in [-m, 0]$; in order to establish the second equality, we used once more, that $h$ is harmonic and $\lambda$-periodic in $q$:
\[
0 = \int_{-\lambda/2}^{\lambda/2} (h_{pp}h_{pq})_q \, dq = \int_{-\lambda/2}^{\lambda/2} [h_{pp}^2 + h_{pp}h_{ppp}] \, dq = \int_{-\lambda/2}^{\lambda/2} [h_{pp}^2 - h_{pp}h_{ppp}] \, dq.
\]
Combining (4.42) and (4.43) yields $\theta' \geq 0$ for all $p \in [-m, 0]$. Finally we show that the inequality is strict on $(-m, 0]$. Assuming the contrary, namely that $\theta'(p_0) = 0$ for some $p_0 \in (-m, 0]$, monotonicity of $\theta'$ implies that $\theta'(p) = 0$ for all $p \in [-m, p_0]$, yielding that $\theta'' = 0$ on $[-m, 0]$. Then (4.43) implies that $h_{pq} = h_{pp} = 0$ in $[-\lambda/2, \lambda/2] \times [-m, p_0]$, and thus $h_p$ is constant throughout this rectangle. By recalling that $h_p = (c-u)^{-1}$ on the lower boundary $p = -m$ by (2.28), this contradicts Proposition 3.3.

**Definition 4.5.** The particle drift $\mathcal{D} = \mathcal{D}(Y_0)$ is the net horizontal distance that a particle moves in the physical frame until the elapsed time passes by:

\[
\mathcal{D} := X(\tau) - X(0).
\]

**Proposition 4.6.** A particle path is closed if and only if $\mathcal{D} = 0$.

**Proof.** If a particle trajectory is closed, there exists some time $T > 0$ such that $(X(T), Y(T)) = (X_0, Y_0)$. From $Y(T) = Y_0$ it follows that $T = n\theta$ for some nonzero integer $n$. Then it holds that $0 = X(n\theta) - X(0) = n(X(\tau) - X(0))$ which implies that $\mathcal{D}(Y_0) = 0$.

If on the other hand $\mathcal{D}(Y_0) = 0$, then $X(\tau) = X(0)$, $Y(\tau) = Y(0)$ and $\tau = L/c$. By $x$-periodicity both $t \mapsto (X(t), Y(t))$ and $t \mapsto (X(t+\tau), Y(t+\tau))$ solve (4.32) with the initial condition $(X(0), Y(0)) = (X_0, Y_0)$. Since $(U, V)$ is real analytic and bounded, thus in particular Lipschitz continuous, we have uniqueness of solutions, i.e. $X(t+\tau) = X(t)$ and $Y(t+\tau) = Y(t)$ for all $t \in \mathbb{R}$. \hfill \Box

**Proposition 4.7.** The particle drift $\mathcal{D}$ satisfies
\[
\mathcal{D}(p) = c\theta(p) - L > \frac{cL}{c - \kappa} - L \quad \text{for all} \quad p \in [-m, 0]. \quad (4.44)
\]
In particular, the drift is strictly positive for each fluid particle, if $\kappa \geq 0$. 

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Proof. We have that \( X(T) = x(t) + ct \), and in time \( \tau \) the solution \((x(t), y(t))\) of (4.33) moves by definition from the trough line \( x = L/2 \) to the next trough line \( x = -L/2 \) in the moving frame. Hence \( X(0) = x(0) = L/2 \), \( X(\tau) = x(\tau) + c\tau \) and thus \( X(\tau) - X(0) = -L + c\tau = c\theta - L \). The estimate in (4.44) is due to Proposition 4.3.

As an immediate consequence of Propositions 4.6 and 4.7, we obtain the following

**Corollary 4.8.** If \( \kappa \geq 0 \), the particle drift \( \mathcal{D}(p) \) is positive for all \( p \in [-m, 0] \); in particular, there are no closed particle paths.

Moreover, Proposition 4.4 and Proposition 4.7 yield

**Corollary 4.9.** The particle drift \( \mathcal{D}(p) \) decreases strictly with depth \( p \in [-m, 0] \).

We collect all possible scenarios concerning the sign of the particle drift in case of a negative \( \kappa \) in the following

**Remark 4.10** (Sign of the drift in the case of an adverse current). If \( \kappa < 0 \), the particle drift \( \mathcal{D} \) can either have the same (positive or negative) sign everywhere in \( D \), or there exists a streamline \( \mathcal{Y} \) of zero drift in \( D \), where \( \mathcal{D} \) changes sign (from positive to negative, when going from the surface \( S \) to the bed \( B \) across \( \mathcal{Y} \), according to Corollary 4.9). To be more precise, let \( \kappa_B \) and \( \kappa_S \) be the values of the adverse current, for which the drift vanishes on \( B \) and \( S \) respectively. In view of the notation introduced in Remark 3.8 and the formula (4.44) for the particle drift, we have that

\[
\kappa_B = \frac{L}{\theta(-m)} - c_0 \quad \text{and} \quad \kappa_S = \frac{L}{\theta(0)} - c_0, \tag{4.45}
\]

where

\[
\theta(-m) = \int_{-L/2}^{L/2} \frac{dx}{c - u(x, -d)} = \int_{-L/2}^{L/2} \frac{dx}{c_0 - u_0(x, -d)} > \frac{L}{c_0}, \tag{4.46}
\]

and

\[
\theta(0) = \int_{-L/2}^{L/2} \frac{dx}{c - u(x, \eta(x))} = \int_{-L/2}^{L/2} \frac{dx}{c_0 - u_0(x, \eta(x))} > \frac{L}{c_0}, \tag{4.47}
\]

due to Proposition 4.3. Therefore, as desired, both \( \kappa_B \) and \( \kappa_S \) are negative. Additionally, Corollary 4.9 tells us that

\[\kappa_S < \kappa_B < 0.\]

So in view of the strict monotonicity of the particle drift \( \mathcal{D}(p) \), see Corollary 4.9, we end up with the following cases.

1. \( \kappa_B \leq \kappa < 0 \). The drift is positive from the surface \( S \) down to the bed \( B \). In the borderline case \( \kappa = \kappa_B \), we have that \( \mathcal{D}(p) > 0 \) for \( p \in (-m, 0] \) and \( \mathcal{D}(-m) = 0 \).

2. \( \kappa_S < \kappa < \kappa_B \). In this situation, \( \mathcal{D}(0) > 0 \) and \( \mathcal{D}(-m) < 0 \), thus Corollary 4.9 yields the existence of a unique streamline \( \mathcal{Y} \) in \( D \) where \( \mathcal{D} \) vanishes (along which particles move in circles due to Proposition 4.6), and the drift becomes negative for particles below \( \mathcal{Y} \), see Fig. 6.

3. \( -\frac{g^2}{4\pi} + c \leq \kappa \leq \kappa_S \). The drift is negative from the surface to the bed. In the borderline situation \( \kappa = \kappa_S \) it holds that \( \mathcal{D}(p) < 0 \) for \( p \in [-m, 0] \) and \( \mathcal{D}(0) = 0 \).
We exploit now the insights gained from studying the velocity field in section 3 and the particle drift to describe the particle trajectories in a qualitative manner. The discussion contains the three different scenarios of no underlying current ($\kappa = 0$), a favorable current ($\kappa > 0$) and an adverse current ($\kappa < 0$).

4.1 Particle trajectories in flows without underlying currents

In the situation of the absence of an underlying current, we proved (see Proposition 3.7) that the level set $\{u = 0\}$ in $D_+$ consists of a smooth curve $C_+$, which connects $B_+$ with $S_+$ and intersects each streamline $\{\psi = p\}$ with $p \in [-m, 0]$ exactly once. The corresponding curve $C_-$ that represents the level set $\{u = 0\}$ in $D_-$ is obtained by reflecting $C_+$ in the vertical line $\{x = 0\}$.

Let us distinguish between the position $(x(t), y(t))$ of the particle at time $t$ in the moving frame and the corresponding position $(X(t), Y(t))$ in the physical frame. Let $\gamma = \gamma_{Y_0} : [0, \theta] \to \mathbb{R}^2, t \mapsto (x(t), y(t))$ be the trajectory of the particle in the moving frame (Fig. 7) and let $\Gamma = \Gamma_{Y_0} : [0, \theta] \to \mathbb{R}^2, t \mapsto (X(t), Y(t))$ be the corresponding path in the physical frame (Fig. 8). As before we assume that the particle is initially located at $a = (x(0), y(0)) = (L/2, Y_0) = (X(0), Y(0)) = A$.

Let $b, c, d$ be the points, where $\gamma$ intersects $C_+$, the crest line $\{x = 0\}$, and the curve $C_-$ respectively, and let us denote by $e = \gamma(\theta(Y_0)) = (-L/2, Y_0)$ the endpoint of $\gamma$. Let $I_{ab}$ be the open time interval while $(x(t), y(t))$ is located between $a$ and $b$; the time intervals $I_{bc}, I_{cd}$ and $I_{de}$ are defined analogously. Furthermore let $B, C, D$ and $E$ be the corresponding points in the physical frame. We have that $E = \Gamma(\theta) = (-L/2 + e\theta, Y_0)$ lies to the right of $A$ since $D(Y_0) > 0$. Since the images $\gamma(I_{ab})$ and $\gamma(I_{de})$ lie within regions where $u < 0$, the corresponding horizontal displacement $\Gamma(t)$ has to be backwards during these time intervals. The image $\gamma(I_{ad})$ lies in the region where $u > 0$, hence $\Gamma(t)$ moves forward during this time interval. Moreover $\gamma(I_{ac})$ lies in $D_+$, where $v$ is strictly positive, thus $\Gamma(t)$ moves upwards within the time window $I_{ac}$ and $\Gamma(t)$ moves downwards for $t \in I_{ce}$ since $\gamma(I_{ce})$ lies in $D_-$, where $v$ is strictly negative. This description holds for all particles being initially located above the flat bed.
Figure 7: The trajectory $\gamma$ in the moving frame intersects the trough lines in points $a$, $e$, and crest line in $c$, where $v$ changes the sign, and the points $b$ and $d$, where $u$ changes the sign.

Figure 8: The path $\Gamma$ in the physical frame corresponds to the trajectory $\gamma$ in the moving frame, and the points $A, B, C, D, E$ correspond to $a, b, c, d, e$ respectively; cf. Fig. 7.

A particle which is initially located on the flat bed, i.e. $Y_0 = -d$, will always remain there, since $v = 0$. We have that $\Gamma_{-d}(t)$ moves forward at times $t \in I_{bd}$, it moves backwards when $t$ lies in the time intervals $I_{ab}$ and $I_{de}$. This means that the particle oscillates backward-forward-backward, mirroring the projection of the loops of $\Gamma_{Y_0}$, $Y_0 > -d$ to the flat bed.

4.2 Particle trajectories in favorable currents

As in the previous case, the drift is positive for all particles; see Proposition 4.7. We recall that the sign of the vertical velocity component $v$ is not effected by the presence of an underlying current, see Proposition 3.1. We only have to take into account the qualitative changes of the horizontal velocity component $u$ for an increasing mean current $\kappa$ in order to give a qualitative description of the motion of fluid particles. According to Remark 3.8 we distinguish three different cases, regarding the sign of $u$ in $D$.

In the case of a small favorable current, where $\kappa \leq -u_0(L/2, -d)$, the particle path pattern is the same as in the case $\kappa = 0$. Only in the borderline case $\kappa = -u_0(L/2, -d)$ the situation changes for particles at the flat bed: there is no backward motion at all.

In the case of a moderate favorable current, where one finds that $-u_0(L/2, -d) < \kappa < -u_0(L/2, \eta(L/2))$, the qualitative picture of the particle paths depends on the depth: a critical streamline $\mathcal{Y}$ separates $D$ into two layers. Particles in the upper layer follow
the $\kappa = 0$ pattern, whereas particles on or below $\mathcal{Y}$ do not move backwards.

If $\kappa \geq -u_0(L/2, \eta(L/2))$, we say that a strong favorable current is present and all particles experience a pure forward motion.

Particles which are located at the flat bed move to the right with a periodic change of velocity in the latter two sub-cases.

We visualized our considerations in Fig. 9.

Figure 9: Particle paths in the presence of a favorable current: the first trajectory corresponds to that of a particle experiencing a small favorable current or that of a particle in a moderate one above the critical streamline $\mathcal{Y}$ (the loops are smaller in comparison with Fig. 8), the second describes the wavelike path on or below $\mathcal{Y}$, or that of the particles experiencing a strong favorable current.

4.3 Particle trajectories in adverse currents

In the situation of an adverse current, the possibility of a negative particle drift has to be taken additionally into account in order to describe particle paths. We combine the results in Remark 3.8 and Remark 4.10 to classify the particle path pattern.

Figure 10: A closed particle trajectory on a zero-drift streamline.
Figure 11: The trajectories of two different particles looping to the left: the distance from $A$ to $E$ depends on $\kappa$ and the depth of the particle’s initial position.

Figure 12: A fluid particle moving wavelike to the left.

First we observe, that Proposition 3.4 on the monotonicity of $u$ together with (4.45), (4.46) and (4.47) allow us to relate $\kappa_S, \kappa_B, -u_0(0, \eta(0))$ and $-u_0(0, -d)$:

$$-u_0(0, \eta(0)) < \kappa_S \quad \text{and} \quad -u_0(0, -d) < \kappa_B;$$

we keep also in mind that $\kappa_S < \kappa_B$ as well as $-u_0(0, \eta(0)) < -u_0(0, -d)$. Note that in general we can not determine a priori, whether $\kappa_S < -u_0(0, -d)$ or $-u_0(0, -d) \leq \kappa_S$. Therefore the following scenarios for the motion of particles might occur when an adverse current is present.

1. $\kappa_B \leq \kappa$. The drift is still positive (it vanishes on the bottom, if $\kappa = \kappa_B$), and $u$ changes its sign periodically along every streamline. The qualitative picture is the same as in the $\kappa = 0$ case, cf. Fig. 8.

2. $\kappa_S < \kappa < \kappa_B$ and $-u_0(0, -d) \leq \kappa$. The sign of the drift alters across a certain streamline $\mathcal{Y}$ and $u$ still changes sign along every streamline (if $\kappa = -u_0(0, -d)$, $u$ is negative on $B \setminus \{(0, -d)\}$, entailing a pure backward motion which stagnates only in bottom points directly below the crest). The particles on $\mathcal{Y}$ move clockwise on a closed path, cf. Fig. 10. Particles above $\mathcal{Y}$ are looping to the right, below $\mathcal{Y}$ they are looping to the left, cf. Fig. 11.

3. $-u_0(0, -d) \leq \kappa \leq \kappa_S$. The drift is negative for all particles (except for surface particles in the case $\kappa = \kappa_S$, in which $\mathcal{Y}(0) = 0$ and surface particles move on closed paths) and $u$ behaves as in the previous scenario, hence particles are looping to the left.
4. \( \kappa_S < \kappa < -u_0(0, -d) \). There is a zero drift streamline \( Y_1 \), that separates \( D \) into an upper region of positive drift and a lower region of negative drift, along which particles move on closed paths. And there is a streamline \( Y_2 \) which splits \( D \) into an upper region, where \( u \) changes signs, and a lower region, where \( u \) is negative. Observe, that it can not happen, that \( Y_1 \) lies below \( Y_2 \), since this would entail particles with a pure backward motion but a positive drift. Moreover it is not possible, that \( Y_1 \) and \( Y_2 \) coincide: closed loops enforce a periodic occurrence of backward and forward motion. Therefore \( Y_1 \) lies above \( Y_2 \), and \( D \) splits into three different layers. In the lowest one, particles move wavelike to the left, cf. Fig. 12. Particles in the middle layer are looping to the left, and particles in the top layer are looping to the right.

5. \( \kappa < \kappa_S \) and \(-u_0(0, \eta(0)) < \kappa < -u_0(0, -d) \). The drift is negative for all particles and there exists a streamline \( Y \) that splits \( D \) into an upper layer, where \( u \) changes sign along streamlines, and a lower layer, where \( u \) is negative. Particles in the upper layer loop to the left, whereas particles below \( Y \) move as depicted in Fig. 12.

6. \( c - \frac{g}{2\Omega} < \kappa \leq -u_0(0, -d) \). Here both \( D \) and \( u \) are negative throughout \( D \). All particles move wavelike to the left.

Remark 4.11. We have demonstrated that the movement of particles within irrotational flows beneath periodic traveling equatorial surface waves follows the same qualitative pattern as it is the case for Stokes waves; cf. the results in [2, 10]. We could essentially reproduce these results for irrotational equatorial waves. Let us point out once again, that the bounds we have imposed on \( c \) and \( \kappa \) are not restrictive from a physical point of view.

Nevertheless, it is remarkable, that even in the presence of Coriolis effects in the \( f \)-plane, we can preserve the analysis without any restrictions on the wave amplitude; in particular our results do not rely on approximations.

Let us finally point out, that the effects of underlying (uniform) currents on particle paths in our setting is quite different from that in the explicit equatorial waves obtained in [3, 5, 13, 16, 17, 21] for flows with non-constant vorticity. These papers start by imposing a circular particle path in the absence of a current, changed to a trochoid by an underlying current, whereas in our setting for irrotational flows, closed particle paths are only possible, if there is a suitable adverse current.

Appendices

A  Vorticity equation for equatorial waves

We provide a derivation of the vorticity equation for three-dimensional flows beneath equatorial waves, which describes the evolution of the vorticity in a flow.

Let us for this purpose denote the velocity field by

\[ u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \]

and the (scalar) pressure field by \( P(x, t) \), where \( x = (x_1, x_2, x_3) \) are the spatial coordinates and \( t \) is the time in the physical frame. Henceforth the gradient reads \( \nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \).
furthermore the symbol "×" denotes the cross product in \( \mathbb{R}^3 \) and "." denotes the scalar product.

With this notation and by setting the density \( \rho = 1 \) for notational convenience, the governing equations for equatorial waves in the \( f \)-plane read

\[
\begin{align*}
\frac{u_t}{u_2} + (u \cdot \nabla)u + 2\Omega(u_3, 0, -u_1) &= -\nabla(P + gz) \\
\nabla \cdot u &= 0.
\end{align*}
\] (A.1)

We apply the vector identity \((u \cdot \nabla)u = \nabla(1/2 u \cdot u) + (\nabla \times u) \times u\) on the first equation of (A.1) and denote by \( \omega = (\nabla \times u) \) the vorticity to obtain

\[
\frac{u_t}{u_2} + \omega \times u + 2\Omega(u_3, 0, -u_1) = -\nabla(1/2 u \cdot u + P + gz).
\] (A.2)

Next we apply \( \nabla \times \) on (A.2), use the identity

\[
\nabla \times (a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + a(\nabla \cdot b) - b(\nabla \cdot a)
\]

for continuously differentiable vector fields \( a, b \) in \( \mathbb{R}^3 \), and get

\[
\omega_t + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u + \omega(\nabla \cdot u) - u(\nabla \cdot \omega) - 2\Omega \partial_{x_2}u = 0.
\]

Since the divergence of \( \omega \) is generally zero and due to the second equation (conservation of mass) in (A.1), we end up with the vorticity equation

\[
\frac{D\omega}{Dt} = (\omega \cdot \nabla)u + 2\Omega \partial_{x_2}u.
\] (A.3)

Here, \( D/Dt = \partial/\partial t + (u \cdot \nabla) \) denotes the so-called material derivative.

For the special situation of a two-dimensional flow, which can be identified with a three dimensional flow satisfying \( u_2 = 0 \) with \( u_1 \) and \( u_3 \) being independent of the \( x_2 \)-variable, it holds that \( (\omega \cdot \nabla)u = \omega_2 \partial_{x_2}u = 0 \), and therefore equation (A.3) reads

\[
\frac{D\omega}{Dt} = 0.
\]

This means, that the local spin of each fluid particle is preserved, as it moves with the flow. In particular, particles having no local spin at the initial time, will never acquire it.

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**References**


On the existence of benthic storms
Ronald Quirchmayr


Abstract

We study a model for the wind-induced current field of the Pacific ocean in order to demonstrate that currents in the surface layer are carried down to the deepest regions above the abyssal sea floor, which indicates the existence of the phenomenon of comparably strong currents in bottom regions as a result of wind-stress forces at the surface, also known as benthic storms.

1 Introduction

A longtime assumption in oceanography was that the water in the benthic boundary layer (BBL) above the abyssal sea floor, which develops as a result of friction at the bed, would be rather still. Former theory of the BBL with focus on mean-flow properties relied on models assuming horizontally homogeneous stationary currents (see [3], [4]). The consideration of such models stemmed from the insufficient availability of empirical data for the hydrodynamics (but also the chemical and biological processes, etc. and their interactions) in the BBL, due to its inaccessibility in the open ocean. The perception changed fundamentally as a consequence of observations that have been made during long term projects such as the High Energy Benthic Boundary Layer Experiment (HEBBLE Program), which was primarily initiated for practical reasons such as the extraction of minerals from the sea floor, anti submarine defense, navigation, the selection of waste burial sites, etc. It turned out that the BBL is not static at all. There are dramatic velocity increases in the benthic current, which occur periodically, the so-called benthic storms. Their importance springs from the fact that they stir up bottom sediments, which are then captured and transported over large distances by weaker but stable currents. Unlike the stationary weak currents, benthic storms seem to be controlled not only by forces resulting e.g. from thermohaline and tidal phenomena (also the Coriolis force plays a role in equatorial regions), but results from wind-stress at the surface layer, which is carried down to the BBL via mesoscale eddies having diameters up to 200km. This wind-driven mechanism as origin of benthic storms particularly applies to equatorial regions in the Pacific (see [9]); there are regions where wind plays no or merely a minor role in the generation of benthic storms (see [11]).

The purpose of these notes is to prove - on the basis of a simple (static) model - that wind-stress forces at the surface indeed propagate down to the sea floor and thus may generate benthic storms. Let us point out, that such a basic model is not capable to describe the complex dynamics of the currents in the BBL. However it tells us that currents at the surface layer influence the flow in deep regions directly above the bottom.

2 A note on currents in the equatorial region

Our considerations rely on a linear eddy viscosity model in [4] for the wind-induced current field of the Pacific ocean in the equatorial region, where stratification is greater
than anywhere else in the ocean (see [6]): a sharp thermocline separates a shallow water layer of warm water from a deep layer of colder water with higher density. The fluid domain, sketched in Fig. 1, consists of the subsurface layer

\[ U_1 := \{(x, z) \in \mathbb{R}^2: -h < z < 0\} \]

with constant water density \( \rho \) between the water surface \( \{(x, z) \in \mathbb{R}^2: z = 0\} \) and the thermocline \( \{(x, z) \in \mathbb{R}^2: z = -h\} \) (with \( h > 0 \)), and a deep layer

\[ U_2 := \{(x, z) \in \mathbb{R}^2: -d < z < -h\} \]

density having a slightly higher density \( \rho(1 + r) \) for some fixed \( r > 0 \).

Figure 1: The two layers are separated by the thermocline at a depth of roughly 120m. Most parts of the sea floor in the Pacific ocean are indeed flat, located at depths exceeding 3500m; so-called abyssal plains.

We consider the linearized equations for a steady state flow with a vanishing vertical-fluid-velocity component in the \( f \)-plane approximation for both frictional layers. That is, presume

\[
0 = -\frac{1}{\rho} \left( \frac{P_x}{P_z} \right) + (\nu u_z)_z, \quad (2.1)
\]

\[
-2\Omega u = -\frac{1}{\rho} P_z - g, \quad (2.2)
\]

\[
u_x = 0 \quad (2.3)
\]

throughout \( U_1 \), and accordingly we require

\[
0 = -\frac{1}{\rho(1 + r)} \left( \frac{P_x}{P_z} \right) + (\nu u_z)_z, \quad (2.4)
\]

\[
-2\Omega u = -\frac{1}{\rho(1 + r)} P_z - g, \quad (2.5)
\]

\[
u_x = 0 \quad (2.6)
\]
to hold within the deep layer $U_2$. The two unknowns - the velocity field $u$ and the pressure $P$ - are assumed to be smooth within $U_1 \cup U_2$. The depth dependent viscosity parameter $\nu = \nu(z)$ is given. It is smooth throughout

$$ U := U_1 \cup U_2 \cup \{ z = -h \}, $$

positive and away from zero; in [5] it is suggested to take a suitable exponential decay to some small positive value as a good approximation for $\nu$. The constants $g$ and $\Omega$ denote the gravitational constant and the rotational speed of the earth around the polar axis toward the east.

In addition to equations (2.1) - (2.6) we consider the following boundary conditions. We impose a no-slip condition on the bottom, a vanishing vorticity of the velocity field on the thermocline and a constant atmospheric pressure on the surface. Furthermore we assume the velocity and the pressure to be continuous across the thermocline:

$$
\begin{align*}
&u = 0 \quad \text{on} \quad z = -d, \\
&u_z = 0 \quad \text{on} \quad z = -h, \\
&P = P_{\text{atm}} \quad \text{on} \quad z = 0, \\
&u, P \in C(\overline{U}).
\end{align*}
\tag{2.7} \tag{2.8} \tag{2.9} \tag{2.10}
$$

The proposition below tells us that an absence of currents in regions above the sea floor would imply zero current up to the water surface. Let us denote such a bottom region by

$$ U_\varepsilon := \{ (x, z) \in \mathbb{R} : -d \leq z < -d + \varepsilon \}; \tag{2.11} $$

cf. Fig. 1.

**Proposition 2.1.** Assume that there is some region $U_\varepsilon$ above the bottom where the velocity field $u$ vanishes. Then $u$ is identically zero from the bottom to the surface.

**Proof.** We have that $(\nu u_z)_{zz} = 0$ in $U_1 \cup U_2$. In order to see this, we differentiate (2.2) with respect to $x$ first, and use (2.3) to infer that $P_zx = 0$ in $U_1$. The claim follows by differentiating (2.1) with respect to $z$ and exploiting the smoothness of $P$ in $U_1$:

$$ 0 = P_{xx} = P_{zz} = (\nu u_z)_{zz} \quad \text{in} \quad U_1. $$

Analogously we get that $(\nu u_z)_{zz} = 0$ in $U_2$. Therefore, $\nu u_z = A_2 z + B$ for $A, B \in \mathbb{R}$, and by (2.8) we find that

$$ \nu u_z = A_2 (z + h) \quad \text{in} \quad U_2. $$

Since $u \equiv 0$ in $U_\varepsilon$, we find that $A_2 = 0$, hence $u_z = 0$ in $U_2$ and from (2.7) and (2.10) we infer that

$$ u \equiv 0 \quad \text{in} \quad U_2. \tag{2.12} $$

We may now infer from (2.4) and (2.5) that $P_z \equiv 0$ and $P_z = -g \rho (1 + r)$, thus

$$ P(x, z) = -\rho (1 + r)gz + C_2 $$

in $U_2$ for some constant of integration $C_2 \in \mathbb{R}$.

Similarly we find that

$$ \nu u_z = A_1 (z + h) \quad \text{in} \quad U_1 \tag{2.13} $$
for some $A_1 \in \mathbb{R}$. Therefore (2.9), (2.10) and (2.12) imply that
\[
u(z) = \int_{-h}^{z} \frac{A_1(\xi + h)}{\nu(\xi)} \, d\xi
\]
for $-h \leq z \leq 0$.

We obtain from (2.1) and (2.13) that $P_x = \rho A_1$ in $U_1$ and from (2.2) that $P_z = \rho(2\Omega u - g)$ in $U_1$, thus
\[
\lim_{z \uparrow -h} P(x, z) = -\rho(1 + r)gz + C_2 = -\rho gz + \rho A_1 x + C_1 = \lim_{z \downarrow -h} P(x, z).
\]
This is only possible, if $A_1 = 0$ is satisfied. We deduce from (2.9) that $C_2 = C_1 - \rho rgh$, and $C_1 = P_{\text{atm}}$. Hence the pressure field equals the hydrostatic pressure:
\[
P(x, z) = \begin{cases}
P_{\text{atm}} - \rho gz & \text{in } U_1 \cup \{z = 0\} \\
P_{\text{atm}} - \rho(1 + r)gz - \rho rgh & \text{in } U_2 \cup \{z = -h\} \cup \{z = -d\}
\end{cases}
\]
Moreover, knowing that $A_1 = 0$, we obtain from (2.13) that $u_z = 0$ throughout $U_1$, hence $u$ is constant within $U_2$. Therefore by (2.12) and the continuity assumption (2.10) we conclude that $u \equiv 0$ throughout $U$.

**Remark 2.2.** Let us point out, that the region $U_\varepsilon$ does not particularly represent the BBL mentioned earlier in the introduction. It stands for some - theoretically arbitrary small - region directly above the bottom.

The continuity assumption (2.10) can not be meaningfully strengthened in the sense of requiring $u$ and $P$ to be continuously differentiable or even smooth across the thermocline. We have seen in the proof of Proposition 2.1 that the hydrostatic pressure is only piecewise smooth; its $z$-derivative has a jump at the thermocline. Furthermore, requiring a smooth current $u$ throughout $U$, would trivialize the model. We would then already obtain that $(\nu u_z)_z \equiv 0$ in $U$ without imposing the additional condition of Proposition 2.1 (on some region $U_\varepsilon$). Hence we would get $u_z \equiv 0$ in $U$ by (2.8) and therefore the bottom condition (2.7) tells us that the only smooth solution of (2.1) - (2.3) and (2.4) - (2.6) with boundary conditions (2.7) - (2.9) is the trivial one.

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**References**

ON THE EXISTENCE OF BENTHIC STORMS


Abstract

In this doctoral thesis we investigate several aspects of nonlinear water waves. The exposition is divided into two parts.

The first part is concerned with a model equation which we derive from the classical water wave problem for unidirectional water waves, where the divergence-free flow beneath the free water surface over a flat bed is governed by the incompressible Euler equations, with gravity acting as the only external force. This model equation describes the evolution of the horizontal velocity component of the flow field evaluated at a specific depth within a shallow water regime, allowing for waves of large amplitude. We study the corresponding Cauchy problem and establish local well-posedness in the natural Sobolev setting by using Kato’s semigroup approach for quasilinear hyperbolic evolution equations. Additionally we prove that the regularity of the solution of the Cauchy problem inherits the regularity of the initial data in terms of the Sobolev exponent. This enables us to derive a blow-up criterion for solutions, which allows a meaningful interpretation from a physical point of view. In a second paper, joint work with Anna Geyer, we investigate the traveling wave solutions of this highly nonlinear model equation. By applying qualitative methods from the theory of dynamical systems, in particular tools from integrable planar systems, we establish a full classification of all traveling waves. Thereby we discover completely new types of traveling waves, e.g. peaked solitary waves with compact support and periodic traveling waves with peaked crests and troughs, which do not appear as solutions of shallow water equations for waves of moderate amplitude such as the well-known Camassa-Holm equation.

In the second part, we present some investigations of ocean flow dynamics. We give a detailed qualitative analysis of the irrotational velocity field beneath smooth symmetric periodic traveling waves in the equatorial region and describe the pattern of the paths of particles beneath such waves. Our analysis, which makes use of methods from complex and harmonic analysis, does – in contrast to the first part – not rely on approximations. Thus we do not need to impose restrictions on the amplitude of the wave. The underlying governing equations differ slightly from the classical water wave problem: the Euler equations are extended by linear terms accounting for the effects of the earth’s rotation near the Equator, which become relevant for large ocean waves. We show that this perturbation does not alter the qualitative properties of the flow field. Also the particle paths are similar to the ones of the flows beneath classical Stokes waves. Finally we discuss some aspects of a stratified wind-induced current field with eddy viscosity in the equatorial region.
Zusammenfassung

Die vorliegende Dissertation, welche sich grob in zwei Teile aufgliedert, beleuchtet verschiedene Aspekte nichtlinearer Wasserwellen.


Im zweiten Teil wird die Dynamik gewisser Ozeanwellen untersucht. Wir analysieren Strömungsfelder ohne Vorzität unterhalb einer symmetrischen, sich periodisch ausbreitenden Wanderwelle im Bereich des Äquators hinsichtlich qualitativther Aspekte und beschreiben die Formen der Partikelbahnen. Im Unterschied zum ersten Teil wird direkt das zugrundeliegende physikalische Modell behandelt, welches im Falle von Ozeanwellen den Einfluss der Erdrotation berücksichtigt. Es werden keine Näherungsmodelle verwendet, wodurch auf eine Beschränkungen der Größenordnung der Wellenamplitude verzichtet werden kann. Wir zeigen, dass sowohl die qualitativen Eigenschaften der Strömungsfelder, als auch die Formen der Partikelbahnen ähnlich sind wie im Falle von klassischen Stokes-Wellen. Schließlich behandeln wir gewisse Aspekte einer geschichteten Meeresströmung im äquatorialen Bereich.
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