DISSERTATION

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Fractional diffusion limits of kinetic transport equations

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Mag. Pedro Aceves-Sánchez

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Kurzfassung

Die Arbeit widmet sich dem Studium des makroskopischen Grenzwert verschiedener kinetischer Gleichungen mit langsam abklingenden Gleichgewichtsverteilungen. Im klassischen Fall ist die Gleichgewichtsverteilung eine Maxwell-Verteilung. Für diesen Fall besagt ein bekanntes Resultat aus der Theorie der kinetischen Gleichungen, dass das asymptotische Verhalten einer linearen kinetischen Transportgleichung mit parabolischer Skalierung durch eine Wärmeleitungsgleichung beschrieben wird. Wird die Maxwell-Verteilung jedoch durch eine langsam abklingende Gleichgewichtsverteilung ersetzt, handelt es sich beim makroskopischen Limes um eine fraktionale Wärmeleitungsgleichung, Die Arbeit besteht aus vier voneinander unabhängigen Teilen, welche für eine Publikation eingereicht bzw. bereits akzeptiert wurden.

Im ersten Teil dieser Arbeit wird eine gestörte lineare kinetische Transportgleichung betrachtet, wobei eine ausgezeichnete Richtung im Konvektionsterm eingeführt wird. Eine mögliche Interpretation dieses Modells ist die Modellierung von Bakterien, welche unter Einfluss von Chemotaxis, in Regionen mit einer höheren Wirkstoffkonzentration gelockt werden. Dabei ist die Wirkstoffkonzentration orts- und zeitabhängig. Betrachtet man daher im Rahmen dieses Modells die Gleichgewichtsverteilung des gesamten Streuoperators, so besteht zusätzlich zur Geschwindigkeitsabhängigkeit eine Abhängigkeit von Ort und Zeit. Trotz dieser Schwierigkeiten sind wir in der Lage A-Priori-Abschätzungen herzuleiten, die es erlauben zum Grenzwert überzugehen und eine fraktionale Drift-Diffusions-Gleichung auf rigorose Weise abzuleiten.

Im zweiten Teil betrachten wir den Fall einer linearen Vlasov-Boltzmann Gleichung mit langsam abklingender Gleichgewichtsverteilung auf dem Ganzraum sowie einer orts- und zeitabhängigen äußeren Kraft. Es ist bekannt, dass der lineare Boltzmann-Streuoperator eine 'spectral gap' besitzt. Jedoch erhalten wir ebenfalls einen koerzitiven Operator, wenn wir eine externe Kraft hinzufügen. Dieses Ergebnis liefert uns geeignete A-priori-Abschätzungen, welche es ermöglichen, den makroskoptischen Limes auf rigorose Weise herzuleiten. Die Gleichung, die den Grenzwert beschreibt, ist eine fraktionale Wärmeleitungsgleichung mit einem advektiven Term. Für ein bestimmtes Abklingverhalten der Gleichgewichtsverteilung betrachten wir auch den Grenzwert im Fall starker Felder welcher als mikroskopischer Grenzwert eine Drift-Gleichung liefert.

Der dritte Teil behandelt die Berechnung des makroskopischen Limes einer linearen fraktionalen Vlasov-Fokker-Planck-Gleichung. Dabei ist die externe Kraft nicht mit einer Poisson-Gleichung gekoppelt. Vielmehr handelt es sich um eine gegebene orts- und zeitabhängige Funktion. Es ist eine wohlbekannte Tatsache, dass der Fokker-Planck-Operator koerzitiv ist. Allerdings können wir mit Hilfe der fraktionalem-Poincaré-Ungleichung auch die Koerzitivität für den fraktionalen Fokker-Planck-Operator nachweisen. Zusätzlich können wir die Koerzitivität auch für einen Operator zeigen, der sich aus dem fraktionalen Fokker-Planck-Operator und der externen Kraft zusammensetzt. Diese Eigenschaft sowie der Einsatz quadratischer Entropien erlaubt es, A-Priori-Abschätzungen herzuleiten. Mit diesen sowie einer geeignet gewählten Testfunktion gelingt es uns schließlich, den makroskopischen Limes rigoros zu bestimmen.

Im letzten Teil der Arbeit wird eine lineare kinetische Transportgleichung mit langsam abklingender Gleichgewichtsverteilung in einem glatt-berandeten Gebiet mit 'zero inflow'-Randbedingungen betrachtet. Zunächst werden A-Priori-Abschätzungen mit Hilfe quadratischer Entropien, welche aus der Koerzitivität des Streuoperators folgen, abgeleitet. Mit Hilfe einer Technik, die auch als Methode der Momente bezeichnet wird, kann schließlich der makroskopische Limes bestimmt werden. Dabei wird diese Methode so angepasst, dass auch die Randbedingungen berücksichtigt werden können. Insbesondere müssen die Testfunktion in der Nähe des Randes quadratisch abfallen. Im Falle

eines beschränkten konvexen Gebiets erinnert der makroskopische Grenzwert an eine fraktionale Wärmeleitungsgleichung mit einem eingeschränkten fraktionalen Laplace-Operator. Für nicht-konvexe Gebiete liefert der makroskopische Limes jedoch überraschenderweise einen vollkommen anderen Operator.

Abstract

This thesis is devoted to the study of macroscopic limits of various kinetic equations featuring a heavy tailed equilibrium distribution. In the classical case in which the equilibrium distribution is a Maxwellian it is a well-known result in kinetic theory that the asymptotic behavior of a linear kinetic equation with a parabolic scaling is governed by a heat equation. However, if the Maxwellian is replaced by an equilibrium distribution having a heavy tail then the macroscopic limit is a fractional heat equation.

This thesis consists of four independent works which have been submitted or accepted for publication. The first part of this work consist in the study of a perturbed linear kinetic transport equation in which a preferred direction is introduced into the perturbation term. One possible interpretation of this model is the modeling of bacteria which under chemotaxis choose to go to regions of higher chemo-attractant. The chemo-attractant concentration is space and time dependent. Therefore if we consider the equilibrium distribution of the whole scattering operator we shall have a space and time dependent equilibrium distribution in addition to the velocity dependence. However, we can overcome this difficulty and obtain a priori estimates which are used to pass to the limit and obtain a fractional-drift-diffusion equation in a rigorous manner.

In the second part we consider the case of a linear Vlasov-Boltzmann equation with a heavy tailed equilibrium distribution in the whole domain and with a given external force depending in space and time. It is a well-known fact that the linear kinetic scattering operator has a spectral gap, however, we also obtain a coercive operator if we add the external force term. This result give us appropriate a priori estimates which enable us to obtain the macroscopic limit in a rigorous manner. The limiting equation is a fractional heat equation with an advective term. In addition, under certain decay behavior of the equilibrium distribution we also consider the high-field limit obtaining as a macroscopic limit a drift equation.

The third part is concerned with obtaining the macroscopic limit of a linear Vlasov-fractional-Fokker-Planck equation where the external force is not coupled to a Poisson equation but it is given and it is space and time dependent. It is well-known that the Fokker-Planck operator is coercive, however, thanks to a fractional Poincaré inequality we also prove that the fractional-Fokker-Planck operator is coercive. In addition, we prove that if we introduce an operator consisting of the fractional-Fokker-Planck operator together with the external force term the coercivity property also holds. This property allow us to obtain a priori estimates using quadratic entropies. Using these a priori estimates and an auxiliary test function method permit us to obtain the macroscopic limit rigorously.

Finally, the fourth part consists in the study of a linear kinetic transport equation with heavy tailed equilibrium distribution in a smooth bounded domain with zero inflow boundary conditions. We obtain a priori estimates thanks to quadratic entropies derived from the coercivity property of the scattering operator. The macroscopic limit is established using a technique known as the moments method. This method is adapted in order to take into account the boundary conditions. In particular, the test functions considered have a quadratic decay at the boundary. The macroscopic limit is reminiscent of a fractional heat equation with a restricted fractional Laplace operator in a bounded convex domain. However, in a non-convex domain we unexpectedly obtain a completely different operator.

Contents

Introduction

> "When nothing seems to help, I go and look at a stonecutter hammering away at his rock, perhaps a hundred times without as much as a crack showing in it. Yet at the hundred and first blow it will split in two, and I know it was not that last blow that did it, but all that had gone before"
>
> Jacob A. Riis

In recent years the concept of anomalous diffusion has become a topic of intense study since it has been used as a tool for the description of many phenomena. Advances in tracking mechanisms have shed light in the way animals, bacteria, charged particles in plasmas, among many other phenomena move (see [49, 35, 34] and the references therein). In this thesis our main goal is in obtaining in a rigorous manner evolution equations with nonlocal operators as hydrodynamic limits of kinetic transport equations with heavy tailed equilibrium distributions. More precisely, when a suitable rescaling of a linear kinetic equation with heavy tailed equilibrium distribution is performed we are able to obtain an evolution equation for the macroscopic density in which a fractional Laplacian appears. The underlying stochastic phenomena of this nonlocal partial differential equations is connected with Lévy flights.

One way to describe the evolution of a large number of interacting particles is through Newton's law of motion. This is the so-called *microscopic* description of the system. Another possibility for the description of such a system is via the so-called *macroscopic* description in which the evolution of the density of the system is described (among other quantities such as the momentum and temperature). Yet another approach is a statistical description or also known as *mesoscopic* description. This approach was first introduced by J. C. Maxwell in [39] and L. Boltzmann [13] in their study of, respectively, the rings of Jupiter and the dynamic of gases, where a large system of interacting particles is involved. The idea consists in introducing a density function $f = f(x, v, t)$ (or probability density function) which gives the density of particles at time $t \geq 0$ and position $x \in \Omega$ (where Ω is a smooth bounded domain of $\mathbb{R}^d$ or the whole space $\mathbb{R}^d$) with a velocity $v \in \mathbb{R}^d$; the quantity $f(x, v, t)\,\mathrm{d}x\,\mathrm{d}v$ represents the number of particles in the volume element $\mathrm{d}x\,\mathrm{d}v$ with center $(x, v) \in \Omega \times \mathbb{R}^d$. Then the evolution of the density function f is described by the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f)$$

where the operator Q accounts for the interaction between the particles in a rarefied gas or as the scaterring by a background material (see *e.g.* [51, 42]). The need for this approach comes from the fact that the microscopic description has a high dimensionality. For instance, if we are interested in the description of the air in a room with typical room temperature we will have to describe the evolution of around 10^{20} molecules per $1cm^3$ and therefore it is intractable from a numerical point of view even for the highly efficient supercomputers of our modern era.

There are several phenomena in which the mesoscopic description has been applied with great success. For instance in the environment of space shuttles reentering earth the air is rarefied and therefore the Knudsen number is not small; thus the Navier-Stokes equations do not apply. However, kinetic transport equations have played an important role in the modeling of this phenomena (see [16]). Another example coming from biology is the study of bacterial movement in tubes filled with nutrients. In [47], the authors based on experimental data were able to identify the characteristic paramenters and proposed a kinetic equation that describes in a very accurate way not only the shape of the traveling waves observed in the experiments but the velocity of them as well. Many other applications as well as an up-to-date account on kinetic theory can be found in [51], [42], [29], [45] and the references therein.

Let us consider the following simple linear Boltzmann transport equation

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= \int_{\mathbb{R}^d} M(v) f(v') - M(v') f(v) \, \mathrm{d}v' \\ &= Q(f) \end{aligned} \tag{0.1}$$

where the function M is a Maxwellian, namely, $M(v) = C\mathrm{e}^{-|v|^2}$ and $C > 0$ is a normalization constant. To the best of our knowledge the first works dealing with hydrodynamic limits of (0.1) are [37], [30], [8] and [52]. These papers treat different aspects and generalizations of (0.1) and, in particular, they proved that under a parabolic scaling, namely, $(x, t) \mapsto (x/\varepsilon, t/\varepsilon^2)$, (0.1) takes the form

$$\varepsilon^2 \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \int_{\mathbb{R}^d} M(v) f_\varepsilon(v') - M(v') f_\varepsilon(v) \, \mathrm{d}v' \tag{0.2}$$

and when $\varepsilon \to 0$ the distribution function f_ε converges to ρM, in some sense, where ρ solves the heat equation

$$\partial_t \rho = \Delta \rho.$$

This limit can be performed in a rigorous way using a Hilbert's expansion, see for instance [24]. A key ingredient for obtaining in the limit the heat equation is the fact that $\int |v|^2 M(v) \, \mathrm{d}v < +\infty$ since the equilibrium distribution is a Maxwellian. However, in the case in which the second moment of the equilibrium distribution of the collision operator in (0.2) is not finite we can still recover a limit if we look at a different time-scale. This was done for the first time simultanously in [41] using analytic techniques and in [32] via a stochastic analysis approach. The technique introduced in [41] consists in taking the Fourier transform in the space variable and Laplace transform in the time variable. More precisely, in [41] it is assumed that the equilibrium distribution function $M(v)$ of (0.1) is such that

$$M(v) = M(-v), \quad M(v) > 0, \quad \int M \, \mathrm{d}v = 1, \qquad \text{for all } v \in \mathbb{R}^d, \tag{0.3}$$

$$M(v) \sim \frac{\kappa}{|v|^{d+\alpha}}, \qquad \text{as } |v| \to \infty, \tag{0.4}$$

where $\alpha \in (0, 2)$ and κ is a positive constant. Then the second moment is not finite and the parabolic scaling does not give a meaningful limit. However, it was proven in [41] that if instead we rescale as $(x, t) \mapsto (x/\varepsilon, t/\varepsilon^\alpha)$ then (0.1) takes the form

$$\varepsilon^\alpha \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \int_{\mathbb{R}^d} M(v) f_\varepsilon(v') - M(v') f_\varepsilon(v) \, \mathrm{d}v' \tag{0.5}$$

and we can identify the limit of f_ε, when $\varepsilon \to 0$, as $\rho(x, t) M(v)$ where ρ solves in the distributional sense

$$\partial_t \rho = -(-\Delta)^{\alpha/2} \rho.$$

The fractional Laplacian $(-\Delta)^{\alpha/2}$ is defined as

$$(-\Delta)^{\alpha/2}\rho(x) = c_{d,\alpha}\text{P.V.}\int_{\mathbb{R}^d} \frac{\rho(x)-\rho(y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y.$$

Therefore on the limit we obtain a nonlocal partial differential equation. The method used in [41] has the drawback of not being able to handle kinetic equations in which the cross-section is space or time dependent. In order to overcome this difficulty, in [5] a Hilbert expansion approach was introduced and in [40] a moments type method was used. The latter showed to be a very powerful tool in the study of hydrodynamic limits and it will be one of the essential techniques used in this thesis.

Another way to obtain a fractional heat equation as a macroscopic limit is by considering linear Boltzmann equations in which the collision frequency is degenerate. This was explored in [6] where a fractional heat equation is obtained after an appropriate rescaling in space and time. On the other hand, there have also been recent developments in the numerical solution of the rescaled linear Boltzmann equation (0.5). In [21] and [22] asymptotic preserving schemes were introduced.

Nonlocal operators have been recently the object of intense study due to its applications in areas such as Financial Mathematics [23], dislocation dynamics [14], spreading of tracers in water currents [7], criminology [20], among many others. Another reason why they have become popular is due to the fact that they offer new interesting mathematical problems which have deep connections with stochastic analysis. The difference between a nonlocal partial differential equation and a local one strives in the fact that for the local one only values in an arbitrarily small neighborhood need to be computed in contrast with a nonlocal one where points far out of the point of computation play a role. In this thesis we derive nonlocal evolution equations of drift-diffusion type where the diffusion is given by a fractional Laplacian. This type of equations play an important role for example in the study of conservation equations with fractional diffusion of the form

$$\partial_t u + D_x f(u) + (-\Delta)^{\alpha/2} u = 0$$

since in order to get a better understanding of the solutions its linearized version needs to be analyzed.

This thesis is divided into five Chapters, each one from the second to the fifth one consist of a publication or a submitted preprint. Therefore the notation is only consistent within each Chapter. Chapter 2 is a collaboration with Prof. C. Schmeiser in which we study a perturbation of a linear transport equation with the aid of two techniques: The Fourier-Laplace transform method introduced in [41] and the moments method introduced in [40]. More precisely, we consider the equation

$$\begin{aligned} \varepsilon^\alpha\,\partial_t f_\varepsilon + \varepsilon v\cdot\nabla_x f_\varepsilon &= Q_0(f_\varepsilon) + \varepsilon^{\alpha-1}Q_1(f_\varepsilon) \\ &=: Q_\varepsilon(f_\varepsilon) \end{aligned} \tag{0.6}$$

where

$$Q_0(f) := \int_{\mathbb{R}^d} Mf' - M'f\,\mathrm{d}v'$$

and

$$Q_1(f) := \int_{\mathbb{R}^d} [\Phi(v,v',c)Mf' - \Phi(v',v,c)M'f]\,\mathrm{d}v'.$$

We assume on M the following:

$$M > 0, \quad M \text{ is rotationally symmetric}, \quad \int_{\mathbb{R}^N} M(v)dv = 1,$$
$$M(v) = \frac{\gamma}{|v|^{N+\alpha}}, \qquad \text{for } |v| \geq 1, \quad 1 < \alpha < 2,\ \gamma > 0. \tag{0.7}$$

On the other hand, we assume that $c \in \left(W^{1,\infty}(\mathbb{R}^d \times [0,\infty))\right)^d$ and on Φ we make several assumptions depending on the method used, see Theorem 1.1 of Chapter 3. For instance, for the moments method we assume that $(1 + |v| + |v'|)\Phi$ is bounded and Lipschitz continuous with respect to c. A possible motivation for the collision kernel of the linear transport equation (0.6) could be the modeling via kinetic transport equations of the chemotactic motility of microorganisms driven by gradients of chemo-attractors. The modeling of the movement of microorganisms through kinetic equations was started by the pioneering works [2] and [44]. For a recent account of different approaches in the modeling of microorganisms consult [38]. One of the most well studied microorganisms is the bacterium Escherichia coli, whose swimming pattern can be described as a run-and-tumble process [9, 10]. This processes are characterized by periods of straight running alternated with periods of instantaneous reorientation (or tumbling). In the presence of a spatial chemo-attractant gradient, this stochastic process is biased upwards the gradient, albeit E. coli is too small it is believed that it measures gradients in time along its path and increases its tumbling frequency if it experiences decreasing chemo-attractant concentrations. This mechanism is believed to cause the desired drift. In recent years, the progress in the mechanisms for tracking particle trajectories [4, 33, 3] suggests that certain microorganisms behave following a so-called Lévy flight. In particular, there is evidence that the bacteria E. coli adopts a Lévy flight type movement when there is scarcity of food resources [53]. The Lévy flight type movement is encoded in the kinetic equation through the heavy tailed function M, whereas the drift caused by the chemo-attractant gradient is described by the function $\Phi(v, v', c)$, depending on the velocities v' and v before and, respectively, after the jump, and on the gradient c.

In order to find the appropriate a priori estimates needed to pass to the limit in (0.6) we use the coercivity of the collision operator $-Q_\varepsilon$ defined in (0.6). This property of the operator $-Q_\varepsilon$ is proven following a similar line of arguments as in [26]. Let us note that in contrast to [19] where only the coercivity of the operator $-Q_0$ is enough to obtain a priori estimates, in Chapter 3 we need to use the coercivity of the operator Q_ε.

Since we assume that the gradient c is space and time dependent the equilibrium distribution function denoted as F_ε is also expected to be space and time dependent in addition to the v dependence. The existence and uniqueness of the equilibrium distribution function F_ε such that $\int F_\varepsilon \,\mathrm{d}v = 1$ and $F_\varepsilon > 0$ is proven using either a fix point argument or the Krein-Rutman Theorem. Multiplying (0.6) by $f_\varepsilon/F_\varepsilon$ and integrating by parts we obtain

$$\begin{aligned}\frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon\|^2_{L^2(dv\,\mathrm{d}x/F_\varepsilon)} &= -\frac{\varepsilon^\alpha}{2}\int\int \frac{\partial_t F_\varepsilon}{F_\varepsilon}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x + \frac{\varepsilon}{2}\int\int \frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x \\ &\quad + \int\int Q_\varepsilon(f_\varepsilon)\frac{f_\varepsilon}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Using the coercivity of the collision operator $-Q_\varepsilon$ together with the bounds

$$\left\|\frac{\partial_t F_\varepsilon}{F_\varepsilon}\right\|_\infty, \left\|\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon}\right\|_\infty \leq \varepsilon^{\alpha-1}\lambda,$$

where $\lambda > 0$ and independent of ε, imply uniform bounds on the quadratic entropy: $\sup_{t\in[0,T]}\|f_\varepsilon(\cdot,\cdot,t)\|_{L^2(\,\mathrm{d}v\,\mathrm{d}x/F_\varepsilon)} \leq C(T) < \infty$ for every $T > 0$ arbitrarily big but finite.

However, the norm is ε dependent through the equilibrium distribution F_ε, thus noting that there exists $\mu_1, \mu_2 > 0$ such that $\mu_1 M \leq F_\varepsilon \leq \mu_2 M$, uniformly in x, v and t, we obtain the a priori estimates with the ε independent norm $\|\cdot\|_{L^2_{M^{-1}}(\mathbb{R}^{2d})}$. On the other hand, thanks to the coercivity of $-Q_\varepsilon$ and the estimate $|F_\varepsilon - M| \leq \varepsilon^{\alpha-1}CM$, where $C > 0$, we obtain a priori estimates also on $(r_\varepsilon := \varepsilon^{1-\alpha}(f_\varepsilon - \rho_\varepsilon M))$. From the a priori estimate on (f_ε) we can easily derive a priori estimates for (ρ_ε) and using the moments method of [40] we pass to the limit and prove that $f_\varepsilon \rightharpoonup^\star \rho(x,t)M(v)$, as $\varepsilon \to 0$ in $L^\infty(0,T; L^2_{M^{-1}}(\mathbb{R}^{2d}))$ where ρ solves in the distributional sense

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u(c)) + A(-\Delta)^{\alpha/2}\rho = 0, \\ &\rho(x,0) = \rho^{in}(x) := \textstyle\int_{\mathbb{R}^d} f^{in}(x,v)\,\mathrm{d}v, \end{aligned} \tag{0.8}$$

with

$$A = \gamma \int_{\mathbb{R}^d} \frac{w_1^2 |w|^{-d-\alpha}}{1 + w_1^2}\,\mathrm{d}w, \qquad u(c) = \int Q_1(M) v\,\mathrm{d}v,$$

and γ is defined in (0.7). Let us note that in contrast to (0.11) the drift term in (0.8) is due to the bias of the vector field c which appers inside the cross-section of the scattering operator Q_ε.

The third Chapter consist of the preprint [1] and is a joint work with Prof. Antoine Mellet. We deal with the question of obtaining the macroscopic limit of a rescaled Vlasov-Boltzmann equation. More precisely, let us consider the following rescaled kinetic transport equation

$$\varepsilon^\alpha \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon + \varepsilon^{\alpha-1} E \cdot \nabla_v f_\varepsilon = \int_{\mathbb{R}^d} \sigma(v,v')M(v)f_\varepsilon(v') - \sigma(v',v)M(v')f_\varepsilon(v)\,\mathrm{d}v' \tag{0.9}$$

where $1 \leq \alpha < 2$, the cross-section σ is such that

$$0 < \sigma_1 \leq \sigma(v,v') = \sigma(v',v) \leq \sigma_2,$$

the equilibrium distribution M satisfies (0.3)-(0.4) and the external force E belongs to the space $[W^{1,\infty}([0,\infty) \times \mathbb{R}^d)]^d$. In [40] the problem of studying the asymptotic behavior of the solution f_ε was posed and left open; in [1] we solved this problem. One of the key ingredients is to introduce the operator

$$\mathcal{T}_\varepsilon(f) = -Q(f) + \varepsilon^{\alpha-1} E \cdot \nabla_v f. \tag{0.10}$$

It is a well known fact that the operator Q is a coercive operator with a spectral gap in the space $L^2_{M^{-1}}$ (see [26]). Moreover, we proved that the operator $\mathcal{T}_\varepsilon$ is also coercive and has a spectral gap in the space $L^2_{F_\varepsilon^{-1}}(\mathbb{R}^d)$ where $F_\varepsilon > 0$, has mass one and spans the kernel of the operator $\mathcal{T}_\varepsilon$. Using this property of the operator $\mathcal{T}_\varepsilon$ we are able to obtain appropriate a priori estimates on the quadratic entropy of the solution f_ε. More precisely, we obtain that (f_ε) is uniformly bounded in ε in $L^\infty((0,T); L^2_{F_\varepsilon^{-1}}(\mathbb{R}^{2d}))$ for finite but arbitrarily big $T > 0$. Another key observation is to note that F_ε is bounded above and below by a constant times M and therefore the uniform estimate also holds in the space $L^\infty((0,T); L^2_{M^{-1}}(\mathbb{R}^{2d}))$. Then up to a subsequence we show that $f_\varepsilon \rightharpoonup^\star \rho(x,t)M(v)$ for the noncritical case $1 < \alpha < 2$, whereas for the critical case $\alpha = 1$ we get that $f_\varepsilon \rightharpoonup^\star \rho(x,t)F(v - E(x,t))$ in $L^\infty(0,T; L^2_{M^{-1}}(\mathbb{R}^{2d}))$. On the other hand, for the case $\alpha = 2$ it turns out that the correct scaling is

$$\varepsilon^2 \ln(\varepsilon^{-1})\partial_t f_\varepsilon + \varepsilon v \cdot \nabla_v f_\varepsilon + \varepsilon^{\alpha-1} \ln(\varepsilon^{-1}) E \cdot \nabla_v f_\varepsilon = Q(f_\varepsilon).$$

Then the asymptotic behavior of f_ε gives on the limit the following drift-diffusion equation for the macroscopic density ρ:

$$\partial_t \rho + \nabla_x (E\rho) - \Delta \rho = 0. \tag{0.11}$$

Finally another result in Chapter 3 is that if we choose $\alpha \in (1, 2)$ to be the asymptotic behavior of M as given in (0.4) then we study the high field asymptotic regime:

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon} E \cdot \nabla_v f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon).$$

In this case we obtain that $f_\varepsilon \rightharpoonup^\star \rho(x,t)F(v-E(x,t))$ where ρ solves in the distributional sense

$$\partial_t \rho + \operatorname{div}(E\rho) = 0.$$

On Chapter 4 we deal with the macroscopic limit of the Vlasov-fractional-Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = -(-\Delta)^{\alpha/2}(f) + \nabla_v (vf) \tag{0.12}$$

where $E \in (W^{1,\infty}([0,\infty) \times \mathbb{R}^d))^d$ and $\alpha \in [1, 2]$. Let us note that the case $\alpha = 2$ corresponds to the case of the Vlasov-Fokker-Planck equation where the external force is a given function. We adapt the techniques introduced in [15] and [25] to prove that there exists a unique solution f of (0.12) in $L^2((0,T) \times \mathbb{R}^d; H_v^{\alpha/2}(\mathbb{R}^d))$ for any finite but arbitrarily big $T > 0$. One of the main difficulties in comparison to the works [15, 25] is due to the nonlocal character of the fractional Laplacian.

Then we study the asymptotic behavior as $\varepsilon \to 0$ of the solution f_ε of

$$\varepsilon^\alpha \, \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon + \varepsilon^{\alpha-1} E \cdot \nabla_v f_\varepsilon = -(-\Delta)^{\alpha/2}(f_\varepsilon) + \nabla_v (v f_\varepsilon). \tag{0.13}$$

Let us note that the collision operator defined as

$$\mathcal{L}^\alpha(f) := -(-\Delta)^{\alpha/2}(f_\varepsilon) + \nabla_v (v f_\varepsilon) \tag{0.14}$$

has an equilibrium distribution that we denote by G_α, see [11]. Moreover, the operator $-\mathcal{L}^\alpha$ has a spectral gap thanks to the fractional Poincaré inequality (or also known as Φ-entropy inequality, see [28])

$$-\int_{\mathbb{R}^d} \mathcal{L}^\alpha(f) \frac{f}{G_\alpha} \, \mathrm{d}v \geq C \int_{\mathbb{R}^d} (f - \rho_f G_\alpha) \frac{\mathrm{d}v}{G_\alpha}. \tag{0.15}$$

Introducing the operator

$$\mathcal{B}_\varepsilon(f) := -\mathcal{L}^\alpha(f) + \varepsilon^{\alpha-1} E \cdot \nabla_v f$$

we prove that there exists a unique function $F_\varepsilon > 0$ such that

$$\mathcal{B}_\varepsilon(F_\varepsilon) = 0, \qquad \int_{\mathbb{R}^d} F_\varepsilon \, \mathrm{d}v = 1,$$

and $F_\varepsilon(x,v,t) = G_\alpha(v - \varepsilon^{\alpha-1}E(x,t))$. Moreover, the operator $\mathcal{B}_\varepsilon$ also has a spectral gap:

$$\int_{\mathbb{R}^d} \mathcal{B}_\varepsilon(f) \frac{f}{F_\varepsilon} \, \mathrm{d}v \geq C \int_{\mathbb{R}^d} (f - \rho_f F_\varepsilon)^2 \frac{\mathrm{d}v}{F_\varepsilon}.$$

One of the main difficulties in comparison to [18] is that the equilibrium distribution of $\mathcal{B}_\varepsilon$ depends on x, v and t, and therefore, we need to estimate $\partial_t F_\varepsilon$ and $\nabla_x F_\varepsilon$ in order to get a priori estimates. These estimates strive on the fact that the density function G_α is the probability density function of an α-stable process and using results of [12] we can control F_ε in terms of G_α and the terms $\partial_t F_\varepsilon$, $\nabla_v F_\varepsilon$ and $\nabla_x F_\varepsilon$ can be estimated by $\nabla_v G_v$. Thus multiplying (0.13) by f/F_ε, using the coercivity and spectral gap property of the operator $\mathcal{B}_\varepsilon$ we obtain thanks to Gronwall's inequality the necessary a priori estimates which imply that $f_\varepsilon \rightharpoonup^\star \rho(x,t)G_\alpha(v)$ in $L^\infty(0,T;L^2_{G_\alpha^{-1}}(\mathbb{R}^{2d}))$ where ρ is the weak-$\star$ limit of a subsequence of (ρ_ε) in $L^\infty(0,T;L^2(\mathbb{R}^d))$ for $\alpha \in (1,2]$. Moreover, we also obtain that $(r_\varepsilon := \varepsilon^{-\alpha/2}[f_\varepsilon - \rho_\varepsilon F_\varepsilon])$ is uniformly bounded with respect to ε in $L^2_{G_\alpha^{-1}}((0,T)\times\mathbb{R}^{2d})$. Next we use the auxiliary test function method introduced in [18] consisting in considering the auxiliary equation

$$\begin{aligned} \varepsilon v\cdot\nabla_x\psi_\varepsilon - v\cdot\nabla_v\psi_\varepsilon &= 0 && \text{in } \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^+,\\ \psi(x,v,0) &= \varphi(x,t) && \text{in } \mathbb{R}^d\times\mathbb{R}^+, \end{aligned} \tag{0.16}$$

where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d\times[0,\infty))$, which can be readily integrated via the method of characteristics yielding $\psi_\varepsilon(x,v,t) = \varphi(x+\varepsilon v,t)$. Multiplying (0.13) by ψ_ε and using the a priori estimates obtained for (f_ε), (ρ_ε) and the residue (r_ε) we obtain that ρ satisfies in the distributional sense

$$\begin{aligned} \partial_t\rho + \nabla_x\cdot(E\rho) + (-\Delta)^{\alpha/2}\rho &= 0 && \text{in } \mathbb{R}^d\times\mathbb{R}^+,\\ \rho(x,0) &= \rho^{in}(x) && \text{in } \mathbb{R}^d. \end{aligned} \tag{0.17}$$

On the other hand, for the case $\alpha = 1$ we obtain that $f_\varepsilon \rightharpoonup^\star \rho(x,t)G_1(v-E(x,t))$ in $L^\infty(0,T;L^2_{G_1^{-1}}(\mathbb{R}^{2d}))$ and the limiting equation satisfied by the density function ρ in the sense of distributions is

$$\begin{aligned} \partial_t\rho + \nabla_x(E\rho) + (-\Delta)^{1/2}\rho &= 0 && \text{in } \mathbb{R}^d\times\mathbb{R}^+,\\ \rho(x,0) &= \rho^{in}(x) && \text{in } \mathbb{R}^d. \end{aligned} \tag{0.18}$$

Finally, in Chapter 5 we consider the derivation of a fractional heat equation from a linear transport equation, both equations posed spatially in a smooth bounded domain with zero inflow boundary conditions on the kinetic equation. There are several equivalent definitions of the fractional Laplacian in the whole domain (see for instance [36] where ten equivalent definitions are given). However, in a bounded domain there are several possibilities. Each definition is linked with a stochastic process. For instance, let us consider the so-called restricted fractional Laplacian denoted as $(-\Delta\mid_\Omega)^{\alpha/2}$ and defined as

$$(-\Delta\mid_\Omega)^{\alpha/2}f(x) := c_{d,\alpha}\text{P.V.}\int_{\mathbb{R}^d}\frac{f(x)-f(y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y$$

where $c_{d,\alpha} > 0$, $f \in C_c^\infty(\Omega)$ and $f \equiv 0$ in $\mathbb{R}^d \setminus \Omega$ where Ω is a smooth bounded subset of $\mathbb{R}^d$. For an up-to-date account of fractional Laplacian operators see [43]. For the operator $(-\Delta\mid_\Omega)^{\alpha/2}$ the stochastic process underlying it is an α-stable stochastic process taking place inside Ω and killed upon leaving Ω. Another possibility is the so-called spectral fractional Laplacian, see for instance [54] for its definition and [50] for the stochastic process connected with it. In [48] it was proven that this two definitions are different since, for instance, the eigenfunctions of the restricted fractional Laplacian are no better than Hölder continuous up to the boundary whereas for the spectral fractional Laplacian the eigenfunctions are smooth up to the boundary. This result is of paramount importance since in many articles both definitions were used interchangeably leading to false results. In Chapter 5 we introduce a new definition of fractional Laplacian in a bounded domain. This definitions arises naturally as the macroscopic limit of a kinetic equation posed in a bounded domain with zero inflow boundary conditions and it coincides with restricted fractional Laplacian in bounded convex domains. The method to obtain the macroscopic limits relies in the moments method introduced in [40] with a particular choice of the test functions and a priori estimates based on quadratic entropies. We conclude with a discussion relating our result with [27] were the behavior of the solution of the fractional heat equation with restricted fractional Laplacian is discussed. Finally, let us mention that along the lines of this work the only other work that we are aware of, to the best of our knowledge, is [17] where a fractional heat equation is obtained as a macroscopic limit of a kinetic fractional-Fokker-Planck equation in a circle with specular and absorption boundary conditions. The macroscopic limit for the absorption case corresponds to the restricted fractional Laplacian and thus is different from the macroscopic limit obtained by us.

Declaration of authorship

Chapter 2 is a joint work with Prof. Christian Schmeiser and it has been accepted for publication in the SIAM Journal for Mathematical Analysis which will be issued in 2016. Chapter 3 consists of a joint work with Prof. Antoine Mellet from the University of Maryland and it has been submitted for publication. On the other hand, Chapter 4 was the result of a joint project with Ludovic Cesbron from Cambridge University and it has been submitted. Finally, Chapter 5 is a collaboration with Prof. Christian Schmeiser and it has also been submitted. All of these works are based on discussions and exchange of ideas with my co-authors.

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Fractional-diffusion-advection limit of a kinetic model

Pedro Aceves-Sánchez and Christian Schmeiser

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Abstract

A fractional diffusion equation with advection term is rigorously derived from a kinetic transport model with a linear turning operator, featuring a fat-tailed equilibrium distribution and a small directional bias due to a given vector field. The analysis is based on bounds derived by relative entropy inequalities and on two recently developed approaches for the macroscopic limit: a Fourier-Laplace transform method for spatially homogeneous data and the so called moment method, based on a modified test function.

1 Introduction

The goal of this paper is to study the limit as $\varepsilon \to 0$ of the distribution function $f_\varepsilon(x,v,t)$ (depending on position $x \in \mathbb{R}^N$, velocity $v \in \mathbb{R}^N$, and time $t \geq 0$), solving the kinetic Cauchy problem

$$\begin{aligned} \varepsilon^\alpha \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon &= Q_\varepsilon(f_\varepsilon) \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^+ , \\ f_\varepsilon(t=0) &= f^{in} \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N , \end{aligned} \tag{1.1}$$

where the linear collision operator is given by

$$Q_\varepsilon(f) = \int_{\mathbb{R}^N} (T_\varepsilon(v' \to v, x, t) f' - T_\varepsilon(v \to v', x, t) f) dv' , \tag{1.2}$$
$$\text{with } T_\varepsilon(v' \to v, x, t) = \left(1 + \varepsilon^{\alpha-1} \Phi(v, v', c(x,t))\right) M(v) ,$$

with the prescribed vector field $c(x,t) \in \mathbb{R}^N$, and with the equilibrium distribution $M(v)$ with the properties

$$M > 0 , \quad M \text{ is rotationally symmetric,} \quad \int_{\mathbb{R}^N} M(v) dv = 1 ,$$
$$M(v) = \frac{\gamma}{|v|^{N+\alpha}} , \qquad \text{for } |v| \geq 1 , \quad 1 < \alpha < 2 , \ \gamma > 0 . \tag{1.3}$$

The decay property is responsible for the choice of the scaling in (1.1), which will turn out to be significant in the following. Note that M has finite first order but not second order moments. As usual in kinetic theory, f' denotes evaluation at v'.

The collision operator can be written as $Q_\varepsilon = Q_0 + \varepsilon^{\alpha-1} Q_1$ with the dominating, directionally unbiased relaxation operator

$$Q_0(f) = \rho_f M - f , \qquad \rho_f := \int f \, dv ,$$

and the turning operator

$$Q_1(f) = \int [\Phi(v, v', c) M f' - \Phi(v', v, c) M' f] dv' ,$$

supposed to bias velocity changes towards the direction given by c. Here and in the following, $\mathrm{d}v$, $\mathrm{d}v'$, and $\mathrm{d}x$ denote the Lebesgue measure on $\mathbb{R}^N$ and $\mathrm{d}t$ the Lebesgue measure on $\mathbb{R}^+$, which always have to be understood as the integration domains, except stated otherwise. In the scaling process, the ratio of characteristic times between the biased and unbiased velocity jump mechanisms has been denoted by $\varepsilon^{\alpha-1}$, and then macroscopic length and time scales have been introduced.

A possible motivation for the model (1.1) is the description of ensembles of motile microorganisms, subject to a chemical signal encoded in the vector field c, which might be interpreted as the spatial gradient of a chemo-attractant. One of the best studied microorganisms is the bacterium *Escherichia coli*, whose swimming pattern can be described as a run-and-tumble process [5, 6], characterized by periods of straight running alternated with (much shorter) periods of reorientation (or tumbling). Under the idealizing assumption of instantaneous velocity jumps, this can be described stochastically by kinetic transport equations, which have been introduced as models for microorganisms in the pioneering works [1] and [18]. In the presence of a chemo-attractant gradient, the velocity jump process is biased, which is described by the function $\Phi(v, v', c)$, depending on the velocities v' and v before and, respectively, after the jump, and on the gradient c.

From a macroscopic point of view (where length and time scales are large compared to individual runs), a standard description of the resulting motility is by Brownian motion with a drift. On the other hand, the recent progress in tracking individual trajectories [2, 3, 9, 12, 16] allowed to show that the movement of certain microorganisms is better described by a so-called Lévy flight. In particular, there is evidence that *E. coli* adopts a Lévy flight type movement when there is scarcity of food resources [23]. Macroscopically, Lévy flights show a scaling behavior different from Brownian motion, where the average displacement scales with the square root of time, a behavior called fractional Brownian motion. In the model considered here, this kind of behavior is described by a high probability of larger velocities encoded in the fat tail (1.3) of the equilibrium distribution M.

For an equilibrium distribution M with finite second order moments, the scaling with $\alpha = 2$ would be appropriate, and the macroscopic limit $\varepsilon \to 0$ would lead to a convection diffusion equation for the limit of the macroscopic density ρ_f (see, e.g., [7]). On the other hand it has been shown in [4, 14, 15] that with the assumption (1.3) and with $Q_1 = 0$ the macroscopic limit leads to a fractional diffusion equation; see also [11], where this has been carried out via a probabilistic approach. Fractional diffusion equations with advection have been the object of intense study in recent years. Issues such as regularity have been addressed by many authors, most notably by Silvestre and co-workers [20, 21]. The problem of a rigorous derivation of a fractional diffusion equation with convection from kinetic models has been posed, but left open in [14]. This is the purpose of the present work.

Two different methods will be used, leading to results with slightly different assumptions on the data. The Laplace-Fourier transform approach of [15] can only be used in the case of constant c. On the other hand, it requires milder assumptions on the turning rate Φ than the moment method of [14]. In these works, the coercivity properties of the leading order collision operator Q_0 are the essential ingredient for obtaining estimates uniform with respect to ε. The important contribution of the present work is to employ the equilibrium distribution of the full collision operator Q_ε and a corresponding entropy dissipation property. The latter holds although detailed balance is not required, as has first been shown in [8] and actually is known now as a general result for generators of Markov processes [10], like Q_ε, which is obviously preserving positivity and conserving mass:

$$\int Q_\varepsilon(f)\,\mathrm{d}v = 0\,.$$

Fractional diffusion is generated by a fractional power of the Laplacian (*fractional Laplacian*), which can be defined via the Fourier transform $\mathcal{F}$ as a multiplication operator in Fourier coordinates,

$$\mathcal{F}((-\Delta)^{\alpha/2}\rho)(k) := |k|^\alpha \mathcal{F}(\rho)(k)\,, \tag{1.4}$$

or as a singular integral,

$$(-\Delta)^{\alpha/2}\rho(x) = c_{N,\alpha}\ \text{P.V.}\int_{\mathbb{R}^N} \frac{\rho(x)-\rho(y)}{|x-y|^{N+\alpha}}\,\mathrm{d}y\,, \tag{1.5}$$

where P.V. denotes the Cauchy principal value, and

$$c_{N,\alpha} = \Gamma(\alpha+1)\left(\int_{\mathbb{R}^N} \frac{w_1^2|w|^{-N-\alpha}}{1+w_1^2}\,\mathrm{d}w\right)^{-1}\,,$$

with the Gamma function Γ. The value of $c_{N,\alpha}$ will be verified by our results below. Note that for $\alpha > 1$ a principal value can be avoided by the equivalent representation

$$(-\Delta)^{\alpha/2}\rho(x) = c_{N,\alpha}\int_{\mathbb{R}^N} \frac{\rho(x)-\rho(y)-(x-y)\cdot\nabla_x\rho(x)}{|x-y|^{N+\alpha}}\,\mathrm{d}y\,.$$

For a detailed discussion of the properties of the fractional Laplacian consult [13, 17, 22]. We only note that it is formally self-adjoint, which is a straightforward consequence of both representations (1.4) and (1.5).

The main result of this work is the rigorous validity of the macroscopic limit $\varepsilon \to 0$:

Theorem 1.1. *Let* $f^{in} \in L^2(\mathrm{d}v\,\mathrm{d}x/M)$, $(1+|v|)f^{in} \in L^1_+(\mathrm{d}v\ \mathrm{d}x)$, *and either*

Assumption A: $c = \text{const} \in \mathbb{R}^N\,,\quad (1+|v|+|v'|)\Phi$ *is bounded, or*

Assumption B: $c = \text{const} \in \mathbb{R}^N\,,\quad \Phi$ *is bounded,* $\int \Phi(v',v,c)M'dv' = 0\,,$ *or*

Assumption C: $c \in W^{1,\infty}(\mathrm{d}x\,\mathrm{d}t)^N\,,\quad (1+|v|+|v'|)\Phi$ *is bounded and Lipschitz continuous with respect to* $c\,,$

Then there exists $\rho \in L^\infty_{loc}(\mathrm{d}t; L^2(\mathrm{d}x))$, *such that the solution* f_ε *of (1.1) converges, as* $\varepsilon \to 0$, *to* ρM *in* $L^\infty_{loc}(\mathrm{d}t; L^2(dv\,dx/M))$ *weak*, and* ρ *solves in the distributional sense the Cauchy problem*

$$\begin{aligned}&\partial_t\rho + \nabla_x\cdot(\rho u(c)) + A(-\Delta)^{\alpha/2}\rho = 0\,,\\ &\rho(t=0) = \rho^{in} := \textstyle\int f^{in}dv\,.\end{aligned} \tag{1.6}$$

with

$$A = \gamma\int_{\mathbb{R}^N} \frac{w_1^2|w|^{-N-\alpha}}{1+w_1^2}\,\mathrm{d}w\,,\qquad u(c) = \int Q_1(M)v\,\mathrm{d}v\,.$$

The main parts of the proof will be given in Sections 4 (Assumptions A and B) and 5 (Assumption C), after presentation of the formal macroscopic limit for a simple model problem in Section 2, and the derivation of several uniform (in the small parameter ε) bounds on the solution of (1.1) in Section 3.

2 Formal asymptotics of a simple model

In this section the Cauchy problem (1.1), (1.2) is considered with constant $c \in \mathbb{R}^N$ and with the turning kernel

$$\Phi(v,v',c) = c\cdot\frac{v}{|v|}\,. \tag{2.7}$$

This means that the turning rate is independent of the incoming velocity v' and prefers outgoing velocities v in the direction of c. The Fourier-Laplace approach to the macroscopic limit will be carried out formally, deferring a rigorous justification for the general form of the turning kernel satisfying Assumption A or B of Theorem 1.1 to Section 6. Note that (2.7) does not satisfy Assumption A, but Assumption B by its oddness with respect to the first variable.

We introduce the Fourier transformation $\mathcal{F}$ with respect to x and the Laplace transformation $\mathcal{L}$ with respect to t,

$$(\mathcal{F}f)(k) := \int \mathrm{e}^{-ik\cdot x} f(x)\,\mathrm{d}x\,, \qquad (\mathcal{L}f)(p) := \int_0^\infty \mathrm{e}^{-pt} f(t)\,\mathrm{d}t\,, \qquad p>0\,, \quad k\in\mathbb{R}^N\,,$$

and define the Fourier-Laplace transform

$$\widehat{f_\varepsilon} := \mathcal{L}\mathcal{F} f_\varepsilon\,.$$

Taking the Fourier-Laplace transform of (1.1) with the turning kernel (2.7) yields

$$\varepsilon^\alpha p\widehat{f_\varepsilon} - \varepsilon^\alpha \mathcal{F}f^{in} + \varepsilon i v\cdot k\widehat{f_\varepsilon} = M\left(1+\varepsilon^{\alpha-1}c\cdot v/|v|\right)\widehat{\rho_\varepsilon} - \widehat{f_\varepsilon}\,, \tag{2.8}$$

with $\rho_\varepsilon := \rho_{f_\varepsilon}$, where the evenness of M has been used. This can be rewritten as

$$\widehat{f_\varepsilon} = \frac{\varepsilon^\alpha\mathcal{F}f^{in}}{1+\varepsilon^\alpha p+\varepsilon i v\cdot k} + \frac{M\left(1+\varepsilon^{\alpha-1}c\cdot v/|v|\right)\widehat{\rho_\varepsilon}}{1+\varepsilon^\alpha p+\varepsilon i v\cdot k}\,. \tag{2.9}$$

Integration with respect to v leads to a closed equation for $\widehat{\rho_\varepsilon}$ (a consequence of the simple form of the model problem), which can be written in the form

$$\left(\int \frac{\varepsilon^\alpha p + \varepsilon^{2\alpha}p^2 + \varepsilon^2(v\cdot k)^2 + \varepsilon i v\cdot k}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2} M\,\mathrm{d}v \right.$$
$$\left. - \int \frac{\varepsilon^{\alpha-1}c\cdot v/|v|(1+\varepsilon^\alpha p - \varepsilon i v\cdot k)}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2} M\,\mathrm{d}v\right)\widehat{\rho_\varepsilon} = \int \frac{\varepsilon^\alpha\mathcal{F}f^{in}}{1+\varepsilon^\alpha p+\varepsilon i v\cdot k}\,\mathrm{d}v$$

Again by the evenness of M, the imaginary part of the first integral and the real part of the second integral on the left hand side vanish. Therefore, division by ε^α gives

$$\left(\int \frac{p+\varepsilon^\alpha p^2+\varepsilon^{2-\alpha}(v\cdot k)^2}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2} M\,\mathrm{d}v + i\int \frac{c\cdot v/|v|(v\cdot k)}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2} M\,\mathrm{d}v\right)\widehat{\rho_\varepsilon}$$
$$= \int \frac{\mathcal{F}f^{in}}{1+\varepsilon^\alpha p+\varepsilon i v\cdot k}\,\mathrm{d}v \tag{2.10}$$

Now we are prepared for formally passing to the limit, which is easy for all terms except one (because of the nonexistence of second order moments of M):

$$\int \frac{\varepsilon^{2-\alpha}(v\cdot k)^2}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2} M\,\mathrm{d}v = O(\varepsilon^{2-\alpha}) + \varepsilon^{2-\alpha}\gamma\int_{|v|>1} \frac{(v\cdot k)^2|v|^{-N-\alpha}}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2}\,\mathrm{d}v$$

With the coordinate transformation $v = (w_1 k/|k| + w^\perp)/(\varepsilon|k|)$ (a stretching and a rotation), it becomes clear that the right hand side converges as $\varepsilon\to 0$ to

$$A|k|^\alpha\,, \qquad \text{with } A = \gamma\int_{\mathbb{R}^N} \frac{w_1^2|w|^{-N-\alpha}}{1+w_1^2}\,\mathrm{d}w > 0\,.$$

For the computation of the limit of the imaginary term in (2.10), the rotational symmetry of M is used. For the formal limit ρ of ρ_ε we obtain

$$(p + A|k|^\alpha + iBc\cdot k)\,\widehat{\rho} = \mathcal{F}\rho^{in} = \int \mathcal{F}f^{in}\,\mathrm{d}v\,, \qquad \text{with } B = \frac{1}{N}\int |v|M(v)\,\mathrm{d}v > 0\,.$$

This is the Fourier-Laplace transformed version of the Cauchy problem

$$\partial_t\rho + \nabla_x\cdot(\rho Bc) + A(-\Delta)^{\alpha/2}\rho = 0\,, \qquad \rho(t=0) = \rho^{in}\,,$$

i.e. (0.8) with

$$u(c) = Bc = \int Q_1(M)v\;\mathrm{d}v\,, \qquad Q_1(M) = c\cdot\frac{v}{|v|}M\,.$$

3 Uniform bounds

The derivation of bounds uniform in the small parameter ε is based on the equilibrium distribution $F_\varepsilon(v;x,t)$ of the full collision operator $Q_\varepsilon = Q_0+\varepsilon^{\alpha-1}Q_1$, defined as solution of the problem

$$Q_\varepsilon(F_\varepsilon) = 0\,, \qquad \int F_\varepsilon\,\mathrm{d}v = 1\,. \tag{3.11}$$

In this problem, x and t play the role of parameters, present through the dependence of Q_1 on the vector field $c(x,t)$.

Lemma 3.1. *Let the assumptions of Theorem 1.1 hold. Then, for $\varepsilon > 0$ small enough, the problem* (3.11) *has a unique solution F_ε satisfying*

$$\frac{1-\varepsilon^{\alpha-1}\overline{\Phi}}{1+\varepsilon^{\alpha-1}\overline{\Phi}} \le \frac{F_\varepsilon}{M} \le \frac{1+\varepsilon^{\alpha-1}\overline{\Phi}}{1-\varepsilon^{\alpha-1}\overline{\Phi}}\,, \tag{3.12}$$

where $\overline{\Phi}$ is an upper bound for the modulus $|\Phi|$ of the turning kernel. Furthermore,

$$\left|\frac{\partial_t F_\varepsilon}{F_\varepsilon}\right|, \left|\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon}\right| \le \varepsilon^{\alpha-1}\lambda\,, \tag{3.13}$$

with the constant λ independent of ε, and $\lambda = 0$ under Assumptions A and B.

Proof. The existence and uniqueness result follows from the appendix of [8] or from [19], but it can also be easily derived by contraction, using the fixed point formulation for $G_\varepsilon = \frac{F_\varepsilon}{M} \in L^\infty(\,\mathrm{d}v)$.

$$G_\varepsilon(v) = \frac{1+\varepsilon^{\alpha-1}\int\Phi(v,v',c)M'G_\varepsilon'\,\mathrm{d}v'}{1+\varepsilon^{\alpha-1}\int\Phi(v',v,c)M'\,\mathrm{d}v'} = \varepsilon^{\alpha-1}\mathcal{F}[G_\varepsilon](v) + \frac{1}{1+\varepsilon^{\alpha-1}\int\Phi(v',v,c)M'\,\mathrm{d}v'}\,,$$

which implies the estimates (3.12) in a straightforward way (using the normalization of M and F_ε).

For the time derivative, we get

$$\partial_t G_\varepsilon = \varepsilon^{\alpha-1}\left(\mathcal{F}[\partial_t G_\varepsilon] + \frac{\left(G_\varepsilon\int\nabla_c\Phi(v',v,c)M'\,\mathrm{d}v' + \int\nabla_c\Phi(v,v',c)M'G_\varepsilon'\,\mathrm{d}v'\right)\cdot\partial_t c}{1+\varepsilon^{\alpha-1}\int\Phi(v',v,c)M'\,\mathrm{d}v'}\right)\,,$$

which is of course only relevant in the case of Assumption C in Theorem 1.1. As a consequence of this assumption (boundedness of $\nabla_c\Phi$ and of $\partial_t c$) and of the uniform

L^∞ bound (3.12) for G_ε, the inhomogeneity is $O(\varepsilon^{\alpha-1})$, uniformly in (x,t,v). This implies, again by contraction, a uniform L^∞ bound of $O(\varepsilon^{\alpha-1})$ for $\partial_t G_\varepsilon$, giving the bound on $\partial_t F_\varepsilon/F_\varepsilon$ in (3.13), again as a consequence of (3.12).

Analogously, the components of $\nabla_x G_\varepsilon$ are shown to be $O(\varepsilon^{\alpha-1})$. Finally, multiplication of the equation for $\nabla_x G_\varepsilon$ by v and using the boundedness of $(|v|+|v'|)\Phi$ and $(|v|+|v'|)\nabla_c\Phi$ leads to an $O(\varepsilon^{\alpha-1})$ bound on $v\cdot\nabla_x G_\varepsilon$ and therefore also for $v\cdot\nabla_x F_\varepsilon/F_\varepsilon$. □

Remark 3.2. As consequences of (3.12),

(a) $\mu_1 M \le F_\varepsilon \le \mu_2 M$,

(b) $|F_\varepsilon - M| \le \varepsilon^{\alpha-1}\mu_3 M$,

hold with ε-independent constants μ_1,μ_2,μ_3, which has already been used in the above proof and will be used in the following.

Entropy decay properties for collision operators with detailed balance have been a classical tool in kinetic theory. The detailed balance assumption has been dispensed with in [8] (see also [10]), where a proof of the following result can be found.

Lemma 3.3. *Let the assumptions of Theorem 1.1 hold and let ε be small enough such that $1+\varepsilon^{\alpha-1}\Phi \ge \nu\mu_2 > 0$. Then the collision operator Q_ε satisfies the coercivity inequality*

$$-\int Q_\varepsilon(f)\frac{f}{F_\varepsilon}\,\mathrm{d}v \ge \nu\|f-\rho_f F_\varepsilon\|^2_{L^2(\mathrm{d}v/F_\varepsilon)} \qquad \textit{for all } f\in L^2(\mathrm{d}v/F_\varepsilon)\,. \tag{3.14}$$

The existence and uniqueness of a nonnegative solution f_ε of (1.1) for small enough ε is a classical result of kinetic theory and will be assumed here. The coercivity result Lemma 3.3 will be used for the derivation of bounds for the solution. The dependence of the equilibrium distribution F_ε on x and t destroys entropy decay, but fortunately uniform bounds on finite time intervals will still be possible.

Lemma 3.3 suggests the use of L^2-norms with weight $1/F_\varepsilon$:

$$\begin{aligned}\frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon\|^2_{L^2(dv\,\mathrm{d}x/F_\varepsilon)} \quad &= \quad -\frac{\varepsilon^\alpha}{2}\int\!\!\int\frac{\partial_t F_\varepsilon}{F_\varepsilon}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x + \frac{\varepsilon}{2}\int\!\!\int\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x \\ &\quad + \int\!\!\int Q_\varepsilon(f_\varepsilon)\frac{f_\varepsilon}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x\,,\end{aligned}$$

where the second term on the right hand side is the result of an integration by parts. Now we use (3.13) and (3.14):

$$\frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon\|^2_{L^2(\mathrm{d}v\,\mathrm{d}x/F_\varepsilon)} \le \varepsilon^\alpha\lambda\|f_\varepsilon\|^2_{L^2(\mathrm{d}v\,\mathrm{d}x/F_\varepsilon)} - \nu\|f_\varepsilon-\rho_\varepsilon F_\varepsilon\|^2_{L^2(\mathrm{d}v\,\mathrm{d}x/F_\varepsilon)}\,, \tag{3.15}$$

Theorem 3.4. *Let the assumptions of Theorem 1.1 hold. Then for small enough $\varepsilon>0$*

(i) *f_ε is uniformly (with respect to ε) bounded in $L^\infty\left(\mathrm{d}t; L^1(\mathrm{d}v\,\mathrm{d}x)\right)$ and in $L^\infty\left(e^{-\lambda t}\,\mathrm{d}t; L^2(\mathrm{d}v\,\mathrm{d}x/M)\right)$ with λ from* (3.15)*, vanishing under Assumptions A and B,*

(ii) *ρ_ε is uniformly bounded in $L^\infty\left(e^{-\lambda t}\,\mathrm{d}t; L^2(\mathrm{d}x)\right)$ and in $L^\infty\left(\mathrm{d}t; L^1(\mathrm{d}x)\right)$,*

(iii) *$r_\varepsilon := \varepsilon^{1-\alpha}(f_\varepsilon - \rho_\varepsilon M)$ is uniformly bounded in $L^2\left(e^{-2\lambda t}\,\mathrm{d}t; L^2(\mathrm{d}v\,\mathrm{d}x/M)\right)$.*

Remark 3.5. By uniform boundedness of f_ε in $L^\infty\left(e^{-\lambda t}\,\mathrm{d}t; L^2(\mathrm{d}v\,\mathrm{d}x/M)\right)$, we mean that $e^{-\lambda t}\|f_\varepsilon\|_{L^2(\mathrm{d}v\,\mathrm{d}x/M)}$ is bounded uniformly in ε and in $t\in\mathbb{R}^+$.

Proof. The first result (i) is a consequence of the Gronwall lemma, after neglecting the last term in (3.15) and of the conservation of total mass. Note that by Remark 1 (a) the weights $1/M$ and $1/F_\varepsilon$ are equivalent. Then (ii) follows from the inequality

$$\rho_\varepsilon \le \|f_\varepsilon\|_{L^2(dv/M)}\,,$$

derived from the Cauchy-Schwarz inequality and the normalization of M. Finally, (iii) is a consequence of (3.15) after integration with respect to t, using

$$|f_\varepsilon - \rho_\varepsilon M| \le |f_\varepsilon - \rho_\varepsilon F_\varepsilon| + \rho_\varepsilon |F_\varepsilon - M| \le |f_\varepsilon - \rho_\varepsilon F_\varepsilon| + \varepsilon^{\alpha-1}\mu_3 \rho_\varepsilon M\,,$$

by Remark 1 (b). Note that $\varepsilon^{\alpha/2} < \varepsilon^{\alpha-1}$. □

Since M has moments of any order smaller than α, existence of these moments is propagated by the kinetic equation. We shall need the first order moment:

Lemma 3.6. *Let the assumptions of Theorem 1.1 and Proposition 1 be satisfied and let ε be small enough. Then $\int\int |v| f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x$ is bounded uniformly with respect to t and ε.*

Proof. Since for ε small enough, $1 + \varepsilon^{\alpha-1}\Phi$ is uniformly bounded from above and away from zero, after multiplication of (1.1) by $|v|$ and integration with respect to x and v, we estimate

$$\varepsilon^\alpha \frac{\mathrm{d}}{\mathrm{d}t}\int\int |v| f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x \le C_1 - C_2 \int\int |v| f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,,$$

implying

$$\int\int |v| f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x \le \max\left\{\frac{C_1}{C_2}, \int\int |v| f^{in}\,\mathrm{d}v\,\mathrm{d}x\right\}.$$

□

4 Rigorous asymptotics for constant c

In this section, we shall prove Theorem 1.1 under Assumption A or B, following the strategy of [15]. Analogously to the derivation of (2.10) in Section 2, Fourier-Laplace transformation of (1.1) yields

$$\begin{aligned}&\widehat{\rho_\varepsilon}\int \frac{p + \varepsilon^\alpha p^2 + \varepsilon^{2-\alpha}(v\cdot k)^2}{(1+\varepsilon^\alpha p)^2 + \varepsilon^2 (v\cdot k)^2} M\,\mathrm{d}v - \int \frac{\mathcal{F} f^{in}}{1 + \varepsilon^\alpha p + \varepsilon i v\cdot k}\,\mathrm{d}v \\ &= \frac{1}{\varepsilon}\int \frac{Q_1\left(\widehat{f_\varepsilon}\right)}{1+\varepsilon^\alpha p + \varepsilon i v\cdot k}\,\mathrm{d}v\,. \qquad (4.16)\end{aligned}$$

The rigorous passage to the limit in the left hand side of this equation has already been carried out in [15]. For completeness, we repeat the essential arguments and start with the second term, whose convergence as $\varepsilon \to 0$ to $\mathcal{F}\rho^{in}$ follows from the dominated convergence theorem, noting $|1 + \varepsilon^\alpha p + \varepsilon i v\cdot k| \ge 1$ and

$$|\mathcal{F} f^{in}| \le \int f^{in}\,\mathrm{d}x \in L^1(\,\mathrm{d}v)\,.$$

The dominated convergence theorem also implies

$$\lim_{\varepsilon\to 0}\int \frac{p + \varepsilon^\alpha p^2}{(1+\varepsilon^\alpha p)^2 + \varepsilon^2(v\cdot k)^2} M\,\mathrm{d}v = p\,, \qquad \forall p > 0\,,\ k \in \mathbb{R}^N\,.$$

Furthermore we have

$$\int_{|v|<1} \frac{\varepsilon^{2-\alpha}(v\cdot k)^2}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2} M\,\mathrm{d}v \le \varepsilon^{2-\alpha}|k|^2 \to 0 \quad \text{as } \varepsilon\to 0\,.$$

In the integral over $|v|>1$ we use (1.3) and carry out the coordinate transformation $v=(w_1 k/|k|+w^\perp)/(\varepsilon|k|)$ (a stretching and a rotation):

$$\int_{|v|>1} \frac{\varepsilon^{2-\alpha}(v\cdot k)^2}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2} M\,\mathrm{d}v = \gamma|k|^\alpha \int_{|w|>\varepsilon|k|} \frac{w_1^2|w|^{-N-\alpha}}{(1+\varepsilon^\alpha p)^2+w_1^2}\,\mathrm{d}w$$

Again by dominated convergence, the right hand side converges as $\varepsilon\to 0$ for all $p>0$, $k\in\mathbb{R}^N$, to

$$A|k|^\alpha\,, \qquad \text{with } A=\gamma\int_{\mathbb{R}^N}\frac{w_1^2|w|^{-N-\alpha}}{1+w_1^2}\,\mathrm{d}w>0\,.$$

As a consequence of Theorem 3.4, there exists $\rho\in L^\infty\left(\mathrm{d}t;L^2(dx)\right)\cap L^\infty\left(\mathrm{d}t;L^1(dx)\right)$, such that

$$\rho_\varepsilon \rightharpoonup \rho \qquad \text{in } L^\infty\left(\mathrm{d}t;L^2(dx)\right) \text{ weak*}.$$

Since

$$|\widehat{\rho_\varepsilon}(k,p)| \le \frac{1}{p}\|\rho_\varepsilon\|_{L^\infty(\mathrm{d}t;L^1(dx))}\,,$$

$\widehat{\rho_\varepsilon}$ is uniformly bounded in $L^\infty\left((a,\infty)\times\mathbb{R}^N\right)$ for $a>0$, implying

$$\widehat{\rho_\varepsilon} \rightharpoonup \widehat{\rho} \qquad \text{in } L^\infty\left((a,\infty)\times\mathbb{R}^N\right) \text{ weak*}. \tag{4.17}$$

Our results so far imply distributional convergence of the left hand side of (4.16) to

$$(p+A|k|^\alpha)\widehat{\rho}-\mathcal{F}\rho^{in}\,.$$

Now we turn to the right hand side and observe that due to mass conservation $\left(\int Q_1(f)\,\mathrm{d}v=0\right)$ and with the notation of Theorem 3.4 (iii)

$$\begin{aligned}\frac{1}{\varepsilon}\int\frac{Q_1\left(\widehat{f_\varepsilon}\right)}{1+\varepsilon^\alpha p+\varepsilon iv\cdot k}\,\mathrm{d}v &= -\frac{ik}{1+\varepsilon^\alpha p}\cdot\int\frac{vQ_1\left(\widehat{f_\varepsilon}\right)}{1+\varepsilon^\alpha p+\varepsilon iv\cdot k}\,\mathrm{d}v\\ &= -\frac{ik}{1+\varepsilon^\alpha p}\cdot\left(\widehat{\rho_\varepsilon}\int\frac{vQ_1(M)}{1+\varepsilon^\alpha p+\varepsilon iv\cdot k}\,\mathrm{d}v\right.\\ &\qquad\left.+\,\varepsilon^{\alpha-1}\int\frac{vQ_1\left(\widehat{r_\varepsilon}\right)}{1+\varepsilon^\alpha p+\varepsilon iv\cdot k}\,\mathrm{d}v\right)\end{aligned} \tag{4.18}$$

holds. With $\overline{\Phi}:=\sup_{v,v',c}\Phi$, it is straightforward to show $|Q_1(M)|\le\overline{\Phi}M$. Since the first order moments of M are finite, the dominated convergence theorem implies

$$\lim_{\varepsilon\to 0}\int\frac{vQ_1(M)}{1+\varepsilon^\alpha p+\varepsilon iv\cdot k}\,\mathrm{d}v=\int vQ_1(M)\,\mathrm{d}v=u(c)\,.$$

For the last integral in (4.18), we start with the case of Assumption B (satisfied by the example treated in Section 2), whence $Q_1(f)=M\int\Phi(v,v',c)f'\,\mathrm{d}v'$. This implies

$$|Q_1(\widehat{r_\varepsilon})|\le\overline{\Phi}M\int|\widehat{r_\varepsilon}'|\,\mathrm{d}v'\le\overline{\Phi}M\|\widehat{r_\varepsilon}\|_{L^2(\mathrm{d}v/M)} \tag{4.19}$$

such that

$$\left|\int\frac{vQ_1\left(\widehat{r_\varepsilon}\right)}{1+\varepsilon^\alpha p+\varepsilon iv\cdot k}\,\mathrm{d}v\right|\le\overline{\Phi}\int|v|M\,\mathrm{d}v\,\|\widehat{r_\varepsilon}\|_{L^2(\mathrm{d}v/M)}$$

is, by Theorem 3.4, uniformly bounded in $L^\infty((a,\infty); L^2(\mathrm{d}k))$ by the estimate (using the Cauchy-Schwarz inequality and the Plancherel identity)

$$\begin{aligned}\|\widehat{r_\varepsilon}\|^2_{L^\infty((a,\infty);L^2(\mathrm{d}k\,\mathrm{d}v/M))} &= \sup_{p\ge a}\int\int\left|\int_0^\infty \mathrm{e}^{-pt}\mathcal{F}r_\varepsilon\,\mathrm{d}t\right|^2\frac{\mathrm{d}k\,\mathrm{d}v}{M}\\ &\le \sup_{p\ge a}\int\int\left(\int_0^\infty \mathrm{e}^{-2pt}\,\mathrm{d}t\right)\left(\int_0^\infty |\mathcal{F}r_\varepsilon|^2\,\mathrm{d}t\right)\frac{\mathrm{d}k\,\mathrm{d}v}{M}\\ &= \frac{1}{2a}\|\mathcal{F}r_\varepsilon\|^2_{L^2(\mathrm{d}t\,\mathrm{d}k\,\mathrm{d}v/M)} = C\|r_\varepsilon\|^2_{L^2(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v/M)}\,. \qquad (4.20)\end{aligned}$$

In the case of Assumption A, i.e. $\sup_{v,v',c}(|v|+|v'|)\Phi(v,v',c) =: \overline{\Phi}_1 < \infty$, we estimate

$$|vQ_1(\widehat{r_\varepsilon})| \le \overline{\Phi}_1\left(M\int|\widehat{r_\varepsilon}'|\,\mathrm{d}v' + |\widehat{r_\varepsilon}|\right)$$

which, by (4.20) and by an estimate like in (4.19), is uniformly bounded in $L^\infty((a,\infty);$ $L^2(\mathrm{d}k\,\mathrm{d}v/M))$. Thus, under both Assumptions A and B, the last term in (4.18) is $O(\varepsilon^{\alpha-1})$ in $L^\infty((a,\infty); L^2(\mathrm{d}k\,\mathrm{d}v/M))$.

Finally, using again (4.17), we can pass to the limit also in the right hand side of (4.16) and obtain

$$(p + ik\cdot u(c) + A|k|^\alpha)\widehat{\rho} = \mathcal{F}\rho^{in}\,,$$

the Fourier-Laplace transform of (1.6), concluding the proof.

5 Rigorous asymptotics for non-constant c

In this section we use a completely different approach, introduced by Mellet [14] and called the *moment method*. For a test function $\varphi(x,t) \in C_0^\infty(\mathbb{R}^N \times [0,\infty))$, we denote by $\chi_\varepsilon(x,v,t)$ the unique bounded solution of the auxiliary equation

$$\chi_\varepsilon - \varepsilon v\cdot\nabla_x\chi_\varepsilon = \varphi\,, \qquad (5.21)$$

which can be computed explicitly via the method of characteristics:

$$\chi_\varepsilon(x,v,t) = \int_0^\infty \mathrm{e}^{-z}\varphi(x+\varepsilon vz, t)\,\mathrm{d}z\,. \qquad (5.22)$$

The operator on the left hand side of (5.21) is the adjoint of a part of the operator appearing in (1.1), consisting only of the transport operator and of the loss term of the leading order collision operator Q_0. Some properties of χ_ε are collected in the following lemma, mostly proven already in [14].

Lemma 5.1. *Let $\varphi \in \mathcal{D}(\mathbb{R}^N \times [0,\infty))$, and let χ_ε be defined by* (5.22)*. Then χ_ε, $\partial_t\chi_\varepsilon$, and $\nabla_x\chi_\varepsilon$ are bounded in $L^\infty(\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t)$ and in $L^2(M\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t)$ uniformly in ε. Furthermore*

$$\int M|\chi_\varepsilon-\varphi|\,\mathrm{d}v\,,\ \int M|\partial_t\chi_\varepsilon - \partial_t\varphi|\,\mathrm{d}v\,,\ \int M|\nabla_x\chi_\varepsilon-\nabla_x\varphi|\,\mathrm{d}v = O(\varepsilon)\,,$$

uniformly in x and t.

Proof. The boundedness statements in L^∞ are a straightforward consequence of the boundedness of φ and of its derivatives. Because of $\int_0^\infty e^{-z}\,\mathrm{d}z = 1$,

$$\chi_\varepsilon(x,v,t)^2 \le \int_0^\infty e^{-z}\varphi(x+\varepsilon vz,t)^2\,\mathrm{d}z$$

holds, and therefore

$$\|\chi_\varepsilon\|^2_{L^2(M\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t)} \le \int_0^\infty \int\int\int_0^\infty M(v)e^{-z}\varphi(x+\varepsilon vz,t)^2\,\mathrm{d}z\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = \|\varphi\|^2_{L^2(\,\mathrm{d}x\,\mathrm{d}t)}\,,$$

and the same argument for the derivatives.

On the other hand, with the Lipschitz constant L of φ,

$$\begin{aligned}\int M|\chi_\varepsilon-\varphi|\,\mathrm{d}v &= \int M\left|\int_0^\infty e^{-z}(\varphi(x+\varepsilon vz,t)-\varphi(x,t))\,\mathrm{d}z\right|\,\mathrm{d}v\\ &\le \varepsilon L\int |v|M\,\mathrm{d}v\int_0^\infty ze^{-z}\,\mathrm{d}z\,, \qquad (5.23)\end{aligned}$$

implying the desired result by the finiteness of the first order moments of M. The proof of the remaining two statements is analogous. □

Multiplication of the kinetic equation (1.1) by χ_ε, integration with respect to x, v, and t, and using (5.21), gives

$$\begin{aligned}&\int_0^\infty\int\int f_\varepsilon\partial_t\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \int\int f^{in}\chi_\varepsilon(t=0)\,\mathrm{d}v\,\mathrm{d}x + \varepsilon^{-\alpha}\int_0^\infty\int\int \rho_\varepsilon M(\chi_\varepsilon-\varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &= -\frac{1}{\varepsilon}\int_0^\infty\int\int Q_1(f_\varepsilon)\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \qquad (5.24)\end{aligned}$$

The rest of the proof is concerned with the passage to the limit $\varepsilon\to 0$ in each of the terms of (5.24). Similarly to the preceding section, we outline the arguments for the terms on the left hand side, which have already been treated in [14].

Rewriting the first term on the left hand side leads to

$$\int_0^\infty\int\int f_\varepsilon\partial_t\varphi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty\int\int f_\varepsilon(\partial_t\chi_\varepsilon-\partial_t\varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \to \int_0^\infty\int \rho\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t\,,$$

as $\varepsilon\to 0$, where $f_\varepsilon\to\rho M$ in the sense of distibutions as a consequence of Theorem 3.4. The second term above vanishes in the limit by an argument analogously to (5.23), since, by Lemma 3.6, f_ε has first order moments in v, integrable with respect to x and bounded in t. In the same way

$$\lim_{\varepsilon\to 0}\int\int f^{in}\chi_\varepsilon(t=0)\,\mathrm{d}v\,\mathrm{d}x = \int \rho^{in}\varphi(t=0)\,\mathrm{d}x$$

is proven, using the integrability with respect to x of the first v-moments of f^{in}, as assumed in Theorem 1.1.

The third term in (5.24) leads to the fractional diffusion operator. By the rotational symmetry of M, we have

$$\varepsilon^{-\alpha}\int M(\chi_\varepsilon-\varphi)\,\mathrm{d}v = \varepsilon^{-\alpha}\int\int_0^\infty Me^{-z}(\varphi(x+\varepsilon vz)-\varphi(x)-\varepsilon vz\cdot\nabla_x\varphi(x))\,\mathrm{d}z\,\mathrm{d}v\,.$$

This implies

$$\left|\varepsilon^{-\alpha}\int_{|v|<1} M(\chi_\varepsilon-\varphi)\,\mathrm{d}v\right| \le \varepsilon^{2-\alpha}C\int_{|v|<1}|v|^2M\,\mathrm{d}v\int_0^\infty z^2e^{-z}\,\mathrm{d}z\,.$$

In the integral over $|v|>1$, we introduce the coordinate transformation $v\leftrightarrow w=\varepsilon vz$ to obtain

$$\begin{aligned}&\varepsilon^{-\alpha}\int_{|v|>1} M(\chi_\varepsilon-\varphi)\,\mathrm{d}v = \gamma\int_0^\infty\int_{|w|>\varepsilon z} z^\alpha e^{-z}\frac{\varphi(x+w)-\varphi(x)-w\cdot\nabla_x\varphi(x)}{|w|^{N+\alpha}}\,\mathrm{d}w\,\mathrm{d}z\\ &\to -A(-\Delta)^{\alpha/2}\varphi\,.\end{aligned}$$

The limit is uniform in x and t, due to the estimate

$$\left|\int_0^\infty \int_{|w|<\varepsilon z} z^\alpha e^{-z} \frac{\varphi(x+w)-\varphi(x)-w\cdot\nabla_x\varphi(x)}{|w|^{N+\alpha}}\,\mathrm{d}w\,\mathrm{d}z\right|$$
$$\le C\int_0^\infty \int_{|w|<\varepsilon z} z^\alpha e^{-z}|w|^{2-N-\alpha}\,\mathrm{d}w\,\mathrm{d}z \le \varepsilon^{2-\alpha} C\int_0^\infty z^2 e^{-z}\,\mathrm{d}z\,.$$

As a consequence, uniform integrability and weak convergence of ρ_ε are sufficient for passing to the limit in the third term of (5.24). Collecting our results so far, the left hand side of (5.24) converges to

$$\int_0^\infty \int \rho\left(\partial_t\varphi - (-\Delta)^{\alpha/2}\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int \rho^{in}\varphi(t=0)\,\mathrm{d}x\,.$$

Finally we consider the right hand side of (5.24), and use the mass conservation property of Q_1, the properties of χ_ε, and the macro-micro decomposition of f_ε:

$$\begin{aligned}\frac{1}{\varepsilon}\int Q_1(f_\varepsilon)\chi_\varepsilon\,\mathrm{d}v &= \int Q_1(f_\varepsilon)v\cdot\nabla_x\chi_\varepsilon\,\mathrm{d}v\\ &= \rho_\varepsilon u(c)\cdot\nabla_x\varphi + \rho_\varepsilon\int Q_1(M)v\cdot(\nabla_x\chi_\varepsilon - \nabla_x\varphi)\,\mathrm{d}v\\ &\quad + \varepsilon^{\alpha-1}\int Q_1(r_\varepsilon)v\cdot\nabla_x\chi_\varepsilon\,\mathrm{d}v \end{aligned} \tag{5.25}$$

After integration with respect to x and t, we can pass to the limit in the first term on the right hand side by the weak convergence of ρ_ε. By Assumption C we can use again $\sup_{v,v',c}(|v|+|v'|)\Phi(v,v',c) = \overline{\Phi}_1 < \infty$ to obtain

$$|Q_1(f)v| \le \overline{\Phi}_1\left(M\int |f'|\,\mathrm{d}v' + |f|\right)\,. \tag{5.26}$$

In particular, the consequence $|Q_1(M)v| \le 2\overline{\Phi}_1 M$ implies by Lemma 5.1 that the integral in the second term on the right hand side of (5.25) is $O(\varepsilon)$ uniformly in x and t. For the last term in (5.25) we have, after integration with respect to x and t, by (5.26), by the Cauchy-Schwarz inequality, and by Lemma 5.1

$$\left|\int_0^\infty \int\int Q_1(r_\varepsilon)v\cdot\nabla_x\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\le C\left(\left\|M\int |r_\varepsilon'|\,\mathrm{d}v'\right\|_{L^2((0,T);L^2(\,\mathrm{d}v\,\mathrm{d}x/M))} + \|r_\varepsilon\|_{L^2((0,T);L^2(\,\mathrm{d}v\,\mathrm{d}x/M))}\right)$$
$$\le 2C\|r_\varepsilon\|_{L^2((0,T);L^2(\,\mathrm{d}v\,\mathrm{d}x/M))}\,,$$

where $T<\infty$ denotes an upper bound for t in the support of φ. The right hand side is uniformly bounded by Theorem 3.4. Combining our results, the limit of (5.24) as $\varepsilon\to 0$ reads

$$\int_0^\infty \int \rho\left(\partial_t\varphi + u(c)\cdot\nabla_x\varphi - (-\Delta)^{\alpha/2}\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int \rho^{in}\varphi(t=0)\,\mathrm{d}x = 0\,,$$

which is the distributional formulation of (1.6).

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Asymptotic analysis of a Vlasov-Boltzmann equation with anomalous scaling

Pedro Aceves-Sánchez and Antoine Mellet

Submitted

Abstract

This paper is devoted to the approximation of the linear Boltzmann equation by fractional diffusion equations. Most existing results address this question when there is no external acceleration field. The goal of this paper is to investigate the case where a given acceleration field is present. The main result of this paper shows that for an appropriate scaling of the acceleration field, the usual fractional diffusion equation is supplemented by an advection term. Both the critical and supercritical case are considered.

1 Introduction

1.1 Introduction

The goal of this paper is to study the asymptotic behavior of the solution of the following equation as ε tends to zero:

$$\begin{cases} \varepsilon^{\alpha-1}\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \dfrac{1}{\varepsilon^{2-\alpha}}E\cdot\nabla_v f_\varepsilon &= \dfrac{1}{\varepsilon}Q(f_\varepsilon) \quad \text{in } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d, \\ f_\varepsilon(\cdot,\cdot,0) &= f^{in} \quad \text{in } \mathbb{R}^d\times\mathbb{R}^d, \end{cases} \tag{1.1}$$

where $E \in \left[W^{1,\infty}(\mathbb{R}^d\times[0,\infty))\right]^d$ is a given acceleration field and Q is the linear Boltzmann operator defined as

$$Q(f) := \int \sigma(v,v')M(v)f(v') - \sigma(v',v)M(v')f(v)\,\mathrm{d}v'. \tag{1.2}$$

Typically, $f_\varepsilon(x,v,t)$ denotes the distribution function of some particles in a dilute gas, subject to an external acceleration field $E(x,t)$. The small parameter ε can be interpreted as the Knudsen number, which measures the relative importance of the scattering phenomenon (described here by the collision operator Q) compared to the transport of particles (ε is often introduced in the literature as the ratio of the mean free path over some typical macroscopic length, such as the length of the device being studied). The coefficient α determines the relative order of the various terms in (1.1) and it will be fixed by the properties of the thermodynamical equilibrium $M(v)$ appearing in the operator Q. One possible definition for α is

$$\alpha = \sup\left\{\beta \le 2\,;\ \int_{\mathbb{R}^d} |v|^\beta M(v)\,dv < \infty\right\}. \tag{1.3}$$

However, we will make stronger assumptions on the behavior of M for large $|v|$ which will make the definition of α simpler. Concerning the particular choice of scaling in (1.1), we note that the $\varepsilon^{\alpha-1}$ in front of the time derivative corresponds to a particular choice of a time scale at which we know that diffusion will be observed ([16, 15]), while the $\frac{1}{\varepsilon^{2-\alpha}}$ in front of the force term correspond to a strong field assumption (we will always have $\alpha < 2$ and so $\frac{1}{\varepsilon^{2-\alpha}} \gg 1$). Obviously other choices of scaling for this force term are possible (see Remark 1.3), but this particular scaling is exactly the one for which the diffusion process (due to the scattering phenomenon of Q) and the advection process (due to the acceleration term E) are of the same order in the limit (see equations (1.15) and (1.19)).

When $M(v)$ is a Maxwellian distribution function, or more generally when $M(v)$ satisfies

$$\int_{\mathbb{R}^d} |v|^2 M(v)\, dv < \infty,$$

then (1.3) gives $\alpha = 2$ and, we recognize in (1.1) the classical drift-diffusion scaling. If we assume further that $E = 0$, then such limits were first investigated in the pioneering works [11], [5], [25] and [14]. In all these papers, it is assumed that M is a Maxwellian distribution function; In [9], Degond-Goudon-Poupaud extended these results to a more general distribution M, but always under the assumption of finite second moment. The case $E \neq 0$ is addressed for example by Poupaud in [18] when M is a Maxwellian. It is shown in particular that the addition of the force field E leads to a drift term in the limiting equation for the density of particles.

The object of this paper is to investigate what happens when $M(v)$ has a so-called heavy tail distribution function with $\alpha < 2$. To be more precise, we will assume that

$$M(v) \sim \frac{\gamma}{|v|^{d+\alpha}} \text{ as } |v| \to \infty$$

for some $\alpha < 2$. The α describing the large velocity behavior of $M(v)$ is then the same as the α appearing in (1.1) (this is consistent with (1.3)). When $E = 0$, such limits have been the object of several recent works (see for example [16], [15], and [4]), and it has been shown that the limiting behavior of f_ε is described by a fractional diffusion equation.

The main contribution of the paper is thus to consider the case $E \neq 0$. In view of the scaling in equation (1.1), we immediately note that the cases $\alpha \in (1, 2)$, $\alpha = 1$ and $\alpha \in (0, 1)$ are radically different. Indeed, when $\alpha \in (1, 2)$, all the terms in the left hand side of (1.1) are smaller than ε^{-1} when $\varepsilon \ll 1$. So, assuming that f_ε converges to f (for instance in $\mathcal{D}'$), we immediately get $Q(f) = 0$, that is

$$\lim_{\varepsilon \to 0} f_\varepsilon(x, v, t) = \rho(x, t) M(v).$$

By contrast, when $\alpha = 1$, the force term is of the same order as the collision term, and we will get instead

$$\lim_{\varepsilon \to 0} f_\varepsilon(x, v, t) = \rho(x, t) F(x, v, t)$$

where F is the unique solution of

$$Q(F) - E \cdot \nabla_v F = 0, \qquad \int_{\mathbb{R}^d} F\, dv = 1, \tag{1.4}$$

(see Proposition 2.5 below for the existence of F). Equation (1.4) classically appears in the high field asymptotic limit which has been studied for various operators Q [2, 19, 3] (see also Remark 1.3 below). Finally, when $\alpha \in (0, 1)$, the force term in the left hand side of (1.1) is more singular than the collision term, and the limit $f(x, v, t)$ of $f_\varepsilon(x, v, t)$ satisfies

$$E \cdot \nabla_v f = 0.$$

It is not clear to us what one could expect to prove in this last case. In fact, we will see that we are not able to obtain a priori estimates on f_ε to successfully investigate such a limit (note however that f_ε is always bounded in $L^\infty(0,\infty; L^1(\mathbb{R}^d \times \mathbb{R}^d))$, so some limit always exists). In this paper, we thus focus our attention on the two cases $\alpha \in (1,2)$ and $\alpha = 1$. One of the key observations that allowed us to obtain the hydrodynamic limit in a rigorous manner is to note that not only the operator Q appearing in (1.1) is coercive but also the operator

$$\mathcal{T}(f) := -Q(f) + E \cdot \nabla_v f$$

is coercive in a suitable space (see Proposition 2.9). Our proof is based on analytic methods.

We will show that the limit f of f_ε is of the form $\rho(x,t)M(v)$ (or $\rho(x,t)F(x,v,t)$ when $\alpha = 1$) where ρ solves a fractional diffusion equation of order α with a drift term. In that spirit, the first derivation of a fractional diffusion equation with an advection term starting from a kinetic model was first obtained in [1] as a macroscopic limit of an equation featuring a collision operator with a biased velocity. Note that evolution equations involving a fractional-diffusion term appear in many equations of mathematical physics (consult [24] and [20], and the references therein), for instance in fluid dynamics with the so-called quasi-geostrophic flow model (see [7]) (in that case the equation is non linear since the drift depends on the solution). The study of fractional-diffusion advection equations has been a very active field of research recently, and questions such as the regularity of the solutions have been addressed, see for instance [21] and [22]. It is a classical fact that the case of the half Laplacian ($\alpha = 1$ with our notations) plays a critical role in that case since the diffusion operator has the same order as the advection term. In that sense, it is not surprising that the case $\alpha = 1$ plays a critical role in our study as well.

1.2 Assumptions

We now list our main assumptions. As noted above, the acceleration field $E(x,t)$ is assumed to be given (as opposed to, say, solution of Poisson equation), and satisfies

$$E \in \left[W^{1,\infty}\left(\mathbb{R}^d \times (0,\infty)\right)\right]^d. \tag{1.5}$$

Next, we assume that M satisfies:

$$M > 0, \quad M(v) = M(-v) \text{ for all } v \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} M(v)\,\mathrm{d}v = 1, \tag{1.6}$$

$$|v|^{d+\alpha} M(v) \longrightarrow \gamma > 0\,, \qquad \text{as } |v| \to \infty, \quad \text{where } 1 \le \alpha < 2, \tag{1.7}$$

as well as the following regularity assumptions:

$$|D_v M(v)| \le C\frac{M(v)}{1+|v|}, \quad |D_v^2 M(v)| \le C M(v). \tag{1.8}$$

We note that these assumptions are compatible with the asymptotic behavior of M given by (1.7). They are in particular satisfied by the function

$$M(v) = \left(\frac{1}{1+|v|^2}\right)^{\frac{d+\alpha}{2}}$$

and by the probability density function of the so-called α-stable stochastic processes [12].

The cross section $\sigma(v,v')$ appearing in the operator Q will be assumed to satisfy

$$\sigma(v,v') = \sigma(v',v), \qquad \nu_1 \le \sigma(v,v') \le \nu_2, \quad \text{for all } v, v' \in \mathbb{R}^d \tag{1.9}$$

$$|\nabla_v \sigma(v',v)| \le \frac{C}{1+|v|}, \tag{1.10}$$

where C, ν_1 and ν_2 are positive constants. Let us note that the symmetry condition (1.9) on σ guarantees that $Q(M) = 0$. If we define the collision frequency $\nu(v)$ by

$$\nu(v) = \int \sigma(v',v) M(v')\, \mathrm{d}v'$$

then conditions (1.9) and (1.10) imply

$$\nu_1 \le \nu(v) \le \nu_2, \qquad |\nabla_v \nu(v)| \le \frac{C}{1+|v|} \qquad \text{for all } v \in \mathbb{R}^d. \tag{1.11}$$

In addition, we assume that the collision frequency ν is even, namely,

$$\nu(v) = \nu(-v) \qquad \text{for all } v \in \mathbb{R}^d. \tag{1.12}$$

Finally, we need σ and ν to have a nice behavior as $v \to \infty$. More precisely, we assume:

$$|\sigma(v,v') - \nu_0| \le \frac{C}{1+|v|} \qquad \text{for all } v, v' \in \mathbb{R}^{2d}\ , \tag{1.13}$$

for some ν_0, which implies in particular

$$\nu(v) \to \nu_0, \ \text{ as } |v| \to \infty. \tag{1.14}$$

1.3 Main results

Under assumptions (1.5) and (1.9), the existence and uniqueness of a solution $f_\varepsilon \in \mathcal{C}^0([0,\infty); L^1(\mathbb{R}^d \times \mathbb{R}^d))$ to (1.1) can be proved via a semigroup argument. We do not discuss this issue here and refer instead the interested reader to [18] or [8] for the existence of a mild solution and to the Appendix of [10] where the equivalence between the mild solution and a solution in the sense of distributions is shown.

In this paper we investigate the asymptotic behavior of f_ε as $\varepsilon \to 0$. Our first result concerns the case $\alpha \in (1,2)$:

Theorem 1.1. *Assume $\alpha \in (1,2)$ and let $f_\varepsilon(x,v,t)$ be the solution of* (1.1) *with initial condition $f^{in} \ge 0$ satisfying*

$$f^{in} \in L^2_{M^{-1}}(\mathbb{R}^d \times \mathbb{R}^d) \cap L^1(\mathbb{R}^d \times \mathbb{R}^d).$$

Under Assumptions (1.5)-(1.13) *listed above, the function $f_\varepsilon(x,v,t)$ converges weakly in $\star$-$L^\infty(0,T; L^2_{M^{-1}}(\mathbb{R}^d \times \mathbb{R}^d))$ to the function $\rho(x,t)M(v)$ where ρ solves*

$$\begin{cases} \partial_t \rho + \kappa(-\Delta)^{\alpha/2}\rho + \nabla_x \cdot (DE\rho) &= 0 \quad \text{in } (0,\infty) \times \mathbb{R}^d, \\ \rho(\cdot,0) &= \rho^{in} \quad \text{in } \mathbb{R}^d, \end{cases} \tag{1.15}$$

with $\rho^{in}(x) = \int f^{in}(x,v)\,\mathrm{d}v$ and with the coefficient κ and matrix D defined by

$$\kappa = \frac{\gamma \nu_0^2}{c_{d,\alpha}} \int_0^\infty z^\alpha e^{-\nu_0 z}\, \mathrm{d}z, \tag{1.16}$$

and

$$D = \int \lambda(v) \otimes v\, dv, \qquad Q(\lambda) = \nabla_v M(v). \tag{1.17}$$

Note that the constant $c_{d,\alpha}$ appearing in (1.16) is defined in (1.24) and that the existence of the function $\lambda(v)$ appearing in (1.17) will be proved in Lemma 2.11. When $\sigma(v,v')=1$, we can take $\lambda(v)=-\nabla_v M(v)$, and we can check that D is the identity matrix.

Next, we consider the critical case $\alpha=1$. In that case, Equation (1.1) reads

$$\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \frac{1}{\varepsilon}E\cdot\nabla_v f_\varepsilon &= \frac{1}{\varepsilon}Q(f_\varepsilon) \quad \text{in } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,\\ f_\varepsilon(\cdot,\cdot,0) &= f^{in} \quad \text{in } \mathbb{R}^d\times\mathbb{R}^d, \end{cases}$$

and we recognize the so-called high field asymptotic for the Boltzmann equation. Such asymptotics were first studied by Arlotti and Frosali [2] and Poupaud [19] for the linear Boltzmann operator with Maxwellian equilibrium (see also Ben Aballah-Chaker [3] for a non-linear collision operator). The main difference in this case is that the weak limit of f_ε will be the solution F of (1.4) (which depends on E) rather than $M(v)$. The existence and properties of F will be the object of Theorem 2.2 below. In particular, we will prove that there exists a function $F(v,E)$ defined for $(v,E)\in\mathbb{R}^d\times\mathbb{R}^d$ such that for all $E\in\mathbb{R}^d$, $v\mapsto F(v,E)$ solves

$$Q(F)-E\cdot\nabla_v F=0, \qquad \int_{\mathbb{R}^d} F(v,E)\,dv=1. \tag{1.18}$$

We then have:

Theorem 1.2. *Assume $\alpha=1$ and let $f_\varepsilon(x,v,t)$ be the solution of* (1.1) *with initial condition $f^{in}\geq 0$ satisfying*

$$f^{in}\in L^2_{M^{-1}}(\mathbb{R}^d\times\mathbb{R}^d)\cap L^1(\mathbb{R}^d\times\mathbb{R}^d).$$

Under Assumptions (1.5)-(1.13) *listed above, the solution $f_\varepsilon(x,v,t)$ of* (1.1) *converges weakly in $\star$-$L^\infty(0,T;L^2_{M^{-1}}(\mathbb{R}^d\times\mathbb{R}^d))$ to the function $\rho(x,t)F(v,E(x,t))$ where $\rho(x,t)$ solves*

$$\begin{cases} \partial_t\rho+\kappa(-\Delta)^{1/2}\rho+div_x(\mu(E)\rho) &= 0 \quad \textit{in } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,\\ \rho(\cdot,0) &= \rho^{in} \quad \textit{in } \mathbb{R}^d, \end{cases} \tag{1.19}$$

where $\rho^{in}(x)=\int f^{in}(x,v)\,\mathrm{d}v$,

$$\mu(E):=\int v\,(F(v,E)-M(v))\,\mathrm{d}v, \tag{1.20}$$

and

$$\kappa=\frac{\gamma\nu_0^2}{c_{d,1}}\int_0^\infty z e^{-\nu_0 z}\,\mathrm{d}z.$$

This result should be compared to the classical high-field limit ([2, 19]), which leads to a transport equation. Here the (fractional) diffusion takes place at the same time scale as the transport and thus appears in the limiting equation.

Note that the fact that $\mu(E)$ is well defined by formula (1.20) is not completely obvious since $vM(v)$ is not integrable when $\alpha=1$. However, we will see in Lemma 5.1 that $F(v,E)-M(v)$ decays faster than M and that $\mu(E)$ is indeed well defined.

When σ is constant, we can get explicit formulas for $F(v,E)$ and E. Indeed, if $\sigma=1$ then the operator Q reads

$$Q(f)(v)=\int_{\mathbb{R}^d} f(v')\,\mathrm{d}v' M(v)-f(v)$$

and equation (1.4) can be recast as

$$F + E \cdot \nabla_v F = M$$

which can be explicitly integrated along the characteristics yielding the following formula:

$$F(v,E) = \int_0^\infty \mathrm{e}^{-z} M(v - Ez)\,\mathrm{d}z. \tag{1.21}$$

We can also use the equation above to compute

$$\mu(E) = -\int vE \cdot \nabla_v F \,\mathrm{d}v = E \qquad \text{for all } E \in \mathbb{R}^d$$

(using an integration by part and the fact that $\int F(v)\,\mathrm{d}v = 1$).

Remark 1.3. When M satisfies (1.7) with $\alpha \in (1,2)$, we can also consider the high field asymptotic regime as in [2, 19]. It corresponds to the following scaling of the equation:

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon} E \cdot \nabla_v f_\varepsilon &= \frac{1}{\varepsilon} Q(f_\varepsilon) \quad \text{in } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f_\varepsilon(\cdot,\cdot,0) &= f^{in} \quad \text{in } \mathbb{R}^d \times \mathbb{R}^d, \end{cases}$$

In that case, it is relatively easy to show that f_ε converges to $\rho(x,t)F(v,E(x,t))$ where F is given by (1.18) and ρ solves the transport equation

$$\partial_t \rho + \mathrm{div}_x(\rho E) = 0.$$

Remark 1.4. The case $\alpha = 2$ is also interesting. In this case the scaling in equation (1.1) becomes the usual diffusion scaling, however, the second moment $\int |v|^2 M(v)\,dv$ (and thus the diffusion coefficient) is infinite. This critical case was studied in [16], and it was shown that the time scale must be modified by a logarithmic factor, leading to the following equation:

$$\varepsilon \ln(\varepsilon^{-1}) \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \ln(\varepsilon^{-1}) E \cdot \nabla_v f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon).$$

The limiting equation, on the other hand, will now involve the regular Laplace operator.

1.4 Notations and organization of the paper

We recall that the fractional Laplacian appearing in (1.15) and (1.19) can be defined via the Fourier transform as

$$\mathcal{F}\big((-\Delta)^{\alpha/2} f\big)(k) := |k|^\alpha \mathcal{F}(f)(k), \tag{1.22}$$

where $\mathcal{F}(f)$ denotes the Fourier transform of f and is defined as

$$\mathcal{F}(f) := \int e^{-ik\cdot x} f(x)\,\mathrm{d}x, \tag{1.23}$$

or as a singular integral as

$$(-\Delta)^{\alpha/2} f(x) = c_{d,\alpha} \text{ P.V.} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y, \tag{1.24}$$

where P.V. denotes the Cauchy principal value and

$$c_{d,\alpha} = \frac{\alpha 2^{\alpha-1} \Gamma(\frac{\alpha+N}{2})}{\pi^{N/2} \Gamma(\frac{2-\alpha}{2})},$$

where $\Gamma(x)$ is the Gamma function. When $\alpha > 1$, the principal value can be avoided by using the following formula:

$$(-\Delta)^{\alpha/2} f(x) = c_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(x) - f(y) - \nabla_x f(x)(x-y)}{|x-y|^{d+\alpha}} \,\mathrm{d}y\,.$$

For a detailed discussion on the properties of the fractional Laplacian consult [13], [23], or [17].

We denote by $\mathrm{d}x$, $\mathrm{d}v$ and $\mathrm{d}v'$ the Lebesgue measure on $\mathbb{R}^d$ and by $\mathrm{d}t$ the Lebesgue measure on $[0,\infty)$, where $\mathbb{R}^d$ and $[0,\infty)$ will be the integration domains, respectively, unless stated otherwise. We will denote by $L^2_{M^{-1}}(\mathbb{R}^d)$ (respectively $L^2_{F_\varepsilon^{-1}}(\mathbb{R}^d)$) the space of square integrable function with weight M^{-1} (respectively F_ε^{-1}) equipped with the norm

$$\|f\|_{L^2_{M^{-1}}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(v)|^2 \frac{\mathrm{d}v}{M(v)} \right)^{1/2}.$$

Finally, given a function $f \in L^1\left(\mathbb{R}^d\right)$ we define the mass density ρ_f of f as

$$\rho_f := \int f \,\mathrm{d}v. \tag{1.25}$$

The rest of the paper is organized as follows: In the next section, we prove the existence of F, solution of (1.18), and we investigate its properties. In Section 3, we will derive the a priori estimates on f_ε solution of (1.1) which will be necessary for the proofs of our main results. Finally, Theorem 1.1 is proved in Section 4, while Theorem 1.2 is proved in Section 5.

2 The modified equilibrium function F

Classically, a priori estimates for the solutions of (1.1) are obtained as consequence of the following coercivity property of the Boltzmann collision operator:

Lemma 2.1. *Under assumption* (1.9)*, the operator Q is a bounded operator in $L^2_{M^{-1}}(\mathbb{R}^d)$ which satisfies the following coercivity estimate:*

$$-\int_{\mathbb{R}^d} Q(f) f \frac{dv}{M(v)} \geq \nu_1 \int_{\mathbb{R}^d} |f - \rho_f M|^2 \frac{\mathrm{d}v}{M(v)},$$

for all $f \in L^2_{M^{-1}}(\mathbb{R}^d)$ and with ρ_f given by (1.25).

When $E = 0$, this very classical lemma immediately implies that the solution of (1.1) satisfies

$$f_\varepsilon(x,v,t) = \rho_\varepsilon(x,t) M(v) + \varepsilon^{\alpha/2} r_\varepsilon(x,v,t)$$

where the remainder term r_ε is bounded in some appropriate functional space (such a bound is obtained by multiplying (1.1) by f_ε/M and integrating). Such estimates can be generalized to include the case $E \neq 0$ and $\alpha = 2$. Unfortunately, these computations do not seem to be useful in the case $\alpha < 2$ which we are considering here.

In the next section, we will see that we can instead obtain the following expansion for f_ε:

$$f_\varepsilon(x,v,t) = \rho_\varepsilon(x,t) F_\varepsilon(x,v,t) + \varepsilon^{\alpha/2} r_\varepsilon(x,v,t)$$

where F_ε is the normalized equilibrium function solution of

$$\varepsilon^{\alpha-1}E\cdot\nabla_v F_\varepsilon - Q(F_\varepsilon) = 0\,, \qquad \int F_\varepsilon \,\mathrm{d}v = 1. \tag{2.26}$$

Our goal in this section is to prove the existence and uniqueness of F_ε and study its properties.

But first we note that we can write

$$F_\varepsilon(x,v,t) = F(v,\varepsilon^{\alpha-1}E(x,t))$$

where the function $v\mapsto F(v,E)$ solves (for all $E\in\mathbb{R}^d$):

$$E\cdot\nabla_v F - Q(F) = 0, \qquad \int_{\mathbb{R}^d} F(v,E)\,dv = 1. \tag{2.27}$$

This equation plays a central role in the study of the high field asymptotics for Boltzmann type equations, and has been studied for various operators Q. However, it does not seem that it has been studied under our assumptions on the function $M(v)$ (property (2.28) below, in particular, is very specific to our framework). We will thus study (2.27) in detail in this section. More precisely, gathering all the key results that we will prove in this section, we have the following:

Theorem 2.2.

(i) For all $E\in\mathbb{R}^d$, the exists a unique function $v\mapsto F(v,E)$ solution of (2.27)*.*

(ii) There exist two positive constants $C(R)$ and $c(R)$ such that if $|E|\leq R$ then

$$c(R)M(v)\leq F(v,E)\leq C(R)M(v) \qquad \text{for all } v\in\mathbb{R}^d.$$

(iii) The function $E\mapsto F(v,E)$ is C^1 and for all $R>0$ there exists $C(R)$ such that

$$|\partial_E F(v,E)|\leq C(R)\frac{F(v,E)}{1+|v|} \qquad \text{for all } v\in\mathbb{R}^d \text{ and } |E|\leq R. \tag{2.28}$$

Since we are assuming $\alpha\geq 1$, assumption (1.5) implies that $|\varepsilon^{\alpha-1}E(x,t)|$ is bounded uniformly in ε, x and t, and so the results of this theorem will apply to the function $F_\varepsilon(x,v,t) = F(v,\varepsilon^{\alpha-1}E(x,t))$ (see Propositions 3.1 and 3.2). When $\alpha>1$, the behavior of $F(v,E)$ for $|E|\ll 1$ will play an important role. We will thus prove the following result:

Proposition 2.3. *The following expansion holds:*

$$F(v,E) = M(v) + E\cdot\lambda(v) + G(v,E) \tag{2.29}$$

where $\lambda(v)$ is such that

$$Q(\lambda)(v) = \nabla_v M(v), \qquad \int_{\mathbb{R}^d}\lambda(v)\,dv = 0,$$

and G satisfies:

$$\|G(\cdot,E)\|_{L^2_{M^{-1}}(\mathbb{R}^d)}\leq C|E|^2 \quad \text{for all } |E|\leq 1 \tag{2.30}$$

and

$$|G(\cdot,E)|\leq C|E|^2M(v) \quad \text{for all } v\in\mathbb{R}^d,\ |E|\leq 1. \tag{2.31}$$

2.1 Existence of $F(v, E)$

In this Section, we prove the existence of a unique solution to (2.27) (Theorem 2.2 (i)). The proof follows closely the arguments of Poupaud in [19]. We recall it here for the sake of completeness. Throughout this section, we fix $E \in \mathbb{R}^d$ and we define the operator

$$\mathcal{T}(f) := -Q(f) + E \cdot \nabla_v f. \tag{2.32}$$

We also define the operators $\mathcal{A}$ and $\mathcal{K}$ by

$$\mathcal{A}(f) := E \cdot \nabla_v f + \nu f, \qquad \mathcal{K}(f) := \int \sigma(v, v') f(v') \, \mathrm{d}v' M(v)$$

so that $\mathcal{T} = \mathcal{A} - \mathcal{K}$. We note that $\mathcal{K}$ is a positive compact operator in $L^2_{M^{-1}}(\mathbb{R}^d)$ (it is a Hilbert-Schmidt operator), while $\mathcal{A}$ is an unbounded operator with domain

$$D(\mathcal{A}) := \left\{ f \in L^2_{M^{-1}}(\mathbb{R}^d) \mid E \cdot \nabla_v f \in L^2_{M^{-1}}(\mathbb{R}^d) \right\}. \tag{2.33}$$

Furthermore, we can define the inverse operator $\mathcal{A}^{-1}$ as follows:

$$\mathcal{A}^{-1}(h) := \int_0^\infty \mathrm{e}^{-\int_0^s \nu(v - E\tau) \, \mathrm{d}\tau} h(v - Es) \, \mathrm{d}s. \tag{2.34}$$

Indeed, we have:

Lemma 2.4. *The operator $\mathcal{A}^{-1}$ defined by* (2.34) *is a bounded operator in $L^2_{M^{-1}}$ (with a norm depending on $|E|$) which satisfies*

$$(\mathcal{A} \circ \mathcal{A}^{-1})(f) = f \quad \text{for all } f \in L^2_{M^{-1}}(\mathbb{R}^d),$$

and

$$(\mathcal{A}^{-1} \circ \mathcal{A})(f) = f \quad \text{for all } f \in D(\mathcal{A}).$$

Postponing the proof of this Lemma to the end of this section, we first show that it implies the main result of this section:

Proposition 2.5. *For all $E \in \mathbb{R}^d$, there exists a unique positive solution $v \mapsto F(v, E)$ of* (2.27) *in $L^2_{M^{-1}}(\mathbb{R}^d)$.*

Proof of Proposition 2.5. We can rewrite (2.27) as

$$\mathcal{A}F = \mathcal{K}(F), \qquad \int F \, \mathrm{d}v = 1. \tag{2.35}$$

Formula (2.34) shows that $\mathcal{A}^{-1}$ is a nonnegative operator (if $h \geq 0$ then $\mathcal{A}^{-1}(h) \geq 0$). It follows that the operator $\mathcal{K} \circ \mathcal{A}^{-1}$ is a positive compact operator in $L^2_{M^{-1}}(\mathbb{R}^d)$ and so we can apply Krein-Rutman's Theorem (see [12]) to deduce the existence of a unique simple positive eigenvalue λ with associated positive eigenfunction W satisfying

$$\left(\mathcal{K} \circ \mathcal{A}^{-1}\right) W = \lambda W.$$

We now define $F := \mathcal{A}^{-1} W$ and note that thanks to Lemma 2.4 it satisfies

$$\mathcal{K}(F) = \lambda \mathcal{A} F.$$

Integrating this relation with respect to v and using the definition of ν, we find

$$\int \nu(v) F(v) \, \mathrm{d}v = \lambda \int \nu(v) F(v) \, \mathrm{d}v,$$

from which it follows that $\lambda = 1$. After normalizing F the proposition follows. □

We complete this section with a proof of Lemma 2.4:

Proof of Lemma 2.4. The fact that $(\mathcal{A}\circ\mathcal{A}^{-1})(f) = f$ for all $f \in L^2_{M^{-1}}\left(\mathbb{R}^d\right)$, and $(\mathcal{A}^{-1}\circ\mathcal{A})(f) = f$ for all $f \in D(\mathcal{A})$ can be proved as the Proposition 1 in [19].

To show that $\mathcal{A}^{-1}$ is a bounded operator, we first note (using (1.11)) that

$$|\mathcal{A}^{-1}(h)| \le \int_0^\infty \mathrm{e}^{-\nu_1 s} h(v - Es)\,\mathrm{d}s.$$

We thus have

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathcal{A}^{-1}(h)|^2 \frac{\mathrm{d}v}{M(v)} &\le \frac{1}{\nu_1}\int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-\nu_1 s}\frac{|h(v-Es)|^2}{M(v)}\,\mathrm{d}s\,dv\\
&\le \frac{1}{\nu_1}\int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-\nu_1 s}\frac{|h(v)|^2}{M(v+Es)}\,\mathrm{d}s\,dv\\
&\le \frac{1}{\nu_1}\int_{\mathbb{R}^d}\left(\int_0^\infty \mathrm{e}^{-\nu_1 s}\frac{M(v)}{M(v+Es)}\,\mathrm{d}s\right)\frac{|h(v)|^2}{M(v)}\,dv,
\end{aligned}$$

and we conclude thanks to the following claim: There exists a $C > 0$ such that

$$\left(\int_0^\infty \mathrm{e}^{-\nu_1 s}\frac{M(v)}{M(v+Es)}\,\mathrm{d}s\right) \le C(1 + |E|^{d+\alpha}) \qquad \text{for all } v \in \mathbb{R}^d.$$

This last bound is proved by first noticing that (1.7) implies, in particular, the existence of $\mu_1, \mu_2 > 0$ such that

$$\frac{\mu_1}{1+|v|^{d+\alpha}} \le M(v) \le \frac{\mu_2}{1+|v|^{d+\alpha}} \quad \text{for all } v \in \mathbb{R}^d. \tag{2.36}$$

Therefore, using the elementary inequality $|a + b|^p \le C\left(|a|^p + |b|^p\right)$, valid for $p \ge 1$, we obtain the following estimate:

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\nu_1 s}\frac{M(v)}{M(v+Es)}\,\mathrm{d}s &\le \frac{\mu_2}{\mu_1}\int_0^\infty \mathrm{e}^{-\nu_1 s}\frac{1+|v+Es|^{d+\alpha}}{1+|v|^{d+\alpha}}\,\mathrm{d}s\\
&\le C\frac{\mu_2}{\mu_1}\int_0^\infty \mathrm{e}^{-\nu_1 s}(1+|Es|^{d+\alpha})\,\mathrm{d}s\\
&\le C(1+|E|^{d+\alpha}).
\end{aligned}$$

□

2.2 Properties of $F(v, E)$: Theorem 2.2 (ii)

As noted in the Introduction, in the simpler case where the cross section satisfies

$$\sigma(v, v') = 1 \quad \text{for all } v,\, v' \in \mathbb{R}^d,$$

the equation for F reduces to

$$F + E\cdot\nabla_v F = M(v),$$

and we get the following explicit formula for F:

$$F(v, E) = \mathcal{A}^{-1}M(v) = \int_0^\infty \mathrm{e}^{-z}M(v - Ez)\,\mathrm{d}z. \tag{2.37}$$

In the general case, it does not seem possible to get such an explicit formula. However, Assumption (1.9) and the normalization of F imply

$$\nu_1 M(v) \le \mathcal{K}(F) \le \nu_2 M(v).$$

In particular, F satisfies

$$\nu_1 M(v) \le \nu F + E \cdot \nabla_v F \le \nu_2 M(v). \tag{2.38}$$

As a consequence, we can prove the following proposition (see Theorem 2.2 (ii)):

Proposition 2.6. *There exist constants $C(R)$ and $c(R)$ such that if $|E| \le R$ then*

$$c(R)M(v) \le F(v,E) \le C(R)M(v) \qquad \text{for all } v \in \mathbb{R}^d. \tag{2.39}$$

This proposition follows immediately from (2.38) and the following lemma (which will be used several times in this paper):

Lemma 2.7. *There exist two constants $C(R) > 0$ and $c(R) > 0$ such that if $|E| \le R$ then the following holds:*

(i) If f satisfies

$$\nu f + E \cdot \nabla_v f \le \beta M \tag{2.40}$$

for some $\beta > 0$, then

$$f \le C\beta M.$$

(ii) If f satisfies

$$\nu f + E \cdot \nabla_v f \ge \beta M \tag{2.41}$$

for some $\beta > 0$, then

$$f \ge c\beta M.$$

Remark 2.8. A similar result holds if we replace M by $M(v)/(1+|v|)$ in both inequalities.

Proof of Lemma 2.7. Integrating (2.40) (see the definition of $\mathcal{A}^{-1}$ given by (2.34)), we obtain

$$\begin{aligned} f(v) &\le \beta \int_0^\infty \mathrm{e}^{-\int_0^z \nu(v-E\tau)\,\mathrm{d}\tau} M(v-Ez)\,\mathrm{d}z \\ &\le \beta \int_0^\infty \mathrm{e}^{-\nu_1 z} M(v-Ez)\,\mathrm{d}z, \end{aligned}$$

and the first part of the lemma follows from the following claim: There exists $C(R) > 0$ such that

$$\int_0^\infty \mathrm{e}^{-\nu_1 z} M(v-Ez)\,\mathrm{d}z \le C(R)M(v) \qquad \text{for all } v \in \mathbb{R}^d, \text{ and all } |E| \le R. \tag{2.42}$$

In order to prove (2.42), we first write

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-\nu_1 z} \frac{M(v-Ez)}{M(v)}\,\mathrm{d}z &= \int_0^\eta \mathrm{e}^{-\nu_1 z} \frac{M(v-Ez)}{M(v)}\,\mathrm{d}z + \int_\eta^\infty \mathrm{e}^{-\nu_1 z} \frac{M(v-Ez)}{M(v)}\,\mathrm{d}z \\ &= I_1 + I_2, \end{aligned}$$

where $\eta = |v|/(2|E|)$. The triangle inequality gives $||v| - |E|z| \le |v - Ez|$, which implies

$$\left|\frac{v}{2}\right|^{d+\alpha} \le \Big||v| - |E|z\Big|^{d+\alpha} \le |v - Ez|^{d+\alpha}, \qquad \text{for } 0 \le z \le \eta.$$

Hence, using (2.36) yields

$$\frac{M(v-Ez)}{M(v)} \leq \frac{\mu_2}{\mu_1}\frac{1+|v|^{d+\alpha}}{1+|v-Ez|^{d+\alpha}} \leq \frac{\mu_2}{\mu_1}\frac{1+|v|^{d+\alpha}}{1+|v/2|^{d+\alpha}} \qquad \text{for } 0 \leq z \leq \eta.$$

Therefore we deduce

$$\begin{aligned} I_1 = \int_0^\eta \mathrm{e}^{-\nu_1 z}\frac{M(v-Ez)}{M(v)}\,\mathrm{d}z &\leq \frac{\mu_2}{\mu_1}\int_0^\eta \mathrm{e}^{-\nu_1 z}\frac{1+|v|^{d+\alpha}}{1+|v/2|^{d+\alpha}}\,\mathrm{d}z \\ &\leq \frac{\mu_2}{\mu_1\nu_1}\frac{1+|v|^{d+\alpha}}{1+|v/2|^{d+\alpha}} \\ &\leq C_1, \end{aligned}$$

where $C_1 > 0$ does not depend on v. Next, using (2.36) again, we get

$$\begin{aligned} I_2 = \int_\eta^\infty \mathrm{e}^{-\nu_1 z}\frac{M(v-Ez)}{M(v)}\,\mathrm{d}z &\leq \frac{\mu_2}{\mu_1\nu_1}(1+|v|^{d+\alpha})\mathrm{e}^{-\nu_1|v|/(2|E|)} \\ &\leq \frac{\mu_2}{\mu_1\nu_1}(1+|v|^{d+\alpha})\mathrm{e}^{-\nu_1|v|/(2R)} \\ &\leq C_2, \end{aligned}$$

where $C_2 > 0$ does not depend on v (but depends on R). We thus obtain

$$\int_0^\infty \mathrm{e}^{-\nu_1 z}\frac{M(v-Ez)}{M(v)}\,\mathrm{d}z \leq C_1 + C_2$$

which gives (2.42) and completes the proof of the first part of the lemma.

The second part of the lemma is somewhat easier to show. Indeed, proceeding as above, we check that (2.41) implies

$$\begin{aligned} f(v) &\geq \beta\int_0^\infty \mathrm{e}^{-\int_0^s \nu(v-E\tau)\,\mathrm{d}\tau} M(v-Es)\,\mathrm{d}s \\ &\geq \beta\int_0^1 \mathrm{e}^{-\nu_2 s} M(v-Es)\,\mathrm{d}s. \end{aligned}$$

Furthermore, it is readily seen that there is a constant $c(R)$ such that

$$M(v-w) \geq c\,M(v) \qquad \text{for all } v,\ w \in \mathbb{R}^d, \quad |w| \leq R.$$

We deduce

$$\begin{aligned} f(v) &\geq c\beta\int_0^1 \mathrm{e}^{-\nu_2 s} M(v)\,\mathrm{d}s \\ &\geq c\beta M(v), \end{aligned}$$

and the result follows. □

2.3 Coercivity of the operator $\mathcal{T}$

As a consequence of the results of the previous sections, we can now establish the following coercivity property of $\mathcal{T}$, which will play a crucial role in this paper:

Proposition 2.9. *For all $E \in \mathbb{R}^d$, the operator $\mathcal{T}$ defined by (2.32) satisfies*

$$\int \mathcal{T}(f)(v)\frac{f(v)}{F(v,E)}\,\mathrm{d}v \geq 0.$$

Furthermore, for all $R>0$ there exists a constant $\vartheta(R)>0$ such that for all $|E|\leq R$, there holds

$$\int \mathcal{T}(f)(v)\frac{f(v)}{F(v,E)}\,\mathrm{d}v \geq \vartheta(R)\|f-\rho_f F\|^2_{L^2_{F^{-1}}(\mathbb{R}^d)}, \qquad \textit{for all } f \in L^2_{F^{-1}}(\mathbb{R}^d)\,. \tag{2.43}$$

Proof. Throughout this proof, we use the notation f for $f(v)$ and f' for $f(v')$ (and similar notations for F and M).

Let us start by noting the following

$$\begin{aligned}\int \mathcal{T}(f)\frac{f}{F}\,\mathrm{d}v &= \int E\cdot\nabla_v f\frac{f}{F}\,\mathrm{d}v + \int \nu\frac{f^2}{F}\,\mathrm{d}v - \int\int \sigma(v,v')Mf'\frac{f}{F}\,\mathrm{d}v\,\mathrm{d}v' \\ &= \int \frac{1}{2}E\cdot\frac{\nabla_v f^2}{F}\,\mathrm{d}v + \int \nu\frac{f^2}{F}\,\mathrm{d}v - \int\int \sigma(v,v')MF'\frac{f'}{F'}\frac{f}{F}\,\mathrm{d}v\,\mathrm{d}v'\,.\end{aligned}$$

Integrating by parts and using the identity $E\cdot\nabla_v F = \mathcal{K}(F)-\nu F$ we see that

$$\begin{aligned}\frac{1}{2}\int E\cdot\frac{\nabla_v f^2}{F}\,\mathrm{d}v &= -\frac{1}{2}\int f^2 E\cdot\nabla_v\Big(\frac{1}{F}\Big) \\ &= \frac{1}{2}\int \frac{f^2}{F^2}\left(\mathcal{K}(F)-\nu F\right)\,\mathrm{d}v\,.\end{aligned}$$

Using the fact that M and F are normalized functions and that σ is symmetric, we deduce the following:

$$\begin{aligned}\int \mathcal{T}(f)\frac{f}{F}\,\mathrm{d}v &= \frac{1}{2}\int \nu\frac{f^2}{F}\,\mathrm{d}v + \frac{1}{2}\int\int \sigma(v,v')MF'\frac{f^2}{F^2}\,\mathrm{d}v\,\mathrm{d}v' \qquad (2.44)\\ &\quad - \int\int \sigma(v,v')MF'\frac{f'}{F'}\frac{f}{F}\,\mathrm{d}v\,\mathrm{d}v' \\ &= \frac{1}{2}\int\int \sigma(v',v)M'F\frac{f^2}{F^2}\,\mathrm{d}v\,\mathrm{d}v' + \frac{1}{2}\int\int \sigma(v,v')MF'\frac{f^2}{F^2}\,\mathrm{d}v\,\mathrm{d}v' \qquad (2.45)\\ &\quad - \int\int \sigma(v,v')MF'\frac{f'}{F'}\frac{f}{F}\,\mathrm{d}v\,\mathrm{d}v' \\ &= \frac{1}{2}\int\int \sigma(v,v')\left(MF'\Big(\frac{f'}{F'}\Big)^2 + MF'\frac{f^2}{F^2} - 2MF'\frac{f'}{F'}\frac{f}{F}\right)\mathrm{d}v\,\mathrm{d}v' \\ &= \frac{1}{2}\int\int \sigma(v,v')MF'\Big(\frac{f}{F}-\frac{f'}{F'}\Big)^2\,\mathrm{d}v'\,\mathrm{d}v.\end{aligned}$$

Since the right hand side is clearly non-negative, this gives the first inequality in the proposition.

If we further assume that $|E|\leq R$, then we can use (2.39) and together with assumption (1.9) it yields:

$$\int \mathcal{T}(f)\frac{f}{F}\,\mathrm{d}v \geq \frac{\nu_1}{2C(R)}\int\int FF'\Big(\frac{f}{F}-\frac{f'}{F'}\Big)^2\,\mathrm{d}v'\,\mathrm{d}v.$$

Finally, using the decomposition $f = \rho_f F + g$ and the fact $\int_{\mathbb{R}^d} g \, \mathrm{d}v = 0$ we obtain

$$\begin{aligned}
\int \mathcal{T}(f) \frac{f}{F} \, \mathrm{d}v &\geq \frac{\nu_1}{2C(R)} \int \int F F' \left(\frac{g}{F} - \frac{g'}{F'} \right)^2 \mathrm{d}v' \, \mathrm{d}v \\
&= \frac{\nu_1}{2C(R)} \int \int F \frac{g'^2}{F'} - 2gg' + \frac{g^2}{F} F' \, \mathrm{d}v \, \mathrm{d}v' \\
&= \frac{\nu_1}{C(R)} \int \frac{g^2}{F} \, \mathrm{d}v.
\end{aligned}$$

This completes the proof. □

2.4 Properties of $F(v, E)$: Theorem 2.2 (iii)

This Section is devoted to the proof of the estimate on the derivative of F with respect to E (Theorem 2.2-(iii)).

First, we prove the following result.

Lemma 2.10. *For all $R > 0$ there exists $C(R)$ such that the function $F(v, E)$ solution of (2.27) satisfies*

$$|\nabla_v F(v, E)| \leq C(R) \frac{M(v)}{1 + |v|}, \qquad \text{for all } v \in \mathbb{R}^d, \ |E| \leq R. \tag{2.46}$$

Proof. Differentiating (2.27), with respect to v_i, we obtain:

$$\begin{aligned}
E \cdot \nabla_v (\partial_{v_i} F) + \nu (\partial_{v_i} F) = &\int \sigma(v, v') F(v') \, \mathrm{d}v' \, \partial_{v_i} M(v) \\
&+ \int \partial_{v_i} \sigma(v, v') F(v') \, \mathrm{d}v' M(v) - (\partial_{v_i} \nu) F.
\end{aligned} \tag{2.47}$$

The first term in the right hand side of (2.47) can be bounded by $CM(v)/(1 + |v|)$, thanks to (1.9) and assumption (1.8). The second term in (2.47) can also be bounded by $CM(v)/(1 + |v|)$ thanks to the assumption (1.10) and the normalization of F. Finally, using (1.10) and (2.39), the third term in the right hand side of (2.47) can also be bounded by $CM(v)/(1 + |v|)$. We thus have

$$\left| E \cdot \nabla_v (\partial_{v_i} F) + \nu (\partial_{v_i} F) \right| \leq C \frac{M(v)}{1 + |v|}$$

and we conclude the proof using Lemma 2.7 and Remark 2.8. □

We can now complete the proof of Theorem 2.2:

Proof of Theorem 2.2-(iii). We first prove that $\partial_E F$ is uniformly bounded in $L^2_{F^{-1}}$ for $|E| \leq R$: Differentiating (2.27) with respect to E_i yields:

$$\mathcal{T}(\partial_{E_i} F) = -\partial_{v_i} F. \tag{2.48}$$

Thus multiplying by $\partial_{E_i} F / F$ and using the coercivity inequality (2.43) (assuming $|E| \leq R$) we obtain

$$\vartheta \| \partial_{E_i} F \|^2_{L^2_{F^{-1}}} \leq - \int \partial_{v_i} F \frac{\partial_{E_i} F}{F} \, \mathrm{d}v,$$

where we have used the fact that $\partial_{E_i} \int F \, \mathrm{d}v = 0$. The right hand side can be estimated using (2.46) and (2.39):

$$\left| \int \partial_{v_i} F \frac{\partial_{E_i} F}{F} \, \mathrm{d}v \right| \leq C \int |\partial_{E_i} F| \, \mathrm{d}v \leq C \left(\int \frac{|\partial_{E_i} F|^2}{F} \, \mathrm{d}v \right)^{1/2}.$$

We deduce

$$\vartheta \|\partial_{E_i} F\|_{L^2_{F^{-1}}} \leq C$$

which implies in particular

$$\int |\partial_{E_i} F| \, \mathrm{d}v \leq \left(\int \frac{|\partial_{E_i} F|^2}{F} \, \mathrm{d}v \right)^{1/2} \leq C. \tag{2.49}$$

Finally, in order to obtain (2.28) we rewrite (2.48) as

$$\begin{aligned} E \cdot \nabla_v \partial_{E_i} F + \nu \partial_{E_i} F &= \mathcal{K}(\partial_{E_i} F) - \partial_{v_i} F \\ &=: H(v, E) \end{aligned}$$

and, using the fact that $\int \partial_{E_i} F \, dv = 0$, we note that

$$H(v, E) = \int [\sigma(v, v') - \nu_0] \partial_{E_i} F(v', E) \, \mathrm{d}v' M(v) - \partial_{v_i} F$$

So using (1.13), (2.46) and (2.49), we deduce

$$\begin{aligned} |H(v, E)| &\leq \int |\partial_{E_i} F(v', E)| \, \mathrm{d}v' \frac{M(v)}{1 + |v|} + C \frac{M(v)}{1 + |v|} \\ &\leq C \frac{M(v)}{1 + |v|}. \end{aligned}$$

We can then conclude the proof using Lemma 2.7 (see Remark 2.8) and (2.39). ☐

2.5 Properties of $F(v, E)$: Proposition 2.3

When $\sigma = 1$, we see, using (2.37) that

$$F(v, E) \sim M(v) - E \cdot \nabla_v M(v) \quad \text{ as } |E| \to 0. \tag{2.50}$$

In the general case, we do not have an explicit formula for F which would give us such an expansion. Our goal in this section is thus to prove Proposition 2.3 which gives the require asymptotic behavior of F as E goes to zero.

But first, we need to prove the existence of the auxiliary function $\lambda(v)$ appearing in (1.17) and (2.29):

Lemma 2.11. *Assume* (1.6)-(1.10). *Then there exists a unique function* $\lambda \in (L^2_{M^{-1}}(\mathbb{R}^d))^d$ *satisfying*

$$Q(\lambda)(v) = \nabla_v M(v), \qquad \int_{\mathbb{R}^d} \lambda(v) \, dv = 0. \tag{2.51}$$

Furthermore, it satisfies

$$|\lambda(v)| \leq CM(v), \qquad |\partial_{v_i} \lambda_j(v)| \leq CM(v) \text{ for all } 1 \leq i, j \leq d. \tag{2.52}$$

We will first prove Proposition 2.3 and then go back to Lemma 2.11.

Proof of Proposition 2.3. We define

$$G(v,E) := F(v,E) - M(v) - E\cdot\lambda(v).$$

It solves

$$\begin{aligned}\mathcal{T}(G) &= 0 - \mathcal{T}(M) - E\cdot\mathcal{T}(\lambda)\\ &= -E\cdot\nabla_v M - E\cdot(-Q(\lambda) + E\cdot\nabla_v\lambda)\\ &= -E\cdot(E\cdot\nabla_v\lambda),\end{aligned}\tag{2.53}$$

and thus we obtain in particular

$$\|\mathcal{T}(G)\|_{L^2_{F^{-1}}} \le |E|^2\|D_v\lambda\|_{L^2_{F^{-1}}}.$$

If $|E|\le 1$, then inequalities (2.39) and (2.52) give

$$\|D_v\lambda\|^2_{L^2_{F^{-1}}} \le C\int \frac{M(v)^2}{F(v,E)}\,\mathrm{d}v \le \frac{C}{c}\int M(v)\,\mathrm{d}v \le C$$

and so

$$\|\mathcal{T}(G)\|_{L^2_{F^{-1}}} \le C|E|^2.$$

Using the coercivity inequality (2.43) (recall that $|E|\le 1$), and the fact that $\int_{\mathbb{R}^d} G\,\mathrm{d}v = 0$, we deduce

$$\begin{aligned}\|G\|^2_{L^2_{F^{-1}}} = \int \frac{|G|^2}{F}\,\mathrm{d}v &\le \frac{1}{\vartheta}\int \mathcal{T}(G)\frac{G}{F}\,\mathrm{d}v\\ &\le \frac{1}{\vartheta}\|\mathcal{T}(G)\|_{L^2_{F^{-1}}}\|G\|_{L^2_{F^{-1}}}\end{aligned}$$

and so

$$\|G\|_{L^2_{F^{-1}}} \le \frac{1}{\vartheta}\|\mathcal{T}(G)\|_{L^2_{F^{-1}}} \le \frac{C}{\vartheta}|E|^2,$$

which gives (2.30).

Finally, using (2.53) and the definition of $\mathcal{T}$, we write

$$\nu G + E\cdot\nabla_v G = K(G) - E\cdot(E\cdot\nabla_v\lambda).$$

Thanks to (2.30) we obtain

$$|K(G)| \le \|G\|_{L^2_{F^{-1}}} M(v) \le C|E|^2 M(v),$$

which implies, using (2.52), the following estimate:

$$|\nu G + E\cdot\nabla_v G| \le C|E|^2 M(v).$$

We conclude the proof by applying Lemma 2.7. □

Finally, we end this section with a proof of Lemma 2.11 which states the existence of the function $\lambda(v)$:

Proof of Lemma 2.11. The existence and uniqueness of λ follows from the coercivity of the operator Q (see Lemma 2.1) and the fact that

$$\int_{\mathbb{R}^d} \nabla_v M(v)\,dv = 0.$$

Using Lemma 2.1 together with (1.8) we obtain

$$\|\lambda\|_{L^2_{M^{-1}}} \le \frac{1}{\nu_1}\|\nabla M\|_{L^2_{M^{-1}}} \le \frac{C}{\nu_1}. \tag{2.54}$$

Next, we rewrite (2.51) as

$$\begin{aligned}\lambda(v) &= \frac{1}{\nu(v)}\left(\mathcal{K}(\lambda)(v) - \nabla_v M(v)\right)\\ &= \frac{1}{\nu(v)}\left(\int \sigma(v,v')\lambda(v')\,dv' M(v) - \nabla_v M(v)\right),\end{aligned} \tag{2.55}$$

and use (2.54) together with (1.8) to deduce the first inequality in (2.52).

Finally, differentiating (2.55) with respect to v and using (1.10) and (1.8), we easily deduce the second inequality in (2.52). □

3 A priori estimates

In this section we derive the a priori estimates on f_ε solution of (1.1) which will be necessary for the proofs of Theorems 1.1 and 1.2.

First, we introduce the operator

$$\mathcal{T}_\varepsilon(f) := -Q(f) + \varepsilon^{\alpha-1}E\cdot\nabla_v f, \tag{3.56}$$

and we recall that $F_\varepsilon(x,v,t)$ denotes the solution of

$$\mathcal{T}_\varepsilon(F_\varepsilon) = 0 \qquad \int_{\mathbb{R}^d} F_\varepsilon(x,v,t)\,dv = 1.$$

In view of Theorem 2.2 (i), such a function exists and can be written as

$$F_\varepsilon(x,v,t) = F(v,\varepsilon^{\alpha-1}E(x,t)).$$

When $\alpha \ge 1$ and E satisfies (1.5), Theorem 2.2 (ii) implies:

Proposition 3.1. *Assume that $\alpha \ge 1$. Then there exists two positive constants γ_1 and γ_2 such that for all $0 < \varepsilon \le 1$, the following holds:*

$$\gamma_1 M(v) \le F_\varepsilon(x,v,t) \le \gamma_2 M(v).$$

Under the same conditions, Theorem 2.2 (iii) and the chain rule imply:

Proposition 3.2. *Assume that $\alpha \ge 1$. Then for all $\varepsilon \le 1$, the function F_ε satisfies:*

(i) $\displaystyle\left\|\frac{\partial_t F_\varepsilon}{F_\varepsilon}\right\|_{L^\infty(\mathbb{R}^{2d}\times[0,\infty))} \le C\varepsilon^{\alpha-1},$

(ii) $\displaystyle\left\|\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon}\right\|_{L^\infty(\mathbb{R}^{2d}\times[0,\infty))} \le C\varepsilon^{\alpha-1},$

where C is a positive constant depending on $\|E\|_{W^{1,\infty}}$ but not on ε.

Proof. We only prove the second inequality (the first one is easier): We have

$$v\cdot\nabla_x F_\varepsilon = \partial_E F(v,\varepsilon^{\alpha-1}E(x,t))\varepsilon^{\alpha-1}v\cdot\nabla_x E$$

and so (2.28) and the fact that $\alpha \ge 1$ implies

$$|v\cdot\nabla_x F_\varepsilon| \le CF_\varepsilon\frac{\varepsilon^{\alpha-1}v\cdot\nabla_x E}{1+|v|} \le C\varepsilon^{\alpha-1}\|\nabla E\|_{L^\infty}F_\varepsilon,$$

which proves (ii). □

Finally, Proposition 2.9 implies

Proposition 3.3. *Assume that* $\alpha \geq 1$. *Then for all* $\varepsilon \leq 1$ *there holds*

$$\int \mathcal{T}_\varepsilon(f)(v)\frac{f(v)}{F_\varepsilon}\,\mathrm{d}v \geq \vartheta(R)\|f-\rho_f F_\varepsilon\|^2_{L^2_{F_\varepsilon^{-1}}(\mathbb{R}^d)}, \qquad \text{for all } f\in L^2_{F_\varepsilon^{-1}}(\mathbb{R}^d)\,. \tag{3.57}$$

We can now prove the main result of this section:

Proposition 3.4. *Assume that* $\alpha \in [1,2)$ *and that* (1.5)-(1.10) *hold. Let* f_ε *be the solution of* (1.1) *and let* $\rho_\varepsilon(x,t) = \int_{\mathbb{R}^d} f_\varepsilon(x,v,t)\,\mathrm{d}v$. *Then:*

(i) *The sequence* (f_ε) *is bounded uniformly with respect to* ε *in* $L^\infty\left((0,\infty)\,;L^1\left(\mathbb{R}^d\times\mathbb{R}^d\right)\right)$ *and* (ρ_ε) *is bounded uniformly with respect to* ε *in* $L^\infty\left((0,\infty)\,;L^1\left(\mathbb{R}^d\right)\right)$.

(ii) *For all* $T>0$, (f_ε) *is bounded uniformly with respect to* ε *in* $L^\infty\left((0,T);L^2_{M^{-1}}\left(\mathbb{R}^{2d}\right)\right)$, *and* (ρ_ε) *is bounded uniformly with respect to* ε *in* $L^\infty\left((0,T);L^2\left(\mathbb{R}^d\right)\right)$.

(iii) *The function* f_ε *can be decomposed as* $f_\varepsilon = \rho_\varepsilon F_\varepsilon + g_\varepsilon$ *where* g_ε *satisfies*

$$\|g_\varepsilon\|_{L^2((0,T),L^2_{M^{-1}}(\mathbb{R}^{2d}))} \leq C(T)\varepsilon^{\alpha/2}. \tag{3.58}$$

Proof. Integrating (1.1) with respect to x and v and thanks to the conservation of mass property of the operator Q we obtain that (f_ε) is uniformly bounded in $L^\infty((0,\infty)\,;L^1(\mathbb{R}^{2d}))$. Next using (3.56), we recast (1.1) as

$$\varepsilon^\alpha\partial_t f_\varepsilon + \varepsilon v\cdot\nabla_x f_\varepsilon + \mathcal{T}_\varepsilon(f_\varepsilon) = 0.$$

Multiplying this equation by $f_\varepsilon/F_\varepsilon$ and integrating with respect to x and v we get:

$$\begin{aligned}\frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon\|^2_{L^2_{F_\varepsilon^{-1}}(\mathbb{R}^{2d})} = &-\frac{\varepsilon^\alpha}{2}\int\!\!\int\frac{\partial_t F_\varepsilon}{F_\varepsilon}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x + \frac{\varepsilon}{2}\int\!\!\int\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x\\ &+\int\!\!\int\mathcal{T}_\varepsilon(f_\varepsilon)\frac{f_\varepsilon}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Using (3.57) and Proposition 3.2, we deduce

$$\frac{\varepsilon^\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon\|^2_{L^2_{F_\varepsilon^{-1}}(\mathbb{R}^{2d})} + \vartheta\|f_\varepsilon-\rho_\varepsilon F_\varepsilon\|^2_{L^2_{F_\varepsilon^{-1}}(\mathbb{R}^{2d})} \leq \varepsilon^\alpha C\|f_\varepsilon\|^2_{L^2_{F_\varepsilon^{-1}}(\mathbb{R}^{2d})}. \tag{3.59}$$

In particular this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_\varepsilon\|^2_{L^2(\,\mathrm{d}v\,\mathrm{d}x/F_\varepsilon)} \leq 2C\|f_\varepsilon\|^2_{L^2(\,\mathrm{d}v\,\mathrm{d}x/F_\varepsilon)},$$

and Gronwall's Lemma implies that (f_ε) is uniformly bounded in $L^\infty\Big((0,T);L^2_{F_\varepsilon^{-1}}(\mathbb{R}^{2d})\Big)$ for any $T>0$ and thus in $L^\infty\left((0,T);L^2_{M^{-1}}(\mathbb{R}^{2d})\right)$ thanks to Proposition 3.1. We also deduce that

$$\int\rho_\varepsilon^2\,\mathrm{d}x = \int\left(\int f_\varepsilon\,\mathrm{d}v\right)^2\mathrm{d}x \leq \int\!\!\int\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x \leq C.$$

Finally, integrating (3.59) with respect to t and using Proposition 3.1, we obtain (3.58). □

4 Proof of Theorem 1.1

The proof of our main result relies on the test function method first introduced in [15]. The starting point of the method is the introduction of the following auxiliary test function: Given $\varphi(x,t) \in \mathcal{D}(\mathbb{R}^N \times [0,\infty))$, we denote by $\chi_\varepsilon(x,v,t)$ the unique bounded solution of the auxiliary problem

$$\nu(v)\chi_\varepsilon - \varepsilon v \cdot \nabla_x \chi_\varepsilon = \nu(v)\varphi\,, \tag{4.60}$$

which (integrating (4.60) along the characteristics) yields:

$$\chi_\varepsilon(x,v,t) = \int_0^\infty \mathrm{e}^{-\nu(v)z}\nu(v)\varphi(x+\varepsilon vz,t)\,\mathrm{d}z\,. \tag{4.61}$$

We then have:

Lemma 4.1. *Let f_ε be a weak solution of* (1.1) *and let χ_ε be given by* (4.61). *Then the following weak formulation holds:*

$$\begin{aligned}\int\!\!\int\!\!\int f_\varepsilon\, \partial_t\chi_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \int\!\!\int f^{in}\chi_\varepsilon|_{t=0}\,\mathrm{d}v\,\mathrm{d}x + \varepsilon^{-\alpha}\int\!\!\int\!\!\int \rho_\varepsilon\nu F_\varepsilon(\chi_\varepsilon-\varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ = -\varepsilon^{-1}\int\!\!\int\!\!\int g_\varepsilon(E\cdot\nabla_v\chi_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \varepsilon^{-\alpha}\int\!\!\int\!\!\int K(g_\varepsilon)(\chi_\varepsilon-\varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.62}$$

with

$$g_\varepsilon = f_\varepsilon - \rho_\varepsilon F_\varepsilon, \qquad \rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon\ \mathrm{d}v. \tag{4.63}$$

Proof. Taking χ_ε as a test function in (1.1) and using (4.60), we get

$$\begin{aligned}&-\int\!\!\int\!\!\int f_\varepsilon\,\partial_t\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \int\!\!\int f^{in}\chi_\varepsilon|_{t=0}\,\mathrm{d}v\,\mathrm{d}x \\ &= \varepsilon^{-1}\int\!\!\int\!\!\int f_\varepsilon E\cdot\nabla_v\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{-\alpha}\int\!\!\int\!\!\int K(f_\varepsilon)\chi_\varepsilon - \nu f_\varepsilon\varphi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &= \varepsilon^{-1}\int\!\!\int\!\!\int f_\varepsilon E\cdot\nabla_v\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{-\alpha}\int\!\!\int\!\!\int K(f_\varepsilon)(\chi_\varepsilon-\varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where we used the fact that $\int K(f)\ \mathrm{d}v = \int \nu f\,\mathrm{d}v$ for all f. Using (4.63), we deduce:

$$\begin{aligned}&-\int\!\!\int\!\!\int f_\varepsilon\,\partial_t\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \int\!\!\int f^{in}\chi_\varepsilon|_{t=0}\,\mathrm{d}v\,\mathrm{d}x \\ &= \varepsilon^{-1}\int\!\!\int\!\!\int \rho_\varepsilon F_\varepsilon E\cdot\nabla_v\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{-\alpha}\int\!\!\int\!\!\int \rho_\varepsilon K(F_\varepsilon)(\chi_\varepsilon-\varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \varepsilon^{-1}\int\!\!\int\!\!\int g_\varepsilon E\cdot\nabla_v\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{-\alpha}\int\!\!\int\!\!\int K(g_\varepsilon)(\chi_\varepsilon-\varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Finally, using the definition of F_ε and the fact that $\int K(F)\ \mathrm{d}v = \int \nu F\,\mathrm{d}v$, we find

$$\begin{aligned}&\varepsilon^{-1}\int F_\varepsilon E\cdot\nabla_v\chi_\varepsilon\,\mathrm{d}v + \varepsilon^{-\alpha}\int K(F_\varepsilon)(\chi_\varepsilon-\varphi)\,\mathrm{d}v \\ &\quad = -\varepsilon^{-1}\int (E\cdot\nabla_v F_\varepsilon)\chi_\varepsilon\,\mathrm{d}v + \varepsilon^{-\alpha}\int K(F_\varepsilon)(\chi_\varepsilon-\varphi)\,\mathrm{d}v \\ &\quad = -\varepsilon^{-\alpha}\int (K(F_\varepsilon)-\nu F_\varepsilon)\chi_\varepsilon\,\mathrm{d}v + \varepsilon^{-\alpha}\int K(F_\varepsilon)\chi_\varepsilon - \nu F_\varepsilon\varphi\,\mathrm{d}v \\ &\quad = \varepsilon^{-\alpha}\int \nu F_\varepsilon(\chi_\varepsilon-\varphi)\,\mathrm{d}v\end{aligned}$$

which concludes the proof. □

In order to prove Theorem 1.1 we need to show that the right hand side of (4.62) goes to zero, and to identify the limit of the left hand side. The first point follows from the following result.

Proposition 4.2. *For any test function $\varphi \in \mathcal{D}(\mathbb{R}^N \times [0,\infty))$, let χ_ε be defined by* (4.61). *Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha} \int\int\int K(g_\varepsilon)(\chi_\varepsilon - \varphi)\, dx\, dv\, dt = 0,$$

and

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \int\int\int g_\varepsilon E \cdot \nabla_v \chi_\varepsilon \,\mathrm{d}v\, \mathrm{d}x\, \mathrm{d}t = 0.$$

We will give a proof of this proposition which holds for any $\alpha \in (0,2)$ (and not just $\alpha > 1$), since we will use the result for $\alpha = 1$ in the next section.

Proof. To prove the first convergence, we note that

$$\begin{aligned} |K(g_\varepsilon)(x,t)| &= \left| \int \sigma(v,v') g_\varepsilon(x,v',t)\, dv' \right| M(v) \\ &\leq \nu_2 \left(\int \frac{g_\varepsilon(x,v',t)^2}{M(v')}\, dv' \right)^{1/2} M(v). \end{aligned}$$

Therefore

$$\begin{aligned} \left| \varepsilon^{-\alpha} \int\int\int K(g_\varepsilon)(\chi_\varepsilon - \varphi)\, dx\, dv\, dt \right| &\leq \varepsilon^{-\alpha}\nu_2 \int\int \|g_\varepsilon(x,\cdot,t)\|_{L^2_{M^{-1}}} \int_{\mathbb{R}^d} M(v)|\chi_\varepsilon - \varphi|\, dv\, dx\, dt \\ &\leq \varepsilon^{-\alpha}\nu_2 \|g_\varepsilon\|_{L^2_{M^{-1}}} \left(\int\int \left(\int_{\mathbb{R}^d} M(v)|\chi_\varepsilon - \varphi|\, dv \right)^2 dx\, dt \right)^{1/2}, \end{aligned} \tag{4.64}$$

and we conclude thanks to the following result:

Lemma 4.3. *For all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and all $\eta < \alpha$, there exists a constant C depending on η such that*

$$\left(\int \left(\int_{\mathbb{R}^d} M(v)|\chi_\varepsilon - \varphi|\, dv \right)^2 dx \right)^{1/2} \leq C \|\varphi(\cdot,t)\|_{H^1(\mathbb{R}^d)} \varepsilon^\eta.$$

Postponing the proof of this lemma to the end of this proof, we deduce (using (3.58) and the fact that $\varphi(x,t) = 0$ is compactly supported in t):

$$\begin{aligned} \left| \varepsilon^{-\alpha} \int\int\int K(g_\varepsilon)(\chi_\varepsilon - \varphi)\, dx\, dv\, dt \right| &\leq C\varepsilon^{\eta-\alpha} \|g_\varepsilon\|_{L^2_{M^{-1}}(\mathbb{R}^{2d}\times(0,T))} \|\varphi\|_{L^2(0,\infty;H^1(\mathbb{R}^d))} \\ &\leq C \|\varphi\|_{L^2(0,\infty;H^1(\mathbb{R}^d))} \varepsilon^{\eta-\alpha/2}, \end{aligned}$$

and the result follows by choosing any $\eta \in (\alpha/2, \alpha)$.

To prove the second limit, we first rewrite (4.61) as

$$\chi_\varepsilon(x,v,t) = \int_0^\infty \mathrm{e}^{-s} \varphi\left(x + \varepsilon \frac{v}{\nu(v)} z, t \right) \mathrm{d}s$$

and observe that

$$\frac{1}{\varepsilon} \partial_{v_i} \chi_\varepsilon = \int_0^\infty s\mathrm{e}^{-s} \partial_{x_j} \varphi\left(x + \varepsilon \frac{v}{\nu(v)} z, t \right) \partial_{v_i} \left(\frac{v_j}{\nu(v)} \right) \mathrm{d}z. \tag{4.65}$$

Next let us note that thanks to (1.11) we obtain

$$\left|\partial_{v_i}\left(\frac{v_j}{\nu(v)}\right)\right| \le \frac{1}{\nu(v)} + \frac{|v|\,|\nabla_v \nu(v)|}{\nu(v)^2} \le C,$$

for all $1 \le i,j \le d$. Using Jensen's inequality, we deduce:

$$\begin{aligned}\int\!\!\int \left|\frac{1}{\varepsilon}\,\nabla_v \chi_\varepsilon\right|^2 M(v)\,\mathrm{d}v\,\mathrm{d}x &\le C \int\!\!\int\!\!\int_0^\infty s\mathrm{e}^{-s}\left|\nabla_x\varphi\left(x+\varepsilon\frac{v}{\nu(v)}z,t\right)\right|^2 \mathrm{d}z\,M(v)\,\mathrm{d}v\,\mathrm{d}x\\ &\le C\|\nabla_x\varphi(\cdot,t)\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore, by Cauchy-Schwarz

$$\left|\varepsilon^{-1}\int\!\!\int\!\!\int g_\varepsilon(E\cdot\nabla_v\chi_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right| \le \|g_\varepsilon\|_{L^2_{M^{-1}}}\|E\|_{L^\infty}\|\nabla_x\varphi\|_{L^2(\mathbb{R}^d\times(0,\infty))},$$

which completes the proof thanks to (3.58). □

Proof of Lemma 4.3. For any $\delta > 0$ we can write:

$$\begin{aligned}\left(\int_{\mathbb{R}^d} M(v)|\chi_\varepsilon-\varphi|\,\mathrm{d}v\right)^2 &\le C\left(\int_{\mathbb{R}^d}\frac{1}{(1+|v|)^{d+\alpha}}|\chi_\varepsilon-\varphi|\,\mathrm{d}v\right)^2\\ &\le C\left(\int_{\mathbb{R}^d}\frac{1}{(1+|v|)^{d+\delta}}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d}\frac{1}{(1+|v|)^{d+2\alpha-\delta}}|\chi_\varepsilon-\varphi|^2\,\mathrm{d}v\right)\\ &\le C_\delta\int_{\mathbb{R}^d}\frac{1}{(1+|v|)^{d+2\alpha-\delta}}|\chi_\varepsilon-\varphi|^2\,\mathrm{d}v.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}|\chi_\varepsilon-\varphi| &= \left|\int_0^\infty \mathrm{e}^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz)-\varphi(x)]\,\mathrm{d}z\right|\\ &\le \left(\int_0^\infty \mathrm{e}^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz)-\varphi(x)]^2\,\mathrm{d}z\right)^{1/2}\end{aligned}$$

and so

$$\int_{\mathbb{R}^d}|\chi_\varepsilon-\varphi|^2\,\mathrm{d}x \le \int_0^\infty \mathrm{e}^{-\nu(v)z}\nu(v)\int_{\mathbb{R}^d}[\varphi(x+\varepsilon vz)-\varphi(x)]^2\,\mathrm{d}x\,\mathrm{d}z.$$

Finally, using the inequalities

$$\int_{\mathbb{R}^d}[\varphi(x+\varepsilon vz)-\varphi(x)]^2\,\mathrm{d}x \le 2\|\varphi(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}$$

and

$$\int_{\mathbb{R}^d}[\varphi(x+\varepsilon vz)-\varphi(x)]^2\,\mathrm{d}x \le \|\nabla_x\varphi(\cdot,t)\|^2_{L^2(\mathbb{R}^d)}|\varepsilon vz|^2$$

we note that for any $\eta\in(0,1)$ there exists a constant C such that

$$\int_{\mathbb{R}^d}[\varphi(x+\varepsilon vz)-\varphi(x)]^2\,\mathrm{d}x \le C\|\varphi(\cdot,t)\|^2_{H^1(\mathbb{R}^d)}(\varepsilon|v|z)^{2\eta}.$$

We deduce

$$\int\left(\int_{\mathbb{R}^d} M(v)|\chi_\varepsilon-\varphi|\,\mathrm{d}v\right)^2\mathrm{d}x \le C\varepsilon^{2\eta}\|\varphi(\cdot,t)\|^2_{H^1(\mathbb{R}^d)}\int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-\nu(v)z}\nu(v)\frac{(|v|z)^{2\eta}}{(1+|v|)^{d+2\alpha-\delta}}\,\mathrm{d}z\,\mathrm{d}v$$

where the last integral is finite provided we choose $\eta<\alpha$ and then $\delta<2(\alpha-\eta)$. □

Having proved that the two terms in the right hand side of (4.62) go to zero as $\varepsilon \to 0$, we now prove the following result, which shows how the asymptotic equation appears when passing to the limit in (4.62):

Proposition 4.4. *Let $\mathcal{L}^\varepsilon$ be the operator defined by*

$$\mathcal{L}^\varepsilon(\varphi)(x,t) := \varepsilon^{-\alpha} \int \nu F_\varepsilon(\chi_\varepsilon - \varphi)\, \mathrm{d}v$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d \times (0,\infty))$, where χ_ε is defined by (4.61)*. Then*

$$\mathcal{L}^\varepsilon(\varphi) \longrightarrow \mathcal{L}(\varphi) := -\kappa(-\Delta)^{\alpha/2}(\varphi) - (DE)\cdot\nabla_x\varphi \qquad \text{as } \varepsilon \to 0$$

uniformly and in L^2. The matrix D is defined by (1.17) *and κ is given by* (1.16)*.*

The key to the proof of this proposition is the following immediate consequence of Proposition 2.3:

Proposition 4.5. *When $\alpha > 1$, the function F_ε satisfies*

$$F_\varepsilon(x,v,t) = M(v) + \varepsilon^{\alpha-1} E(x,t)\cdot\lambda(v) + G_\varepsilon(x,v,t) \tag{4.66}$$

where $\lambda(v)$ is given by (2.51) *and G_ε satisfies:*

$$|G_\varepsilon(x,v,t)| \le C\varepsilon^{2(\alpha-1)}|E(x,t)|^2 M(v) \qquad \text{for all } (x,v,t). \tag{4.67}$$

Proof of Proposition 4.4. Using Proposition 4.5 above, we write

$$\mathcal{L}^\varepsilon(\varphi) = L_1^\varepsilon + L_2^\varepsilon + L_3^\varepsilon \tag{4.68}$$

where

$$\begin{aligned} L_1^\varepsilon &= \varepsilon^{-\alpha}\int \nu M(v)(\chi_\varepsilon - \varphi)\,\mathrm{d}v \\ L_2^\varepsilon &= \varepsilon^{-1}\int \nu E(x,t)\cdot\lambda(v)(\chi_\varepsilon - \varphi)\,\mathrm{d}v \\ L_3^\varepsilon &= \varepsilon^{-\alpha}\int \nu G_\varepsilon(\chi_\varepsilon - \varphi)\,\mathrm{d}v. \end{aligned}$$

The first term converges to $-\kappa(-\Delta)^{\alpha/2}(\varphi)$ uniformly and in L^2, as was proved, for instance in [15].

For the second term, we note that

$$L_2^\varepsilon = E(x,t)\cdot\left(\varepsilon^{-1}\int \nu\lambda(v)(\chi_\varepsilon - \varphi)\,\mathrm{d}v\right)$$

and we conclude thanks to the following lemma (which is proved below):

Lemma 4.6. *For any test function φ, we have*

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_{\mathbb{R}^d}\nu\lambda(v)(\chi_\varepsilon - \varphi)\,\mathrm{d}\,v = \int_{\mathbb{R}^d}\lambda(v)(v\cdot\nabla_x\varphi(x,t))\,\mathrm{d}\,v = D^T\nabla_x\varphi$$

where the limit holds uniformly and in L^2.

Finally, for the last term in (4.68), we write

$$\begin{aligned}\chi_\varepsilon - \varphi &= \int_0^\infty \mathrm{e}^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz,t)-\varphi(x,t)]\,\mathrm{d}z\\ &= \int_0^\infty \mathrm{e}^{-\nu(v)z}\nu(v)\int_0^z \varepsilon v\cdot\nabla_x\varphi(x+\varepsilon vs,t)\,\mathrm{d}s\,\mathrm{d}z,\end{aligned} \tag{4.69}$$

which gives:

$$\begin{aligned}L_3^\varepsilon &= \varepsilon^{-\alpha}\int_{\mathbb{R}^d}\nu G_\varepsilon(\chi_\varepsilon-\varphi)\,\mathrm{d}v\\ &= \varepsilon^{1-\alpha}\int_{\mathbb{R}^d}\int_0^\infty\int_0^z \mathrm{e}^{-\nu(v)z}\nu(v)^2 G_\varepsilon v\cdot\nabla_x\varphi(x+\varepsilon vs,t)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}v.\end{aligned}$$

Using (4.67), we deduce

$$|L_3^\varepsilon| \le C\varepsilon^{\alpha-1}|E(x,t)|^2\int_{\mathbb{R}^d}\int_0^\infty\int_0^z \mathrm{e}^{-\nu(v)z}\nu(v)^2 M(v)v\cdot\nabla_x\varphi(x+\varepsilon vs,t)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}v.$$

Next, thanks to the fact that $\int_{\mathbb{R}^d}|v|M(v)\,\mathrm{d}v$ is finite (since $\alpha>1$) we obtain

$$\|L_3^\varepsilon\|_{L^\infty(\mathbb{R}^d\times(0,\infty))} \le C\varepsilon^{\alpha-1}|E(x,t)|^2\|\nabla_x\varphi\|_{L^\infty},$$

and applying Jensen's inequality we get

$$|L_3^\varepsilon|^2 \le C(\varepsilon^{\alpha-1}|E(x,t)|^2)^2\int_{\mathbb{R}^d}\int_0^\infty\int_0^z \mathrm{e}^{-\nu(v)z}\nu(v)^2 M(v)|v||\nabla_x\varphi(x+\varepsilon vs,t)|^2\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}v,$$

hence

$$\|L_3^\varepsilon\|_{L^2(\mathbb{R}^d\times(0,T))} \le C\varepsilon^{\alpha-1}\|E(x,t)\|_{L^\infty}^2\|\nabla_x\varphi\|_{L^2(\mathbb{R}^d\times(0,T))},$$

which completes the proof. □

Proof of Lemma 4.6. First, using (4.69) we obtain

$$\varepsilon^{-1}\int_{\mathbb{R}^d}\nu\lambda_i(v)(\chi_\varepsilon-\varphi)\,\mathrm{d}v = \int_{\mathbb{R}^d}\int_0^\infty\int_0^z \mathrm{e}^{-\nu(v)z}\nu(v)^2\lambda_i(v)v\cdot\nabla_x\varphi(x+\varepsilon vs,t)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}v. \tag{4.70}$$

Next, we note that for any $\delta\in(0,1)$, we have

$$|\nabla_x\varphi(x+\varepsilon vs,t)-\nabla_x\varphi(x,t)| \le C|\varepsilon vs|^\delta,$$

and

$$\int_0^\infty\int_0^z \mathrm{e}^{-\nu(v)z}\nu(v)^2\,\mathrm{d}s\,\mathrm{d}z = 1$$

where the first inequality follows from the two inequalities $|\nabla_x\varphi(x+y)-\nabla_x\varphi(x)|\le C$ (for $|y|\ge 1$) and $|\nabla_x\varphi(x+y)-\nabla_x\varphi(x)|\le C|y|$ (for $|y|\le 1$). Hence, thanks to (2.52) we deduce

$$\begin{aligned}&\left|\varepsilon^{-1}\int_{\mathbb{R}^d}\nu\lambda_i(v)(\chi_\varepsilon-\varphi)\,\mathrm{d}v - \int_{\mathbb{R}^d}\lambda_i(v)v\,\mathrm{d}v\cdot\nabla_x\varphi(x,t)\right|\\ &\qquad\le C\int_{\mathbb{R}^d}\int_0^\infty\int_0^z \mathrm{e}^{-\nu(v)z}\nu(v)^2\lambda(v)|v||\varepsilon vs|^\delta\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}v\\ &\qquad\le C\varepsilon^\delta\int_{\mathbb{R}^d}M(v)|v|^{1+\delta}\,\mathrm{d}v.\end{aligned}$$

The uniform convergence follows by choosing δ such that $0 < \delta < \alpha - 1$.

Finally, going back to (4.70), we also deduce

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \left| \varepsilon^{-1} \int_{\mathbb{R}^d} \nu\lambda(v)(\chi_\varepsilon - \varphi)\,\mathrm{d}v \right| dx\,dt \\
&\quad \leq \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^\infty\int_0^z \mathrm{e}^{-\nu(v)z}\nu(v)^2\lambda(v)|v||\nabla_x\varphi(x+\varepsilon vs, t)|\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}v\,dx\,dt \\
&\quad \leq \|\nabla_x\varphi\|_{L^1(\mathbb{R}^d\times(0,T))}\int_{\mathbb{R}^d}\int_0^\infty\int_0^z \mathrm{e}^{-\nu(v)z}\nu(v)^2\lambda(v)|v|\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}v \\
&\quad \leq C\|\nabla_x\varphi\|_{L^1(\mathbb{R}^d\times(0,T))}.
\end{aligned}$$

So by a simple interpolation, we see that since the quantity under consideration is bounded in L^1 and converges uniformly, it also converges in L^2. □

Gathering the results above, we can now complete the proof of Theorem 1.1:

Proof of Theorem 1.1. In view of Proposition 3.4 and using a diagonal extraction argument, we can assume (up to a subsequence) that there exist two functions $f(x, v, t)$ and $\rho(x, t)$ such that

$$f_\varepsilon \rightharpoonup f \quad \text{in } L^\infty((0,T); L^2_{M^{-1}}(\mathbb{R}^{2d}))\text{-weak} \star$$

and

$$\rho_\varepsilon \rightharpoonup \rho \quad \text{in } L^\infty((0,T); L^2(\mathbb{R}^d))\text{-weak} \star$$

for all $T > 0$. Furthermore, Proposition 3.4 (iii), together with Proposition 4.5 implies

$$\|f_\varepsilon - \rho_\varepsilon M\|_{L^2(0,T;L^2_{M^{-1}}(\mathbb{R}^{2d}))} \leq C(T)\varepsilon^{\alpha-1}$$

and so

$$f(x, v, t) = \rho(x, t)M(v).$$

Next, we recall that Lemma 4.3 gives:

$$\int M[\chi_\varepsilon - \varphi]\,\mathrm{d}v \longrightarrow 0 \text{ in } L^2(\mathbb{R}^d \times (0,\infty)),$$

and we can prove similarly that

$$\int M[\partial_t\chi_\varepsilon - \partial_t\varphi]\,\mathrm{d}v \longrightarrow 0 \text{ in } L^2(\mathbb{R}^d \times (0,\infty)).$$

Using these facts, it is easy to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\int_0^\infty\int\int f_\varepsilon\,\partial_t\chi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \int\int f^{in}\chi_\varepsilon|_{t=0}\,\mathrm{d}v\,\mathrm{d}x\right) \\
&\qquad = \int_0^\infty\int \rho\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int\int \rho^{in}\varphi|_{t=0}\,\mathrm{d}x.
\end{aligned}$$

Finally combining this limit with Propositions 4.2 and 4.4, we can now pass to the limit in (4.62) to deduce:

$$\int_0^\infty\int \rho\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int\int \rho^{in}\varphi|_{t=0}\,\mathrm{d}x + \int_0^\infty\int \rho\left[-\kappa(-\Delta)^{\alpha/2}(\varphi) - (DE)\cdot\nabla_x\varphi\right]\mathrm{d}x\,\mathrm{d}t = 0$$

which is the weak formulation of (1.15). □

5 Proof of Theorem 1.2

Before proving Theorem 1.2, we need to show that $\mu(E)$ defined by (1.20) is well defined:

Lemma 5.1. *The function $R(v,E) = F(v,E) - M(v)$ satisfies*

$$|R(v,E)| \leq C|E| \frac{M(v)}{1+|v|},$$

For some constant $C > 0$. In particular, the quantity $\mu(E)$ defined by (1.20) is well defined for all $E \in \mathbb{R}^d$ and satisfies $|\mu(E)| \leq C|E|$.

Postponing the proof of this lemma to the end of this section, we turn to the proof of Theorem 1.2:

Proof of Theorem 1.2. When $\alpha = 1$, $F_\varepsilon(x,v,t) = F(v, E(x,t)$ is independent of ε (we thus drop the ε subscript below) and the weak formulation (4.62) takes the form

$$\begin{aligned}\int\!\!\int\!\!\int f_\varepsilon\, \partial_t \chi_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \int\!\!\int f^{in} \chi_\varepsilon|_{t=0} \,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\int\!\!\int\!\!\int \rho_\varepsilon \nu F(\chi_\varepsilon - \varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ = -\frac{1}{\varepsilon}\int\!\!\int\!\!\int g_\varepsilon (E\cdot \nabla_v \chi_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\varepsilon}\int\!\!\int\!\!\int K(g_\varepsilon)(\chi_\varepsilon - \varphi)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t. \quad (5.71)\end{aligned}$$

Proceeding as in the proof of Theorem 1.1, we have (see Proposition 3.4):

$$f_\varepsilon \rightharpoonup f \quad \text{in } L^\infty((0,T); L^2_{M^{-1}}(\mathbb{R}^{2d}))\text{-weak} \star$$

and

$$\rho_\varepsilon \rightharpoonup \rho \quad \text{in } L^\infty((0,T); L^2(\mathbb{R}^{d}))\text{-weak} \star$$

for all $T > 0$ and we can write

$$f_\varepsilon = \rho_\varepsilon F + g_\varepsilon$$

where g_ε satisfies

$$\|g_\varepsilon\|_{L^2((0,T), L^2_{M^{-1}}(\mathbb{R}^{2d}))} \leq C(T)\varepsilon^{1/2}. \quad (5.72)$$

This implies in particular that

$$f(x,v,t) = \rho(x,t) F(x,v,t).$$

In order to complete the proof of Theorem 1.2, we need to pass to the limit in the weak formulation (4.62). First, we note that thanks to Proposition 4.2 (which we proved without restriction on α), the right hand side in (5.71) vanishes in the limit. Now let us define the operator $\mathcal{L}^\varepsilon(\varphi)$ as

$$\begin{aligned}\mathcal{L}^\varepsilon(\varphi) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \nu(v) F(v,E)(\chi_\varepsilon - \varphi)\,\mathrm{d}v \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \nu(v) F(v,0)(\chi_\varepsilon - \varphi)\,\mathrm{d}v \\ &\quad + \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \nu(v)\big(F(v,E) - F(v,0)\big)(\chi_\varepsilon - \varphi)\,\mathrm{d}v \\ &= \mathcal{L}_1^\varepsilon(\varphi) + \mathcal{L}_2^\varepsilon(\varphi), \end{aligned} \quad (5.73)$$

where $F(v,0) = M(v)$ thanks to the definition of F given in (2.27).

Proposition 4.4 in [15] gives

$$\mathcal{L}_1^\varepsilon(\varphi) \to \kappa(-\Delta)^{1/2}\varphi \text{ in } L^2\text{-strong}.$$

Furthermore, using formula (4.61) for χ_ε, we can recast $\mathcal{L}_2^\varepsilon(\varphi)$ as follows:

$$\begin{aligned}\mathcal{L}_2^\varepsilon(\varphi) &= \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-\nu(v)z}\nu^2(v)\big(F(v,E)-M\big)(\varphi(x+\varepsilon vz)-\varphi(x))\,\mathrm{d}z\,\mathrm{d}v \qquad (5.74)\\ &= \left(\int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-\nu(v)z}\nu^2(v)\big(F(v,E)-M\big)vz\,\mathrm{d}z\,\mathrm{d}v\right)\cdot\nabla_x\varphi(x,t)\\ &\quad+\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-\nu(v)z}\nu^2(v)\big(F(v,E)-M\big)(\varphi(x+\varepsilon vz)-\varphi(x)-\varepsilon vz\cdot\nabla\varphi(x))\,\mathrm{d}z\;\mathrm{d}v,\\ &= \mu(E)\cdot\nabla_x\varphi(x,t)+\mathcal{R}_\varepsilon\end{aligned}$$

and we can now show that $\mathcal{R}_\varepsilon\to 0$ uniformly in x and t: Indeed, Lemma 5.1 implies

$$|\mathcal{R}_\varepsilon|\le C\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-\nu(v)z}\nu^2(v)\frac{M(v)}{1+|v|}(\varphi(x+\varepsilon vz)-\varphi(x)-\varepsilon vz\cdot\nabla\varphi(x))\,\mathrm{d}z\;\mathrm{d}v$$

and for any $\eta\in[1,2]$, we have

$$|\varphi(x+\varepsilon vz)-\varphi(x)-\varepsilon vz\cdot\nabla\varphi(x)|\le C_\eta(\varepsilon|v|z)^\eta.$$

We deduce

$$|\mathcal{R}_\varepsilon|\le C_\eta\varepsilon^{\eta-1}\int_{\mathbb{R}^d}\int_0^\infty \mathrm{e}^{-\nu(v)z}\nu^2(v)\frac{M(v)}{1+|v|}(|v|z)^\eta\,\mathrm{d}z\;\mathrm{d}v$$

The integral in the right hand side is finite as long as $\eta<2$ so we can take $\eta=3/2$ and deduce

$$\|\mathcal{R}_\varepsilon\|_{L^\infty}\to 0 \text{ as } \varepsilon\to 0$$

We have thus shown that

$$\mathcal{L}_2^\varepsilon(\varphi)\to\mu(E)\cdot\nabla_x\varphi(x,t)$$

uniformly in x and t as $\varepsilon\to 0$, which implies that $\mathcal{L}^\varepsilon(\varphi)$ converges uniformly to

$$-\kappa(-\Delta)^{1/2}(\varphi)(x,t)+\mu(E)\cdot\nabla_x\varphi(x,t).$$

Passing to the limit in (5.71) (the first two terms are handled exactly as in the proof of Theorem 1.1), we deduce

$$\int_0^\infty\int\rho\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\int\rho^{in}\varphi|_{t=0}\,\mathrm{d}x+\int_0^\infty\int\rho\left[-\kappa(-\Delta)^{\alpha/2}(\varphi)-\mu(E)\cdot\nabla_x\varphi\right]\,\mathrm{d}x\,\mathrm{d}t=0$$

which is the weak formulation of (1.19).

□

Proof of Lemma 5.1. First, we note that for any $E\in\mathbb{R}^d$, the function $v\mapsto R(v,E)$ solves

$$\mathcal{T}(R)=-E\cdot\nabla_v M.$$

Using the coercivity property of $\mathcal{T}$ (2.43) and the fact that $\int_{\mathbb{R}^d}R(v,E)\,\mathrm{d}v=0$, we deduce

$$\left(\int\frac{R^2}{F}\,\mathrm{d}v\right)^{1/2}\le C|E|\left(\int\frac{|\nabla M|^2}{F}\,\mathrm{d}v\right)^{1/2}\le C|E|.$$

Next, we rewrite the equation for R as

$$\nu R-E\cdot\nabla_v R=\mathcal{K}(R)-E\cdot\nabla_v M. \qquad (5.75)$$

Using the fact that $\int_{\mathbb{R}^d} R(v,E)\,dv = 0$, we can write

$$\mathcal{K}(R)(v) = \int_{\mathbb{R}^d} (\sigma(v,v') - \nu_0) R(v')\,dv',$$

and so using (1.13), we obtain

$$\begin{aligned} |\mathcal{K}(R)| &\leq \int |\sigma - \nu_0||R(v',E)|\,\mathrm{d}v' M(v) \\ &\leq \frac{CM(v)}{1+|v|} \int |R(v',E)|\,\mathrm{d}v' \\ &\leq \frac{CM(v)}{1+|v|} \left(\int |R(v',E)|^2 \frac{\mathrm{d}v'}{F(v',E)} \right)^{1/2} \\ &\leq C|E| \frac{M(v)}{1+|v|}. \end{aligned} \tag{5.76}$$

Finally, assumptions (1.8) yields

$$|E \cdot \nabla_v M(v)| \leq C|E| \frac{M(v)}{1+|v|}.$$

We thus have

$$|\nu R - E \cdot \nabla_v R| \leq C|E| \frac{M(v)}{1+|v|},$$

which implies (using Remark 2.8)

$$|R(v,E)| \leq C|E| \frac{M(v)}{1+|v|} \qquad \text{for all } v \in \mathbb{R}^d,\ E \in \mathbb{R}^d$$

and the lemma follows. □

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Fractional diffusion limit for a fractional Vlasov-Fokker-Planck equation

Pedro Aceves-Sánchez and Ludovic Cesbron

Submitted

Abstract

This paper is devoted to the rigorous derivation of the macroscopic limit of a Vlasov-Fokker-Planck equation in which the Laplacian is replaced by a fractional Laplacian. The evolution of the density is governed by a fractional heat equation with the addition of a convective term coming from the external force. The analysis is performed by a modified test function method and by obtaining a priori estimates from quadratic entropy bounds. In addition, we give the proof of existence and uniqueness of solutions to the Vlasov-fractional-Fokker-Planck equation.

1 Introduction

1.1 The Vlasov-Lévy-Fokker-Planck equation

In this paper we investigate the long-time/small mean-free-path asymptotic behavior in the low-field case of the solution of the Vlasov-Lévy-Fokker-Planck (VLFP) equation

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = \nabla_v \cdot (vf) - \big(- \Delta_v\big)^{\alpha/2} f \qquad \text{in } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \quad (1.1a)$$

$$f(0,x,v) = f^{in}(x,v) \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^d, \quad (1.1b)$$

where $\alpha \in (1,2)$. This equation describes the evolution of the density of an ensemble of particles denoted as $f(t,x,v)$ in phase space, where $t \geq 0$, $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$ stand for, respectively, time, position and velocity. The operator $(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian and is defined by (1.5). Let us recall that, at a microscopic level, equation (1.1a)-(1.1b) is related to the Langevin equation

$$\begin{aligned} \mathrm{d}x(t) &= v(t)\,\mathrm{d}t, \\ \mathrm{d}v(t) &= -v(t)\,\mathrm{d}t + E\,\mathrm{d}t + \mathrm{d}L_t^\alpha, \end{aligned} \qquad (1.2)$$

where L_t^α is a Markov process with generator $-(-\Delta)^{\alpha/2}$ and $(x(t),v(t))$ describe the position and velocity of a single particle (see [15] and [20]). Therefore, this models describes the position and velocity of a particle that is affected by three mechanisms: a dragging force, an acceleration and a pure jump process.

In the particular case when $\alpha = 2$ the fractional operator $(-\Delta)^{\alpha/2}$ takes the form of a Laplace operator Δ and (1.1a)-(1.1b) reduces to the usual Vlasov-Fokker-Planck equation. In this case the Fokker-Planck operator is known to have an equilibrium distribution function given by a Maxwellian $M(v) = C\exp\left(-|v|^2\right)$ where $C > 0$ is a normalization constant. The Vlasov-Fokker-Planck equation has been used in the modeling of many physical phenomena, in particular, for the description of the evolution of plasmas [20]. However, there are some settings in which particles may have long jumps and an α-stable distribution process is more suitable to describe the phenomenon, see for instance [21].

The case in which $\alpha = 2$ reduces to the classical Vlasov-Fokker-Planck equation for a given external field. This equation is related to the Vlasov-Poisson-Fokker-Planck system (VPFP) in the case in which the electric field is self-consistent. Questions such as existence of solutions, hydrodynamic limits and long time behaviour for the VPFP system has been extensively studied by many authors, see for instance [6], [19], and [14]. In particular, in [13] the low field limit is studied for the VPFP system and a Drift-Diffusion-Poisson system is obtained in a rigorous manner.

Let us note that, although it is classical in the framework of kinetic theory to consider a self-consistence electric fields that expresses how particles repulse one another, one can also, in the VPFP system, consider the case in which particles are attracted by each other and this model is used in the description of galactic dynamics.

In the rest of the paper we shall need the following notation: The fractional (or Lévy) Fokker-Planck operator denoted by $\mathcal{L}^{\alpha/2}$ and defined as

$$\mathcal{L}^{\alpha/2} f = \nabla_v \cdot \big(vf\big) - \big(-\Delta_v\big)^{\alpha/2} f. \tag{1.3}$$

In order to investigate the asymptotic behaviour of the system, we introduce the Knudsen number ε which represent the ratio between the mean-free-path and the observation length scale. In the case when $E = 0$ it was observed in [9] that the time rescaling $t' \to \varepsilon^{\alpha-1} t$ and introducing a factor $1/\varepsilon$ in front of $\mathcal{L}^{\alpha/2}$ is the appropriate scaling at which diffusion will be observed in the limit as ε goes to zero. Moreover, we introduce the factor $1/\varepsilon^{2-\alpha}$ in front of the force field term E corresponding to a low-field limit scaling since we shall consider the case $1 \leq \alpha \leq 2$ and thus the scaling of the collision operator $1/\varepsilon$ is much greater than the scaling of the electric field $1/\varepsilon^{2-\alpha}$. Thus we shall study in this paper the asymptotic behaviour as ε tends to zero of the solutions of following rescaled VLFP equation

$$\varepsilon^{\alpha-1}\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \varepsilon^{\alpha-2} E(t,x) \cdot \nabla_v f^\varepsilon = \frac{1}{\varepsilon}\Big(\nabla_v \cdot (vf) - \big(-\Delta_v\big)^{\alpha/2} f\Big). \tag{1.4}$$

1.2 Preliminaries on the Fractional Fokker-Planck operator

In this paper we denote by $\widehat{f}$ or $\mathcal{F}(f)$ the Fourier transform of f and define it as

$$\widehat{f}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{-ik\cdot x} f(x)\,\mathrm{d}x.$$

There are several equivalent definitions of the fractional Laplacian in the whole domain (see [16] or [18]). It can be defined via a Fourier multiplier as

$$\mathcal{F}\Big((-\Delta)^{\alpha/2}(f)\Big)(k) = |k|^\alpha \mathcal{F}(f)(k).$$

On the other hand, assuming that f is a rapidly decaying function we can define the fractional Laplacian in terms of a hypersingular integral as

$$\big(-\Delta_v\big)^{\alpha/2}(f)(v) = c_{d,\alpha}\,\mathrm{P.V.}\int_{\mathbb{R}^d} \frac{f(v) - f(w)}{|v-w|^{d+\alpha}}\,\mathrm{d}w \tag{1.5}$$

where P.V. denotes the Cauchy principal value and the constant $c_{d,\alpha}$ is given by

$$c_{d,\alpha} = \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{2\pi^{d/2} |\Gamma\left(-\frac{\alpha}{2}\right)|}, \tag{1.6}$$

and $\Gamma(\cdot)$ denotes the Gamma function. In [18] it is proven that for any $d > 1$, $c_{d,\alpha} \to 0$ as $\alpha \to 2$. Thus (1.5) does not make sense if we take $\alpha = 2$. However, we have the following result.

Proposition 1.1. *Let $d > 1$. Then for any $f \in C_0^\infty(\mathbb{R}^d)$ we have*

$$\lim_{\alpha \to 2} (-\Delta)^{\alpha/2} f = -\Delta f.$$

For an account of the properties of the fractional Laplacian consult [18], [25], [24] or [17]. Let us note that due to its dependence on the whole domain, the fractional Laplacian is a nonlocal operator and it has the scaling property $\left(-\Delta_v\right)^{\alpha/2}(f_\lambda)(v) = \lambda^\alpha \left(-\Delta_v\right)^{\alpha/2} f(\lambda v)$, for any $\lambda > 0$ where $f_\lambda(v) = f(\lambda v)$. Since it will be useful later on in our analysis, we also mention that since the fractional Laplacian is an integro-differential operator it satisfies:

$$\int (-\Delta)^{\alpha/2} f \,\mathrm{d}v = 0.$$

In [3] it is proved that the Lévy-Fokker-Planck operator $\mathcal{L}^{\alpha/2}$ defined by (1.3) has a unique normalized equilibrium distribution that we shall denote by G_α. Therefore, the Fourier transformation of G_α denoted as $\widehat{G_\alpha}$ and defined as

$$\widehat{G_\alpha}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-i\xi\cdot v} G_\alpha(v) \,\mathrm{d}v,$$

satisfies

$$\xi \cdot \nabla_\xi \widehat{G_\alpha} + |\xi|^\alpha \widehat{G_\alpha} = 0.$$

Thus yielding

$$\widehat{G_\alpha}(\xi) = \mathrm{e}^{-|\xi|^\alpha/\alpha}. \tag{1.7}$$

In the jargon of stochastic analysis, random variables having a characteristic function of the form (1.7) are called symmetric α-stable random variables, consult [2]. Using the notation of [4] let us note that setting $t = 1/\alpha$, $x = v$, and $y = 0$, we obtain the identity $G_\alpha(v) = p(1/\alpha, v, 0)$. Thus Lemma 3 of [4] states that there exists $C_1 = C_1(d,\alpha) > 0$ such that

$$C_1^{-1}\left(\frac{1}{\alpha|v|^{d+\alpha}} \wedge \frac{1}{\alpha^{d/\alpha}}\right) \le G_\alpha(v) \le C_1\left(\frac{1}{\alpha|v|^{d+\alpha}} \wedge \frac{1}{\alpha^{d/\alpha}}\right), \tag{1.8}$$

for all $v \in \mathbb{R}^d$, where $a \wedge b$ denotes the minimum between a and b. On the other hand, Lemma 5 of [4] states the existence of a positive constant $C_2 = C_2(d,\alpha)$ such that

$$\frac{|v|}{C_2}\left(\frac{1}{\alpha|v|^{d+2+\alpha}} \wedge \alpha^{(d+2)/2}\right) \le \nabla_v\, G_\alpha(v) \le C_2|v|\left(\frac{1}{\alpha|v|^{d+2+\alpha}} \wedge \alpha^{(d+2)/2}\right). \tag{1.9}$$

1.3 Main results

As usually in the framework of fractional Vlasov-Fokker-Planck equations, we use the following definition of weak solutions:

Definition 1.2. Consider f^{in} in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $E \in \left(W^{1,\infty}([0,T) \times \mathbb{R}^d)\right)^d$. We say that f is a weak solution of (1.1a)-(1.1b) if, for any $\varphi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{R}^d \times \mathbb{R}^d)$

$$\begin{aligned}\iiint_{Q_T} f\Big(\partial_t\varphi + v\cdot\nabla_x\varphi + \big(E(t,x)-v\big)\cdot\nabla_v\varphi - (-\Delta)^{\alpha/2}\varphi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\ + \iint_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\varphi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0.\end{aligned} \tag{1.10}$$

Section 2 of this paper is devoted to a well-posedness result for the fractional Vlasov-Fokker-Planck with an external electric field E in the following sense.

Theorem 1.3. *For f^{in} in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $E \in \left(W^{1,\infty}([0,T) \times \mathbb{R}^d)\right)^d$ there exists a unique weak solution f of* (1.1a)-(1.1b) *in the sense of Definition 1.2 and it satisfies*

$$f(t,x,v) \geq 0 \text{ on } Q_T, \tag{1.11a}$$

$$f \in \mathcal{X} := \left\{ f \in L^2(Q_T) : \frac{|f(t,x,v)-f(t,x,w)|}{|v-w|^{\frac{d+\alpha}{2}}} \in L^2(Q_T\times\mathbb{R}^d)\right\}. \tag{1.11b}$$

Remark 1.4. The assumption $E \in \left(W^{1,\infty}([0,T) \times \mathbb{R}^d)\right)^d$ in Theorem 1.3 is not optimal in the sense that we could replace it by $E \in \left(L^\infty([0,T) \times \mathbb{R}^d)\right)^d$ or maybe it could be replaced by even weaker assumptions on E, however, finding the optimal regularity of E is out of the scope of this paper.

The proof of this existence result relies on using the Lax-Milgram theorem for a well chosen associated problem, in the spirit of the proof in [10] and in [7] for the existence of weak solutions of the Vlasov-Fokker-Planck equation. The proof of positivity (1.11a) is given in details as it involves the non-local nature of the fractional operator and, as such, differs from the classical proof.
In Section 3, we consider the electric field as a perturbation of the fractional Fokker-Planck operator and as such we introduce $\mathcal{T}_\varepsilon$:

$$\mathcal{T}_\varepsilon(f) := \nabla_v \cdot \Big[\big(v - \varepsilon^{\alpha-1}E(t,x)\big)f\Big] - \big(-\Delta_v\big)^{\alpha/2} f.$$

We prove existence and uniqueness of a normalized equilibrium F_ε for this perturbed operator in Proposition 3.1. Then, we follow the strategy introduced in [1]; we investigate the decay properties of this equilibrium and its convergence to the equilibrium of the unperturbed operator, G_α, as ε goes to 0 in Proposition 3.2. Finally, we prove that $\mathcal{T}_\varepsilon$ is dissipative with regards to the quadratic entropy, Proposition 3.3, which allows us to establish uniform boundedness results for f_ε, the solution of the rescaled equation (1.4)-(1.1b), as well as its macroscopic density $\rho_\varepsilon = \int f_\varepsilon\,\mathrm{d}v$ and its distance to the kernel of $\mathcal{T}_\varepsilon$ we which write r_ε defined by the expansion $f_\varepsilon = \rho_\varepsilon F_\varepsilon + \varepsilon^{\alpha/2} r_\varepsilon$.

In the last section, we turn to the proof of our main result which is the anomalous advection-diffusion limit of our kinetic model. We follow the method introduced in [9] which consist in choosing a test function $\psi_\varepsilon(t,x,v)$ which is solution, for some $\varphi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{R}^d)$ of the auxiliary problem:

$$\begin{aligned} &\varepsilon v\cdot\nabla_x\psi_\varepsilon - v\cdot\nabla_v\psi_\varepsilon = 0 && \text{in } [0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,\\ &\psi(t,x,0) = \varphi(t,x) && \text{in } [0,\infty)\times\mathbb{R}^d,\end{aligned}$$

and show that the weak formulation of our problem, (2.14), with such test functions converges to the weak formulation of the advection fractional diffusion equation. We first prove this convergence in the non-critical case, *i.e.* when $1 < \alpha < 2$ and then we turn to the critical cases $\alpha = 1$ and $\alpha = 2$. The outline of the proof remains the same in both critical cases but a few differences appear, for $\alpha = 2$ the only difference is technical one in the study of the dissipative property of the perturbed operator whereas, in the case $\alpha = 1$, we show that the equilibrium of the perturbed operator is independent of ε and as such it stays perturbed by the electric field $E(t, x)$ even in the macroscopic limit. In all cases, our main result reads:

Theorem 1.5. *Let α be in $(1, 2]$ and f_ε be the weak solution of (1.4)-(1.1b) in the sense of Definition 1.2 on $[0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ for some $T > 0$ and with $f^{in} \in L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d) \cap L^1_+(\mathbb{R}^d \times \mathbb{R}^d)$. Then, f_ε converges weak-$*$ to $\rho(t, x)\, G_\alpha(v)$ in $L^\infty(0, T; L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$, where ρ is the solution in the distributional sense of*

$$\begin{aligned} &\partial_t \rho + \mathrm{div}\,(E\rho) + (-\Delta)^{\alpha/2}\rho = 0 && \text{in } [0, T) \times \mathbb{R}^d, \\ &\rho(0, x) = \rho^{in}(x) && \text{in } \mathbb{R}^d, \end{aligned} \tag{1.12}$$

where $\rho^{in} = \int f^{in}\, \mathrm{d}v$. In the case $\alpha = 1$ the same anomalous diffusion limit holds but instead of $G_\alpha(v)$ the equilibrium distribution of velocity becomes

$$G_{\alpha,E}(t, x, v) = G_\alpha\big(v - E(t, x)\big) \tag{1.13}$$

The advection fractional-diffusion equation (1.12) describes the evolution of the macroscopic density ρ under the effect of a drift, consequence of the kinetic electric field, and a fractional diffusion phenomenon. The regularity of the solutions of this type of equations has been studied for instance in [22], [23], and [11]. We refer the interested reader to those articles and references within for more details on this macroscopic model.

2 Existence of solution

Throughout this paper, for any $T > 0$ we write $Q_T = [0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ and $\mathcal{C}_c^\infty(Q_T)$ the set of smooth function compactly supported in Q_T. This section is devoted to the proof of the following result of existence and regularity of weak solutions:

Theorem 2.1. *Consider f^{in} in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. There exists a unique weak solution f of (1.1a) on Q_T in the sense that for any $\varphi \in \mathcal{C}_c^\infty(Q_T)$:*

$$\begin{aligned} &\iiint_{Q_T} f\Big(\partial_t\varphi + v \cdot \nabla_x \varphi + \big(E(t, x) - v\big) \cdot \nabla_v \varphi - (-\Delta)^{\alpha/2}\varphi\Big)\, dt\, dx\, dv \\ &\quad + \iint_{\mathbb{R}^d \times \mathbb{R}^d} f^{in}(x, v)\varphi(0, x, v)\, dx\, dv = 0 \end{aligned} \tag{2.14}$$

and this solution satisfies:

$$\begin{aligned} &f(t, x, v) \geq 0 \ \text{ on } Q_T, \\ &f \in \mathcal{X} := \Big\{ f \in L^2(Q_T) : \frac{|f(t, x, v) - f(t, x, w)|}{|v - w|^{\frac{d+\alpha}{2}}} \in L^2(Q_T \times \mathbb{R}^d) \Big\}. \end{aligned} \tag{2.15}$$

Remark 2.2. Let us note that the space $\mathcal{X}$ is equal to $L^2((0, T) \times \mathbb{R}^d; H^{\alpha/2}(\mathbb{R}^d))$.

Proof. We follow the method in [10] and in [7] for the proof of existence and uniqueness of solutions to the linear Vlasov-Fokker-Planck equation. The first part of the proof consists in solving our linear problem in a variational setting, applying a well-known Lax-Milgram theorem of functional analysis. We consider the Hilbert space $\mathcal{X}$ provided with the norm

$$||f||_{\mathcal{X}} = \left(||f||^2_{L^2(Q_T)} + 2c^{-1}_{d,\alpha} ||(-\Delta)^{\frac{\alpha}{4}} f||^2_{L^2(Q_T)} \right)^{\frac{1}{2}} \tag{2.16}$$

where $c_{d,s}$ is defined in (1.6). We refer the reader to [18] for properties of this functional space. Let us denote $\mathcal{T}$ the transport operator, given by

$$\mathcal{T} f = \partial_t f + v \cdot \nabla_x f - \big(v - E(t,x)\big) \cdot \nabla_v f.$$

We define the Hilbert space $\mathcal{Y}$ as:

$$\mathcal{Y} = \left\{ f \in \mathcal{X} : \mathcal{T} f \in \mathcal{X}' \right\} \tag{2.17}$$

where $\mathcal{X}'$ is the dual of $\mathcal{X}$. $(\cdot,\cdot)_{\mathcal{X},\mathcal{X}'}$ stands for the dual relation between $\mathcal{X}$ and its dual. $\mathcal{Y}$ is provided with the norm:

$$||f||^2_{\mathcal{Y}} = ||f||^2_{\mathcal{X}} + ||\mathcal{T} f||^2_{\mathcal{X}'}. \tag{2.18}$$

In order to apply the Lax-Milgram Theorem we consider the associated problem

$$\begin{aligned} &\partial_t \overline{f} + e^{-t} v \cdot \nabla_x \overline{f} + e^t E(t,x) \cdot \nabla_v \overline{f} + e^{\alpha t} (-\Delta)^{\alpha/2} \overline{f} + \lambda \overline{f} = 0 \qquad && (t,x,v) \in Q_T \\ &\overline{f}(0,x,v) = \overline{f}^{in}(x,v) && (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \end{aligned} \tag{2.19}$$

which comes formally by deriving (1.1a) for $\overline{f} = e^{-(\lambda+d)t} f\big(t,x,e^{-t}v\big)$ and $\overline{f}^{in}(x,v) = f^{in}(x, e^{-t}v)$ for some $\lambda \geq 0$. A weak solution of (2.19) is a function $\overline{f} \in \mathcal{X}$ such that for any φ in $\mathcal{C}^\infty_c(Q_T)$:

$$\begin{aligned} \iiint_{Q_T} \Big(-\overline{f}\partial_t \varphi - e^{-t} \overline{f} v \cdot \nabla_x \varphi - e^t \overline{f} E(t,x) \cdot \nabla_v \varphi + e^{\alpha t} \overline{f} (-\Delta)^{\alpha/2} \varphi + \lambda \overline{f} \varphi \Big) \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}v \\ - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \overline{f}^{in} \varphi(0,x,v) \,\mathrm{d}x \,\mathrm{d}v = 0. \end{aligned} \tag{2.20}$$

We first prove existence of a solution in $\mathcal{X}$ of equation (2.19) and we will prove afterwards how this implies existence of a solution of the fractional Vlasov-Fokker-Planck equation with the electric field E.

We know that $\mathcal{C}^\infty_c(Q_T)$ is a subspace of $\mathcal{X}$ with a continuous injection (see, e.g. [18]) and we define the prehilbertian norm:

$$|\varphi|^2_{\mathcal{C}^\infty_c(Q_T)} = ||\varphi||^2_{\mathcal{X}} + \frac{1}{2} ||\varphi(0,\cdot,\cdot)||^2_{L^2(\Omega \times \mathbb{R}^d)}.$$

Now, we can introduce the bilinear form $a : \mathcal{X} \times \mathcal{C}^\infty_c(Q_T) \to \mathbb{R}$ as:

$$a(\overline{f},\varphi) = \iiint_{Q_T} \Big(-\overline{f}\partial_t \varphi - e^{-t} \overline{f} v \cdot \nabla_x \varphi - e^t \overline{f} E(t,x) \cdot \nabla_v \varphi + e^{\alpha t} \overline{f} (-\Delta)^{\alpha/2} \varphi + \lambda \overline{f} \varphi \Big) \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}v$$

and the continuous bounded linear operator L on $\mathcal{C}^\infty_c(Q_T)$ given by:

$$L(\varphi) = - \iint_{\mathbb{R}^d \times \mathbb{R}^d} f^{in}(x,v) \varphi(0,x,v) \,\mathrm{d}x \,\mathrm{d}v.$$

To find a solution $\overline{f}$ in $\mathcal{X}$ of equation (2.20) is equivalent to finding a solution $\overline{f}$ in $\mathcal{X}$ of $a(\overline{f},\varphi) = L(\varphi)$ for any $\varphi \in \mathcal{C}_c^\infty(Q_T)$. Since $\overline{f}$ belongs to $\mathcal{X}$ it is easy to check that $a(\cdot,\varphi)$ is continuous. To verify the coercivity of a we write:

$$-\iiint_{Q_T} \Big(\varphi\partial_t\varphi + e^{-t}\varphi v\cdot\nabla_x\varphi - e^t\varphi E(t,x)\cdot\nabla_v\varphi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\varphi(0,x,v)|^2\,\mathrm{d}x\,\mathrm{d}v$$

and also:

$$\iiint_{Q_T} e^{\alpha t}\varphi(-\Delta)^{\alpha/2}\varphi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \iiint_{Q_T} e^{\alpha t}|(-\Delta)^{\frac{\alpha}{4}}\varphi|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v.$$

Hence, we see that

$$a(\varphi,\varphi) = \iiint_{Q_T}\Big(\lambda\varphi^2 + e^{\alpha t}|(-\Delta)^{\frac{\alpha}{4}}\varphi|^2\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\iint_{\Omega\times\mathbb{R}^d}|\varphi(0,x,v)|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$$

which can be bounded from below as $a(\varphi,\varphi) \geq \min(1,\lambda)|\varphi|^2_{\mathcal{C}_c^\infty(Q_T)}$. Thus, the Lax-Milgram theorem implies the existence of $\overline{f}$ in $\mathcal{X}$ satisfying (2.20). Now, we want to show that this yields existence of a solution of (2.14). To that end, we first consider $\tilde{\varphi}$ in $\mathcal{C}_c^\infty(Q_T)$ such that $\varphi(t,x,v) = e^{\lambda t}\tilde{\varphi}(t,x,e^{-t}v)$. Equation (2.20) becomes (writing $\tilde{\varphi}(e^{-t}v)$ instead of $\tilde{\varphi}(t,x,e^{-t}v)$)

$$\begin{aligned}\iiint_{Q_T} e^{\lambda t}\Big(&-\overline{f}\partial_t\tilde{\varphi}(e^{-t}v) - \overline{f}e^{-t}v\cdot\nabla_x\tilde{\varphi}(e^{-t}v) + \overline{f}e^{-t}v\cdot\nabla_v\tilde{\varphi}(e^{-t}v) - \overline{f}E(t,x)\cdot\nabla_v\tilde{\varphi}(e^{-t}v)\\ &+\overline{f}(-\Delta)^{\alpha/2}\tilde{\varphi}(e^{-t}v)\Big)\,\mathrm{d}t\mathrm{d}x\mathrm{d}v - \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_{in}\tilde{\varphi}(0,x,v)\,\mathrm{d}x\mathrm{d}v = 0.\end{aligned}$$

Hence, if we define $f(t,x,v) = e^{(\lambda+d)t}\overline{f}(t,x,e^t v)$ and change the variable $v \to e^{-t}v$, we recover equation (2.14). It is straightforward to check that f is in $\mathcal{X}$ and it satisfies (2.14) for any $\tilde{\varphi}$ in $\mathcal{C}_c^\infty(Q_T)$. Moreover, since $f \mapsto df - (-\Delta)^{\alpha/2}f$ is a linear bounded operator from $\mathcal{X}$ to $\mathcal{X}'$, the transport term $\mathcal{T}f$ is in $\mathcal{X}'$, hence $f \in \mathcal{Y}$ and (2.14) is verified in $\mathcal{X}'$.

Since the VLFP equation is linear, to show uniqueness it is enough to show that the unique solution with zero initial data is the null function $f \equiv 0$. Let f be a solution of this problem on $\mathcal{Y}$. As before, we define $\overline{f} = e^{-(\lambda+d)t}f(t,x,e^{-t}v)$, which satisfies equation (2.19) with $\overline{f}_{in}$ null. Since $f \in \mathcal{Y}$, we know that $\overline{f}$ belongs to $\mathcal{X}$ and, moreover, that if we define $\widetilde{\mathcal{T}}$ as

$$\widetilde{\mathcal{T}}\overline{f} = \partial_t\overline{f} + e^{-t}v\cdot\nabla_x\overline{f} + e^t E(t,x)\cdot\nabla_v\overline{f} \tag{2.21}$$

then $\widetilde{\mathcal{T}}\overline{f}$ belongs to $\mathcal{X}'$. Through integration by parts we have

$$2\big(\widetilde{\mathcal{T}}\overline{f},\overline{f}\big)_{\mathcal{X}',\mathcal{X}} = \iint_{\mathbb{R}^d\times\mathbb{R}^d}\big(\overline{f}\big)^2(T,x,v)\,\mathrm{d}x\,\mathrm{d}v \geq 0.$$

On the other hand, since $\overline{f}$ satisfies (2.19), $\widetilde{\mathcal{T}}\overline{f} = -\lambda\overline{f} - (-\Delta)^{\alpha/2}\overline{f}$ in the sense of distributions which yields

$$\big(\widetilde{\mathcal{T}}\overline{f},\overline{f}\big)_{\mathcal{X}',\mathcal{X}} = -\iiint_{Q_T}\Big(\lambda\overline{f}^2 + e^{\alpha t}\big|(-\Delta)^{\frac{\alpha}{4}}\overline{f}\big|^2\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \leq 0. \tag{2.22}$$

Hence both expression are null, in particular this means that the integral $\lambda \overline{f}^2$ is null, hence $f = \overline{f} \equiv 0$ a.e. on Q_T: the solution is unique. In order to prove the positivity of the solution consider once again the associated problem (2.19) and its solution $\overline{f}$ for some $\overline{f}^{in} \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ with $\overline{f}^{in} \geq 0$. Next, we define $\overline{f}_+$ and $\overline{f}_-$ the positive and negative parts of $\overline{f}$ given by:

$$\overline{f}_+(t,x,v) = \max(f(t,x,v),0); \qquad \overline{f}_-(t,x,v) = \max(-f(t,x,v),0)$$

so that $\overline{f} = \overline{f}_+ - \overline{f}_-$ and we denote by A_+ and A_- the respective supports of $\overline{f}_+$ and $\overline{f}_-$. Using $\widetilde{\mathcal{T}}$ defined in (2.21) we have through integration by parts

$$\begin{aligned}
(\widetilde{\mathcal{T}}\overline{f}, \overline{f}_-) = \iiint\limits_{Q_T} \Big(&\overline{f}_- \partial_t (\overline{f}_+ - \overline{f}_-) + e^{-t} \overline{f}_- v \cdot \nabla_x (\overline{f}_+ - \overline{f}_-) \\
&+ e^t \overline{f}_- E(t,x) \cdot \nabla_v (\overline{f}_+ - \overline{f}_-) \Big) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v \\
= -\frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} &\Big(\overline{f}_-^2 (T,x,v) - \overline{f}_-^2(0,x,v) \Big) \, \mathrm{d}x \, \mathrm{d}v \\
&+ \iiint\limits_{Q_T} \Big(\overline{f}_- \partial_t \overline{f}_+ + e^{-t} \overline{f}_- v \cdot \nabla_x \overline{f}_+ + e^t \overline{f}_- E(t,x) \cdot \nabla_v \overline{f}_+ \Big) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v.
\end{aligned}$$

By definition of $\overline{f}_+$ and $\overline{f}_-$ we know that $A_+ \cap A_- = \emptyset$, hence wherever $\overline{f}_-$ is not zero, both $\partial_t \overline{f}_+$, $\nabla_x \overline{f}_+$ and $\nabla_v \overline{f}_+$ are naught, and vice-versa. Moreover, we assume $\overline{f}^{in} \geq 0$ which means $\overline{f}_-(0,x,v) = 0$ so that

$$(\widetilde{\mathcal{T}}\overline{f}, \overline{f}_-) = -\frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \overline{f}_-^2 (T,x,v) \, \mathrm{d}x \, \mathrm{d}v \leq 0.$$

Since $\overline{f}$ is solution of (2.19) we know that $\widetilde{\mathcal{T}}\overline{f} = -\lambda \overline{f} - (-\Delta)^{\alpha/2} \overline{f}$ in the sense of distributions which yields

$$(\widetilde{\mathcal{T}}\overline{f}, \overline{f}_-) = \iiint\limits_{Q_T} \Big(-\lambda \overline{f}_- (\overline{f}_+ - \overline{f}_-) - \overline{f}_- (-\Delta)^{\alpha/2} (\overline{f}_+ - \overline{f}_-) \Big) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v$$

where

$$\begin{aligned}
\int\limits_{\mathbb{R}^d} \overline{f}_- (-\Delta)^{\alpha/2} (\overline{f}_+) \, \nabla_v &= \int\limits_{\mathbb{R}^d} \overline{f}_-(v) \, c_{d,\alpha} \text{ P.V.} \int\limits_{\mathbb{R}^d} \frac{\overline{f}_+(v) - \overline{f}_+(w)}{|v-w|^{d+\alpha}} \, \mathrm{d}w \, \mathrm{d}v \\
&= \int\limits_{A_-} \overline{f}_-(v) \, c_{d,\alpha} \text{ P.V.} \int\limits_{A_+} \frac{\overline{f}_+(v) - \overline{f}_+(w)}{|v-w|^{d+\alpha}} \, \mathrm{d}w \, \mathrm{d}v \\
&= -c_{d,\alpha} \int\limits_{A_-} \text{ P.V.} \int\limits_{A_+} \frac{\overline{f}_-(v) \overline{f}_+(w)}{|v-w|^{d+\alpha}} \, \mathrm{d}w \, \mathrm{d}v \leq 0.
\end{aligned}$$

Note that this integral is well defined because $\overline{f} \in \mathcal{X}$. Hence, we have:

$$(\widetilde{\mathcal{T}}\overline{f}, \overline{f}_-) = \iiint\limits_{Q_T} \Big(\lambda \overline{f}_-^2 - \overline{f}_- (-\Delta)^{\alpha/2} \overline{f}_+ + \big| (-\Delta)^{\alpha/4} \overline{f}_- \big|^2 \Big) \, \mathrm{d}t \mathrm{d}x \mathrm{d}v \geq 0.$$

This proves that $\big(\widetilde{\mathcal{T}}\overline{f}, \overline{f}_-\big) = 0$ which, in particular, means $\lambda \overline{f}_-^2 = 0$ and concludes the proof of positivity, and consequently the proof of Theorem 2.1. □

3 A priori estimates

Let us consider the operator $\mathcal{T}_\varepsilon$ a perturbation of the fractional Fokker-Planck operator with an electric field $E(t,x) \in \big(W^{1,\infty}([0,T]\times\mathbb{R}^d)\big)^d$ defined as

$$\mathcal{T}_\varepsilon(f_\varepsilon) = \nabla_v \cdot \Big[\big(v - \varepsilon^{\alpha-1}E(t,x)\big)f_\varepsilon\Big] - \big(-\Delta_v\big)^{\alpha/2} f_\varepsilon. \tag{3.23}$$

We will prove the following:

Proposition 3.1. *For any $\varepsilon > 0$ fixed, there exists a unique positive equilibrium distribution F_ε solution of:*

$$\mathcal{T}_\varepsilon(F_\varepsilon) = \nabla_v \cdot \Big[\big(v - \varepsilon^{\alpha-1}E(t,x)\big)F_\varepsilon\Big] - \big(-\Delta_v\big)^{\alpha/2} F_\varepsilon = 0, \qquad \int_{\mathbb{R}^d} F_\varepsilon \,\mathrm{d}v = 1. \tag{3.24}$$

Proof. The Fourier transform in velocity of the equilibrium equation (3.24) reads

$$\xi \cdot \nabla_\xi \widehat{F_\varepsilon} = -\Big(i\xi \cdot \varepsilon^{\alpha-1}E(t,x) + |\xi|^\alpha\Big)\widehat{F_\varepsilon},$$

for which we can compute the explicit solution:

$$\widehat{F_\varepsilon}(t,x,\xi) = \kappa e^{-i\varepsilon^{\alpha-1}\xi\cdot E(t,x) - |\xi|^\alpha/\alpha}, \tag{3.25}$$

where κ is a positive constant which ensures the normalisation of the equilibrium. Now, although the inverse Fourier transform $\mathcal{F}^{-1}\big(\widehat{F_\varepsilon}\big)(t,x,v)$ is not explicit let us note that F_ε can be expressed as a translation of the equilibrium distribution G_α of the fractional Fokker-Planck operator:

$$F_\varepsilon(t,x,v) = G_\alpha\big(v - \varepsilon^{\alpha-1}E(t,x)\big). \tag{3.26}$$

Hence, the positivity and normalization of F_ε follows from the properties of G_α. □

Proposition 3.2. *Let F_ε be the unique normalized equilibrium distribution of* (3.23)*. Then there exist positive constants μ, c_1, c_2 and c_3 such that:*

(i) $c_1 G_\alpha \le F_\varepsilon \le c_2 G_\alpha$,

(ii) $\displaystyle \left\|\frac{\partial_t F_\varepsilon}{F_\varepsilon}\right\|_{L^\infty(\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t)}, \left\|\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon}\right\|_{L^\infty(\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t)} \le \varepsilon^{\alpha-1}\mu,$

(iii) $|F_\varepsilon - G_\alpha| \le \varepsilon^{\alpha-1} c_3 G_\alpha$.

for $\varepsilon > 0$ small enough.

Proof. We shall start by proving part (i). Let us assume that L is an arbitrary vector in $\mathbb{R}^d$ such that $|L| \le 1$, then is easy to see that there exists $R_1 > 0$ big enough such that

$$\frac{1}{2^{\frac{1}{d+\alpha}}} \le \left|1 - \frac{|L|}{|v|}\right| \le \left|\frac{v}{|v|} - \frac{L}{|v|}\right|,$$

for all $|v| > R_1$. Hence, it follows that

$$\frac{1}{|v-L|^{d+\alpha}} \le \frac{2}{|v|^{d+\alpha}},$$

for all $|v| > R_1$. Thus, using (1.8) we obtain that there exists $\widetilde{C} > 0$ and $R > 0$ big enough such that

$$G_\alpha(v - L) \le \widetilde{C} G_\alpha(v),$$

for all $|v| > R$ and all $L \in \mathbb{R}^d$ with $L \le 1$. Now, let $C_2 > 0$ such that

$$C_2 \left(\min_{v \in B(0,R)} G_\alpha(v) \right) \ge \|G_\alpha\|_\infty,$$

where $B(0,R) \subset \mathbb{R}^d$, is the ball of radius R centered at the origin. Let us note that the minimum exists since G_α is continuous. Thus choosing $\mu_2 = \widetilde{C} \vee C_2$, where $a \vee b$ denotes the maximum between a and b, we obtain

$$G_\alpha(v - L) \le \mu_2 G_\alpha(v).$$

Next, writing $w = v + L$ where $L \in \mathbb{R}^d$ with $|L| \le 1$ we obtain

$$G_\alpha(w) \le \mu_1 G_\alpha(w - L),$$

Thus, taking $\mu_1 = 1/\mu_2$ we obtain

$$\mu_1 G_\alpha(v) \le G_\alpha(v - L),$$

for all $v \in \mathbb{R}^d$ and $|L| \le 1$.

On the other hand, for part (ii), let us start by noting that thanks to (3.26), F_ε satisfies the following identities:

$$\frac{\partial_t F_\varepsilon}{F_\varepsilon} = -\varepsilon^{\alpha-1} \partial_t E(t,x) \cdot \frac{\nabla_v G_\alpha\big(v - \varepsilon^{\alpha-1} E(t,x)\big)}{G_\alpha\big(v - \varepsilon^{\alpha-1} E(t,x)\big)},$$

and

$$\frac{v \cdot \nabla_x F_\varepsilon}{F_\varepsilon} = -\varepsilon^{\alpha-1} \nabla_x E(t,x) \frac{v \cdot \nabla_v G_\alpha\big(v - \varepsilon^{\alpha-1} E(t,x)\big)}{G_\alpha\big(v - \varepsilon^{\alpha-1} E(t,x)\big)}.$$

Hence, thanks to the assumption $E \in \big(W^{1,\infty}([0,T) \times \mathbb{R}^d)\big)^d$ we only need to prove that there exists a $C > 0$ such that

$$|v \cdot \nabla_v G_\alpha(v - L)| \le C G_\alpha(v - L), \tag{3.27}$$

for all $v \in \mathbb{R}^d$, and all $L \in \mathbb{R}^d$ with $|L| \le 1$. This follows via a similar line of reasoning as in the proof of part (i) around the control (1.9).

Finally we prove part (iii). Since G_α is smooth by the mean value theorem we obtain

$$\begin{aligned} |F_\varepsilon(v) - G_\alpha(v)| &= |G_\alpha(v - \varepsilon^{\alpha-1} E) - G_\alpha(v)| \\ &= \varepsilon^{\alpha-1} |E| |\nabla_v G_\alpha(v - \vartheta\, \varepsilon^{\alpha-1} E)|, \end{aligned}$$

where $\vartheta \in (0,1)$. Thus, the result follows thanks to (3.27) and since $E \in \big(W^{1,\infty}([0,T) \times \mathbb{R}^d)\big)^d$.

□

The key ingredient in order to obtain the a priori estimates needed to pass to the limit in (1.4) is the positivity of the dissipation which we state in the following result.

Proposition 3.3. *Let us consider the operator $\mathcal{T}_\varepsilon$ defined by* (3.23). *The associated dissipation, defined bellow, satisfies*

$$\mathcal{D}_\varepsilon(f) := -\iint \mathcal{T}_\varepsilon(f)\frac{f}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x = \iiint \left(\frac{f(v)}{F_\varepsilon(v)} - \frac{f(w)}{F_\varepsilon(w)}\right)^2 \frac{F_\varepsilon(v)}{|v-w|^{d+\alpha}}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x, \quad (3.28)$$

and if we write $\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v$, then for all $f \in L^2_{F_\varepsilon^{-1}}(\mathbb{R}^d \times \mathbb{R}^d)$ we have

$$\mathcal{D}_\varepsilon(f) \geq \int (f - \rho F_\varepsilon)^2 \frac{\mathrm{d}x\,\mathrm{d}v}{F_\varepsilon(v)}. \quad (3.29)$$

Proof. The Poincaré type inequality (3.29) is a particular case of the so-called Φ-entropy inequalites introduced in [12]. For the sake of completeness we shall give a sketch of the proof adapted to the case that we need.

We shall first start proving (3.28). Writing $\Phi_\varepsilon = v - \varepsilon^{\alpha-1}E(t,x)$ and $g = f/F_\varepsilon$, and since F_ε satisfies (3.24) we have:

$$\begin{aligned}
\mathcal{D}_\varepsilon(f) &= -\iint \Big(\nabla_v\cdot(\Phi_\varepsilon g F_\varepsilon)\,g - \big(-\Delta_v\big)^{\alpha/2}(gF_\varepsilon)\,g\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&= -\iint \Big(\Phi_\varepsilon F_\varepsilon \frac{1}{2}\nabla_v(g^2) + \nabla_v\cdot(\Phi_\varepsilon F_\varepsilon)g^2 - \big(-\Delta_v\big)^{\alpha/2}(g)gF_\varepsilon\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&= \iint \Big(\frac{1}{2}g^2\big(-\Delta_v\big)^{\alpha/2}(F_\varepsilon) - g^2\big(-\Delta_v\big)^{\alpha/2}(F_\varepsilon) + g\big(-\Delta_v\big)^{\alpha/2}(g)F_\varepsilon\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&= \iint \Big(g\big(-\Delta_v\big)^{\alpha/2}(g) - \frac{1}{2}\big(-\Delta_v\big)^{\alpha/2}(g^2)\Big)F_\varepsilon\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Hence, using (1.5) we see that:

$$\begin{aligned}
&\iint \Big(g\big(-\Delta_v\big)^{\alpha/2}(g) - \frac{1}{2}\big(-\Delta_v\big)^{\alpha/2}(g^2)\Big)F_\varepsilon\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad = \iiint \left(\frac{g(v)\big(g(v)-g(w)\big)}{|v-w|^{d+\alpha}} - \frac{1}{2}\frac{g^2(v)-g^2(w)}{|v-w|^{d+\alpha}}\right)F_\varepsilon(t,x,v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad = \frac{1}{2}\iiint \frac{\big(g(v)-g(w)\big)^2}{|v-w|^{d+\alpha}}F_\varepsilon(t,x,v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Recall that $F_\varepsilon(t,x,v) = G_\alpha\big(v - \varepsilon^{\alpha-1}E(t,x)\big)$, therefore through a simple change of variable, if we call $h(t,x,v) = g\big(v-\varepsilon^{\alpha-1}E(t,x)\big)$ we have:

$$\mathcal{D}_\varepsilon(f) = \frac{1}{2}\iiint \frac{\big(h(t,x,v)-h(t,x,w)\big)^2}{|v-w|^{d+\alpha}}G_\alpha(v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x.$$

In order to prove the control (3.29) we consider the semigroup associated with $(-\Delta)^{\alpha/2}$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t(h)(v) = -(-\Delta)^{\alpha/2}\Big(P_t(h)\Big)(v) \quad (3.30)$$

with $P_0(h)(v) = h(v)$ and we see, using (3.25), that if we introduce the kernel

$$K_t(v) = \mathcal{F}^{-1}\Big(\kappa e^{-t|\xi|^\alpha/\alpha}\Big)(v)$$

where κ is a constant normalizing K_1, then we have explicitly $P_t(h) = K_t * h$. For $s \in [0,t]$ we consider

$$\psi(s) = P_s(H^2)(v) \tag{3.31}$$

with $H = P_{t-s}(h)$. We then have for $s \in [0,t]$:

$$\begin{aligned}
\psi'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\left[K_s * \Big(K_{t-s} * h\Big)^2\right] \\
&= \Big(\frac{\mathrm{d}}{\mathrm{d}s}K_s\Big) * \Big(K_{t-s} * h\Big)^2 + K_s * \frac{\mathrm{d}}{\mathrm{d}s}\Big[\big(K_{t-s} * h\big)^2\Big] \\
&= P_s\Big(-(-\Delta)^{\alpha/2}H^2\Big) + 2P_s\Big(H(-\Delta)^{\alpha/2}H\Big) \\
&= P_s\Big(\int \frac{\big(H(v) - H(w)\big)^2}{|v-w|^{d+\alpha}}\,\mathrm{d}w\Big)
\end{aligned}$$

Using the integral expression of the convolution and Jensen's inequality it is straightforward to see that $\big(P_{t-s}(h)(v) - P_{t-s}(h)(w)\big)^2 \le P_{t-s}\big(h(v) - h(w)\big)^2$. Therefore, using Fubini's theorem, we have:

$$\psi'(s)(v) \le P_s\Big(P_{t-s}\Big(\int \frac{\big(h(v)-h(w)\big)^2}{|v-w|^{d+\alpha}}\,\mathrm{d}w\Big)\Big) = P_t\Big(\int \frac{\big(h(v)-h(w)\big)^2}{|v-w|^{d+\alpha}}\,\mathrm{d}w\Big).$$

Integrating over $s \in [0,t]$ one gets

$$P_t\big(h^2\big)(v) - \Big(P_t(h)(v)\Big)^2 \le tP_t\Big(\int \frac{\big(h(v)-h(w)\big)^2}{|v-w|^{d+\alpha}}\,\mathrm{d}w\Big).$$

Finally, taking $t = 1$ and evaluating at $v = 0$ we get:

$$\int h^2(w)G_\alpha(w)\,\mathrm{d}w - \Big(\int h(w)G_\alpha(w)\,\mathrm{d}w\Big)^2 \le \iint \frac{\big(h(v)-h(w)\big)^2}{|v-w|^{d+\alpha}}G_\alpha(v)\,\mathrm{d}v\,\mathrm{d}w. \tag{3.32}$$

Through a simple change of variables, inverse of the one we did earlier, we obtain

$$\int g^2(w)F_\varepsilon(w)\,\mathrm{d}w - \Big(\int g(w)F_\varepsilon(w)\,\mathrm{d}w\Big)^2 \le \iint \frac{\big(g(v)-g(w)\big)^2}{|v-w|^{d+\alpha}}F_\varepsilon(v)\,\mathrm{d}v\,\mathrm{d}w. \tag{3.33}$$

Finally, replacing g by f/F_ε, since F_ε is normalized, we recover (3.29).

□

Since the operator $\mathcal{T}_\varepsilon$ is negative semidefinite in $L^2_{F_\varepsilon^{-1}}(\mathbb{R}^d)$ it is natural to look for bounds of the quadratic entropy associated to solutions f_ε of (1.4). We gather the appropriate a priori estimates that we shall need to pass to the limit in (1.4) in the following result.

Proposition 3.4. *Let the assumptions of Theorem 1.5 be satisfied and let f_ε be the solution of* (1.4). *We introduce the residue r_ε through the macro-micro decomposition $f_\varepsilon = \rho_\varepsilon F_\varepsilon + \varepsilon^{\alpha/2} r_\varepsilon$. Then, uniformly in $\varepsilon \in (0,1)$, we have:*

(i) *(f_ε) is bounded in $L^\infty([0,T]; L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$ and in $L^\infty([0,T]; L^1(\mathbb{R}^d \times \mathbb{R}^d))$,*

(ii) *(ρ_ε) is bounded in $L^\infty([0,T]; L^2(\mathbb{R}^d))$,*

(iii) *(r_ε) is bounded in $L^2([0,T]; L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$.*

Proof. Multiplying (1.4) by $f_\varepsilon/F_\varepsilon$, integrations by parts yield

$$\frac{\varepsilon^{\alpha-1}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x+\frac{\varepsilon^{\alpha-1}}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\frac{\partial_t F_\varepsilon}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x-\frac{1}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon^2}\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{\varepsilon}\mathcal{D}_\varepsilon(f^\varepsilon)=0.$$

Thus, thanks to Proposition 3.2, part (i) and (ii), and (3.29) we obtain

$$\frac{\varepsilon^{\alpha}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x+\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(f_\varepsilon-\rho_\varepsilon F_\varepsilon)^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x\le\varepsilon^\alpha\mu\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x. \tag{3.34}$$

Whence, part (i) follows by Gronwall's lemma and the fact that the weights $1/G_\alpha$ and $1/F_\varepsilon$ are equivalent uniformly in ε which follows from Proposition 3.2, part (i). On the other hand, part (ii) follows thanks to the inequality

$$\rho_\varepsilon\le\left(\int\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\right)^{1/2},$$

which is an immediate consequence of Cauchy-Schwarz and the fact $\int F_\varepsilon\,\mathrm{d}v=1$. Finally, part (iii) follows from (4.48) after integrating with respect to t over $(0,T)$ and thanks to Proposition 3.2 part (ii).

□

4 Proof of Theorem 1.5

We shall follow the method introduced in [9]. Let us start by introducing the following auxiliary problem: for $\varphi\in\mathcal{C}_c^\infty([0,T)\times\mathbb{R}^d)$, define ψ_ε the unique solution of

$$\begin{aligned}&\varepsilon v\cdot\nabla_x\psi_\varepsilon-v\cdot\nabla_v\psi_\varepsilon=0 &&\text{in } [0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,\\ &\psi(t,x,0)=\varphi(t,x) &&\text{in } [0,\infty)\times\mathbb{R}^d\end{aligned} \tag{4.35}$$

The function ψ_ε can be obtained readily via the method of characteristics and can be expressed in an explicit manner as follows:

$$\psi_\varepsilon(t,x,v)=\varphi(t,x+\varepsilon v). \tag{4.36}$$

Next, multiplying (1.4) by ψ_ε and through integrations by parts we obtain

$$\iiint\limits_{Q_T}f_\varepsilon\Big(\varepsilon^{\alpha-1}\,\partial_t\psi_\varepsilon+v\cdot\nabla_x\psi_\varepsilon-\frac{1}{\varepsilon}(v-\varepsilon^{\alpha-1}E)\cdot\nabla_v\psi_\varepsilon-\frac{1}{\varepsilon}(-\Delta)^{\alpha/2}\psi_\varepsilon\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$+\varepsilon^{\alpha-1}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}f^{in}(x,v)\psi_\varepsilon(0,x,v)\,\mathrm{d}v\,\mathrm{d}x=0\,. \tag{4.37}$$

Let us note the following

$$(-\Delta_v)^{\alpha/2}\psi_\varepsilon(t,x,v)=\varepsilon^\alpha(-\Delta_v)^{\alpha/2}\varphi(t,x+\varepsilon v), \tag{4.38}$$
$$\nabla_v\psi_\varepsilon(t,x,v)=\varepsilon\nabla\varphi(t,x+\varepsilon v), \tag{4.39}$$

which follows after a simple computation using the definition (1.5) of the fractional Laplacian. Thus using the auxiliary equation (4.35) and plugging (4.38) into (4.37) yields

$$\int_0^\infty \iint f_\varepsilon \Big(\partial_t \varphi(t, x+\varepsilon v) + E \cdot \nabla_x \varphi(t, x+\varepsilon v) - (-\Delta_v)^{\alpha/2} \varphi(t, x+\varepsilon v) \Big) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t + \iint f^{in}(x,v) \varphi(0, x+\varepsilon v) \, \mathrm{d}v \, \mathrm{d}x = 0 \,. \tag{4.40}$$

4.1 The non-critical case: $1 < \alpha < 2$

In order to pass to the limit in this weak formulation, we introduce the following two results.

Lemma 4.1. *Let* (f_ε) *be the sequence of solutions of* (1.4)*, and* ρ *be the limit of* (ρ_ε) *which exists thanks to Proposition 3.4 part (ii), then*

$$f_\varepsilon(t,x,v) \rightharpoonup \rho(t,x) G_\alpha(v) \quad \text{weakly in } L^\infty([0,T]; L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$$

Proof. This lemma follows directly from Proposition 3.4. Since f_ε is uniformly bounded, it converges weakly in $L^\infty([0,T]; L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$. From the bounds on F_ε established in Proposition 3.2 and the boundedness of ρ_ε in $L^\infty([0,T]; L^2(\mathbb{R}^d))$ we see that $\rho_\varepsilon(t,x) F_\varepsilon(v)$ converges to $\rho(t,x) G_\alpha(v)$ weakly in $L^\infty([0,T]; L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$ where ρ is the weak limit of ρ_ε. Finally, since the residue r_ε is bounded, it follows from the micro-macro decomposition $f_\varepsilon = \rho_\varepsilon F_\varepsilon + \varepsilon^{\alpha/2} r_\varepsilon$ that the limit of f_ε is the same as the limit of $\rho_\varepsilon F_\varepsilon$. □

Lemma 4.2. *For all test functions* ψ *in* $C_c^\infty([0,\infty) \times \mathbb{R}^d)$ *we have:*

$$\lim_{\varepsilon \to 0} \iiint_{Q_T} f^\varepsilon(t,x,v) \psi(t, x+\varepsilon v) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v = \iint_{[0,T)\times\mathbb{R}^d} \rho(t,x) \psi(t,x) \, \mathrm{d}x \, \mathrm{d}t. \tag{4.41}$$

Moreover, if $E(t,x) \in W^{1,\infty}([0,T) \times \mathbb{R}^d)^d$ *then for all* $\Psi \in C_c^\infty([0,\infty) \times \mathbb{R}^d; \mathbb{R}^d)$ *the following convergence holds:*

$$\lim_{\varepsilon \to 0} \iiint_{Q_T} f^\varepsilon(t,x,v) E(t,x) \cdot \Psi(t, x+\varepsilon v) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v = \iint_{[0,T)\times\mathbb{R}^d} \rho(t,x) E(t,x) \cdot \Psi(t,x) \, \mathrm{d}x \, \mathrm{d}t. \tag{4.42}$$

Proof. We will give a detailed proof of the convergence in (4.42), the convergence in (4.41) follows as a consequence of (4.42) by taking $\psi(t, x+\varepsilon v) = E(t,x) \cdot \Psi(t, x+\varepsilon v)$ with a smooth E and Lemma 4.1. For (4.42), we write:

$$\begin{aligned} \iiint_{Q_T} f_\varepsilon E(t,x) \cdot \Psi(t, x+\varepsilon v) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t = & \iint_{[0,T)\times\mathbb{R}^d} \rho(t,x) E(t,x) \cdot \Psi(t,x) \, \mathrm{d}x \, \mathrm{d}t \\ & + \iiint_{Q_T} \Big(f_\varepsilon - \rho(t,x) G_\alpha(v) \Big) E(t,x) \cdot \Psi(t,x) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \\ & + \iiint_{Q_T} f_\varepsilon E(t,x) \cdot \Big(\Psi(t, x+\varepsilon v) - \Psi(t,x) \Big) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.43}$$

The second term in the right hand side of (4.43) converges to zero since f_ε converges to ρG_α weakly in $L^\infty([0,T); L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$ thanks to Lemma 4.1. For the third term on the right hand side of (4.43) thanks to Cauchy-Schwarz and Hölder we obtain

$$
\begin{aligned}
&\Big| \iiint_{Q_T} f_\varepsilon E(t,x) \cdot (\Psi(t,x+\varepsilon v) - \Psi(t,x)) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \Big| \\
&\le \int_0^T \Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f_\varepsilon^2}{G_\alpha} \,\mathrm{d}v\,\mathrm{d}x \Big)^{1/2} \Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big[E(t,x)\cdot(\Psi(t,x+\varepsilon v) - \Psi(t,x)) \Big]^2 G_\alpha \,\mathrm{d}v\,\mathrm{d}x \Big)^{1/2} \mathrm{d}t \\
&\le \|f_\varepsilon\|_{L^\infty([0,T);L^2_{G_\alpha^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d))} \\
&\qquad \times \int_0^T \Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d} [E(t,x)\cdot(\Psi(t,x+\varepsilon v) - \Psi(t,x))]^2 G_\alpha \,\mathrm{d}v\,\mathrm{d}x \Big)^{1/2} \mathrm{d}t. \qquad (4.44)
\end{aligned}
$$

Next, let R be an arbitrary positive real number and let us consider the following splitting

$$
\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big[E\cdot(\Psi(t,x+\varepsilon v) - \Psi(t,x)) \Big]^2 G_\alpha(v) \,\mathrm{d}v\,\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^d} \int_{|v|\le R} \Big[E\cdot(\Psi(t,x+\varepsilon v) - \Psi(t,x)) \Big]^2 G_\alpha(v) \,\mathrm{d}v\,\mathrm{d}x \\
&\qquad\quad + \int_{\mathbb{R}^d} \int_{|v|> R} \Big[E\cdot(\Psi(t,x+\varepsilon v) - \Psi(t,x)) \Big]^2 G_\alpha(v) \,\mathrm{d}v\,\mathrm{d}x. \qquad (4.45)
\end{aligned}
$$

We will use the regularity of Ψ to bound the integral on $|v| < R$. To that end, let us consider the εR neighborhood of the support of Ψ denoted as $\Omega(\varepsilon R)$ which consists of the union of all the balls of radius εR having as center a point in $\operatorname{supp}\Psi$. Next, let Λ denote the diameter of $\operatorname{supp}\Psi$ defined as the maximum over all the distances between two points in $\operatorname{supp}\Psi$. Then it is clear that $\Omega(\varepsilon R) \subseteq B(x_0; \Lambda + \varepsilon R)$ where $B(x_0; \Lambda+\varepsilon R)$ denotes the ball with center at x_0 and radius $\Lambda + \varepsilon R$ and x_0 is any arbitrary fix point in $\operatorname{supp}\Psi$. Then for the integral over $|v| < R$ we have the following

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \int_{|v|\le R} [E\cdot(\Psi(t,x+\varepsilon v) - \Psi(t,x))]^2 G_\alpha(v) \,\mathrm{d}v\,\mathrm{d}x \\
&\qquad \le \|G_\alpha\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{|v|\le R} \Big(\sum_{j=1}^d |E_j| \big| \varepsilon v \cdot \nabla_x \Psi_j(t, x+\vartheta_j \varepsilon v) \big| \Big)^2 \mathrm{d}v\,\mathrm{d}x \\
&\qquad \le 2\varepsilon^2 \|G_\alpha\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{|v|\le R} |v|^2 \Big(\sum_{j=1}^d |E_j|^2 \big| \nabla_x \Psi_j(t, x+\vartheta_j \varepsilon v) \big|^2 \Big) \mathrm{d}v\,\mathrm{d}x \\
&\qquad \le 2\varepsilon^2 \|G_\alpha\|_{L^\infty(\mathbb{R}^d)} \|E\|^2_{W^{1,\infty}([0,T)\times\mathbb{R}^d)} \|\nabla_x \Psi\|_{L^\infty(\mathbb{R}^d)} \int_{|v|\le R} \int_{B(x_0,\delta+\varepsilon R)} |v|^2 \,\mathrm{d}x\,\mathrm{d}v \\
&\qquad \le \varepsilon^2 C_2 (\Lambda + \varepsilon R)^d R^{d+2}, \qquad (4.46)
\end{aligned}
$$

where C_2 is a constant depending on $\|E\|^2_{W^{1,\infty}([0,T)\times\mathbb{R}^d)}$, $\|G_\alpha\|_{L^\infty(\mathbb{R}^d)}$ and $\|D_x^2 \varphi\|_{L^\infty(\mathbb{R}^d)}$ but not on ε, and $\vartheta_j \in (0,1)$ for $j = 1,\dots,d$ is such that $\Psi_j(t,x+\varepsilon v) - \Psi_j(t,x) =$

$\varepsilon v \cdot \nabla_x \Psi_j(t, x + \vartheta_j \varepsilon v)$. For the integral on $|v| > R$ we use the decay of the equilibrium $G_\alpha(v)$ to derive the following upper bound:

$$\int_{\mathbb{R}^d} \int_{|v|>R} \Big[E \cdot (\Psi(t, x + \varepsilon v) - \Psi(t, x)) \Big]^2 G_\alpha(v) \, \mathrm{d}v \, \mathrm{d}x$$

$$\leq \|E\|^2_{W^{1,\infty}([0,T)\times\mathbb{R}^d)} \int_{|v|>R} \left(\int_{\mathbb{R}^d} \Big(2|\Psi(t, x + \varepsilon v)|^2 + 2|\Psi(t,x)|^2 \Big) \, \mathrm{d}x \right) G_\alpha(v) \, \mathrm{d}v$$

$$\leq 4\|E\|^2_{W^{1,\infty}([0,T)\times\mathbb{R}^d)} \int_{\mathbb{R}^d} |\Psi(t,x)|^2 \, \mathrm{d}x \int_{|v|>R} G_\alpha(v) \, \mathrm{d}v$$

$$\leq C \int_{|v|>R} G_\alpha(v) \, \mathrm{d}v.$$

Thanks to Proposition 3.1, for any $\eta > 0$ we can choose $R > 0$ big enough such that

$$\left| G_\alpha(v) - \frac{\vartheta}{|v|^{d+\alpha}} \right| \leq \frac{\eta}{|v|^{d+\alpha}}, \qquad \text{for all } |v| \geq R.$$

Thus choosing $\eta = \vartheta$ we have the following estimate:

$$\int_{|v|>R} G_\alpha(v) \, \mathrm{d}v \leq \int_{|v|>R} \left| G_\alpha(v) - \frac{\vartheta}{|v|^{d+\alpha}} \right| \mathrm{d}v + \int_{|v|>R} \frac{\vartheta}{|v|^{d+\alpha}} \, \mathrm{d}v$$

$$\leq 2 \int_{|v|>R} \frac{\vartheta}{|v|^{d+\alpha}} \, \mathrm{d}v$$

$$\leq \frac{C}{R^\alpha}.$$

From which we conclude

$$\int_{\mathbb{R}^d} \int_{|v|>R} \Big[E \cdot (\Psi(t, x + \varepsilon v) - \Psi(t, x)) \Big]^2 G_\alpha(v) \, \mathrm{d}v \, \mathrm{d}x \leq \frac{C_2}{R^\alpha}. \tag{4.47}$$

Next let us note that for any $\delta > 0$ we can choose $\widetilde{R} > 0$ such that $C_2/R^\alpha < \delta/2$ for all $R > \widetilde{R}$ and then choose $\varepsilon > 0$ so that $\varepsilon^2 C_1 (\Lambda + \varepsilon R)^d R^{d+2} < \delta/2$. And thus deduce that for ε small enough we have

$$\varepsilon^2 C_1 (\Lambda + \varepsilon R)^d R^{d+2} + \frac{C_2}{R^\alpha} < \delta.$$

Therefore, plugging (4.46) and (4.47) into (4.44) and using Proposition 3.4, part (i), we obtain that there exists a fixed $C > 0$ such that

$$\left| \iiint_{Q_T} f_\varepsilon E \cdot (\Psi(t, x + \varepsilon v) - \Psi(t, x)) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \right|$$

$$\leq C \left(\varepsilon^2 C_1 (\Lambda + \varepsilon R)^d R^{d+2} + \frac{C_2}{R^\alpha} \right)$$

$$\leq C\delta,$$

for any $\delta > 0$, hence concluding that the third term on the right hand side of (4.43) goes to zero as $\varepsilon \to 0$. □

Using Lemma 4.2 we can now take the limit in (4.40) and conclude that ρ satisfies

$$\iint_{[0,T)\times\mathbb{R}^d} \rho\Big(\partial_t\varphi + E\cdot\nabla_x\varphi - \big(-\Delta_x\big)^{\alpha/2}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_{in}(x)\varphi(0,x)\,\mathrm{d}x = 0,$$

for all $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$. Thus concluding the proof of Theorem 1.5.

4.2 The critical cases $\alpha = 1$ and $\alpha = 2$

In the critical case $\alpha = 2$ we recover the classical Fokker-Planck operator which means, in particular, as mentioned in the Introduction, that its equilibrium is a Maxwellian $M(v) = C\exp\left(-|v|^2\right)$ instead of the heavy-tail distribution G_α. We can still consider the perturbed operator $\mathcal{T}_\varepsilon$ of Proposition 3.1 and its equilibrium will also be a translation of the unperturbed one:

$$F_\varepsilon(t,x,v) = Ce^{-|v-\varepsilon E(t,x)|^2}$$

and since the decay of the Maxwellian is much faster than the decay of the heavy-tail distributions, Proposition 3.2 holds. The dissipative properties of the Fokker-Planck operator are well known, see e.g. [8] [14] or [5], and it is straightforward to check the boundedness results of Proposition 3.4. Hence, Lemma 4.1 holds and we can take the limit in the weak formulation (4.40) to prove that Theorem 1.5 holds in the case $\alpha = 2$.

In the critical case $\alpha = 1$, the perturbed operator $\mathcal{T}_\varepsilon$ of (3.23) and its equilibrium F_ε (3.26) lose their dependence with respect to ε:

$$\begin{aligned}\mathcal{T}_\varepsilon(f_\varepsilon) &= \mathcal{T}_E(f_\varepsilon) = \nabla_v\cdot\Big[\big(v - E(t,x)\big)f_\varepsilon\Big] - \big(-\Delta_v\big)^{\alpha/2}f_\varepsilon,\\ F_\varepsilon(t,x,v) &= G_{1,E}(t,x,v) = G_1\big(v - E(t,x)\big).\end{aligned}$$

In particular, the equilibrium $G_{1,E}$ will remain unchanged in the limit as ε goes to 0 and Proposition 3.2 will hold with $\alpha = 1$ which, in particular, means that the bounds in *(ii)* and *(iii)* do not go to zero. The operator is still dissipative since the dependence on ε does not matter in the proof of Proposition 3.3, hence we still have (3.33) and multiplying (1.4) by $f_\varepsilon/G_{1,E}$ and integrating by parts yields:

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{G_{1,E}}\,\mathrm{d}v\,\mathrm{d}x + \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(f_\varepsilon - \rho_\varepsilon G_{1,E})^2}{G_{1,E}}\,\mathrm{d}v\,\mathrm{d}x \le \varepsilon\mu\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{G_{1,E}}\,\mathrm{d}v\,\mathrm{d}x. \tag{4.48}$$

Since E is in $\left(W^{1,\infty}([0,T)\times\mathbb{R}^d)\right)^d$, if $f_\varepsilon(t,\cdot,\cdot)$ is in $L^2_{G_{1,E}(t,x,v)}(\mathbb{R}^d\times\mathbb{R}^d)$ and bounded independently of time, then it is also in $L^2_{G_1(v)}(\mathbb{R}^d\times\mathbb{R}^d)$. As a consequence, from (4.48) we still have the uniform in ε boundedness of f_ε, $\rho_\varepsilon = \int f_\varepsilon\,\nabla_v$ and the residue r_ε in $L^\infty([0,T);L^2_{G_1(v)}(\mathbb{R}^d\times\mathbb{R}^d))$ as stated in Proposition 3.4. This yields the following modified version of Lemma 1:

Lemma 4.3. *Let $\alpha = 1$, (f_ε) be the sequence of solutions of* (1.4)*, and ρ be the limit of (ρ_ε) which exists thanks to Proposition 3.4 part (ii), then*

$$f_\varepsilon(t,x,v) \rightharpoonup^\star \rho(t,x)G_{1,E}(t,x,v) \quad \text{in } L^\infty([0,T);L^2_{G_1^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d)).$$

Finally, for the proof of convergence of the weak formulation (4.40), i.e. the proof of Lemma 4.2, we proceed essentially the same way. The only slight difference is that in order to control the third term of (4.43) we will use Cauchy-Schwarz as in (4.44) but we multiplying and divide by $G_1(v)^{1/2}$ instead of the natural equilibrium $G_{1,E}$. The rest of the proof remains the same and we can then take the limit in the weak formulation, which concludes the proof of Theorem 1.5 with $\alpha = 1$.

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Fractional diffusion limit of a linear kinetic equation in a bounded domain

Pedro Aceves-Sánchez and Christian Schmeiser

Submitted

Abstract

A version of fractional diffusion on bounded domains, subject to 'homogeneous Dirichlet boundary conditions' is derived from a kinetic transport model with homogeneous inflow boundary conditions. For nonconvex domains, the result differs from standard formulations. It can be interpreted as the forward Kolmogorow equation of a stochastic process with jumps along straight lines, remaining inside the domain.

1 Introduction

This work is an extension to bounded domains of earlier efforts [4, 19, 20] to derive fractional diffusion equations from kinetic transport models. This raises the issue of the inclusion of boundary effects, which can, however, not be reduced to boundary conditions since fractional diffusion is a nonlocal process. Our main result is the derivation of a new way of realizing 'homogeneous Dirichlet boundary conditions', coinciding on convex domains with an already established model, see e.g. [14].

Let $\Omega \subset \mathbb{R}^d$ denote a bounded domain with smooth boundary. We shall study the asymptotic behavior as $\varepsilon > 0$ tends to zero of the kinetic relaxation model

$$\varepsilon^\alpha \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon) := \int_{\mathbb{R}^d} M f'_\varepsilon - M' f_\varepsilon \, \mathrm{d}v' \,, \tag{1.1}$$

with $f_\varepsilon = f_\varepsilon(x,v,t)$, $(x,v,t) \in \Omega\times\mathbb{R}^d\times[0,\infty)$ (where the superscript $'$ denotes evaluation at v'), subject to zero inflow boundary conditions and well prepared initial data:

$$f_\varepsilon(x,v,t) = 0 \qquad \text{for } (x,v) \in \Gamma^- \,, \quad t > 0 \,, \tag{1.2}$$

$$f_\varepsilon(x,v,0) = f^{in}(x,v) := \rho^{in}(x) M(v) \qquad \text{for } (x,v) \in \Omega \,, \tag{1.3}$$

with $\Gamma^\pm = \{(x,v) |\; x \in \partial\Omega,\ \mathrm{sign}(v\cdot\nu(x)) = \pm 1\}$, where ν denotes the unit outward normal along $\partial\Omega$. We assume a 'fat-tailed' equilibrium distribution M, satisfying

$$M(v) = 1/|v|^{d+\alpha} \qquad \text{for } |v| \ge 1 \,, \text{ with } 0 < \alpha < 2 \,, \tag{1.4}$$

$$M(v) > 0, \qquad M(v) = M(-v) \qquad \text{for all } v \in \mathbb{R}^d \,, \tag{1.5}$$

$$M \in L^\infty(\mathbb{R}^d) \,, \qquad \text{and} \qquad \int_{\mathbb{R}^d} M(v)\, \mathrm{d}v = 1 \,. \tag{1.6}$$

Note that these assumptions imply that M does not have finite second order moments.

The translation of homogeneous Dirichlet boundary conditions to fractional diffusion induce a certain behaviour of solutions close to the boundary. The domain of the fractional diffusion operator, we shall derive, contains test functions in

$$\mathcal{D}_\Omega := \{\varphi \in C_0^\infty(\overline{\Omega}\times[0,\infty)) :\, \delta(x)^{-2}\varphi(x,t) \text{ bounded}\} \,, \tag{1.7}$$

where $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ denotes the distance of a point $x \in \Omega$ to the boundary.

A convenient functional analytic setting for the main result of this paper is the L^2-space $L^2_{M^{-1}}(\Omega \times \mathbb{R}^d)$ of functions of (x, v) with weight $1/M(v)$.

Theorem 1.1. *Let $\rho^{in} \in L^2(\Omega)$, and let f_ε be the solution of (1.1)–(1.3). Then, for any $T > 0$, there exists $\rho \in L^\infty(0, T; L^2(\Omega))$ such that $f_\varepsilon(x, v, t) \to \rho(x, t)M(v)$ as $\varepsilon \to 0$, in $L^\infty(0, T; L^2_{M^{-1}}(\Omega \times \mathbb{R}^d))$ weak-$\star$, and ρ satisfies*

$$\int_\Omega \rho^{in}\varphi(t=0)dx + \int_0^\infty \int_\Omega \rho\, \partial_t\varphi\, dx\, dt = \int_0^\infty \int_\Omega \rho(h_\alpha\varphi - \mathcal{L}_\alpha(\varphi))dx\, dt, \tag{1.8}$$

for all $\varphi \in \mathcal{D}_\Omega$, with

$$\mathcal{L}_\alpha(\varphi)(x, t) = \Gamma(\alpha + 1)\mathrm{P.V.} \int_{\{w \in \mathbb{R}^d : [x, x+w] \subset \Omega\}} \frac{\varphi(x + w, t) - \varphi(x, t)}{|w|^{d+\alpha}} \,\mathrm{d}w,$$

and

$$h_\alpha(x) = \int_{\mathbb{R}^d} \frac{1}{|w|^{d+\alpha}} e^{-\frac{|x - x_0(x,w)|}{|w|}} \,\mathrm{d}w\,, \tag{1.9}$$

where $[x, y]$, $x, y \in \mathbb{R}^d$, denotes the straight line segment connecting x and y, and $x_0(x, w)$ is the point closest to x in the intersection of $\partial\Omega$ with the ray starting at x in the direction w.

The function h_α is well defined by (1.9) and converges to ∞ when $x \to \partial\Omega$, see Proposition 4.1 in Section 4.

Remark 1.2. Theorem 1.1 remains true with slightly modified proofs for generalized versions of the model. For example, (1.4) may be replaced by the more general condition

$$M(v) \sim 1/|v|^{d+\alpha} \qquad \text{as } |v| \to \infty. \tag{1.10}$$

An example coming from stochastic analysis is the probability density function of an α-stable process, see [6].

Remark 1.3. Another possible generalization is to permit a more general collision operator, satisfying the micro-reversibility principle:

$$Q(f) = \int_{\mathbb{R}^d} [\sigma(v, v')M(v)f(v') - \sigma(v', v)M(v')f(v)] \,\mathrm{d}v'$$

where the cross-section σ is symmetric, i.e. $\sigma(v, v') = \sigma(v', v)$, v, v' in $\mathbb{R}^d$, and bounded from above and away from zero:

$$0 < \nu_1 \le \sigma(v, v') \le \nu_2 < \infty\,.$$

The derivation of macroscopic limits from kinetic equations when the collision kernel has a Maxwellian as an equilibrium distribution is a classical problem studied in the pioneering works [25], [15], and [18]. Here the essential properties of the equilibrium distribution are vanishing mean velocity and finite second order moments. In the case where the equilibrium distribution is heavy-tailed, the problem was first studied for relaxation type collision operators in [20], [19] and [4], from an analytical point of view and in [16] with a probabilistic approach, obtaining as a macroscopic limit a fractional heat equation. These are results on whole space, and they have recently been extended to collision operators of fractional Fokker-Planck type [8] and to the derivation of fractional diffusion with drift [1, 2, 3]. The proofs of most of these results are based on the moment method introduced in [19], which will also be used here.

To find an appropriate definition of fractional diffusion in a bounded domain is not obvious since it describes the probability distribution of a jump process. The formulation of appropriate models as macroscopic limits of kinetic equations is the subject of this work and of the very recent contribution [7], where the problem of deriving a fractional heat equation from a kinetic fractional-Fokker-Planck equation is tackled with zero inflow and specular reflection boundary conditions, where the spatial domain is a circle. The main differences between this work and [7] are that we use a relaxation type collision operator, we only consider inflow boundary conditions, but we permit general, in particular nonconvex, position domains.

There are several equivalent definitions of the fractional Laplacian in the whole domain (see [17]), however, for bounded domains there are different definitions, depending on the details of the underlying stochastic process. For instance, if we consider the stochastic process consisting of a fractional Brownian motion with an $\alpha/2-$stable subordinator and killed upon leaving the domain it has as infinitesimal generator the restricted fractional Laplacian (see [14])

$$-(-\Delta|_\Omega)^{\alpha/2}\varphi(x) := c_{d,\alpha}\ \text{P.V.}\int_{\mathbb{R}^d}\frac{\varphi(y)\mathbf{1}_\Omega(y)-\varphi(x)}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,, \qquad c_{d,\alpha}>0\,. \tag{1.11}$$

This operator has also been derived in [7] as macroscopic limit of a kinetic equation in a circle, subject to zero inflow boundary conditions. The macroscopic operator of Theorem 1.1 can be written in the similar form,

$$-h_\alpha\varphi + \mathcal{L}_\alpha(\varphi) = \Gamma(\alpha+1)\,\text{P.V.}\int_{\mathbb{R}^d}\frac{\varphi(y)\mathbf{1}_{\mathcal{S}_\Omega(x)}(y)-\varphi(x)}{|x-y|^{d+\alpha}}dy\,, \tag{1.12}$$

where $\mathcal{S}_\Omega(x)$ denotes the biggest star-shaped subdomain of Ω with center in x. Obviously, (1.11) and (1.12) coincide for convex Ω (the situation of [7]). The difference in the stochastic process interpretations of (1.11) and (1.12) is that in the latter jumps are only permitted along straight lines, which do not leave the domain.

For completeness we also mention the spectral fractional Laplacian defined as follows: The operator $-\Delta$ subject to homogeneous Dirichlet boundary conditions along $\partial\Omega$ has positive eigenvalues $0<\lambda_1\le\lambda_2\ldots$ with corresponding normalized eigenfunctions $\{e_k\}_{k\ge1}$. The spectral fractional Laplacian (subject to homogeneous Dirichlet boundary conditions) is defined by

$$(-\Delta_\Omega)^{\alpha/2}\varphi(x) := \sum_{i=1}^{\infty}\lambda_i^{\alpha/2}e_i(x)\int_\Omega e_i(y)\varphi(y)dy\,. \tag{1.13}$$

It can also be interpreted as generating a stochastic process (see [9]). A representation formula similar to (1.11) and (1.12) has been derived in [23]:

$$(-\Delta_\Omega)^{\alpha/2}\varphi(x) = c_{d,\alpha}\ \text{P.V.}\int_\Omega[\varphi(x)-\varphi(y)]J(x,y)\,\mathrm{d}y + c_{d,\alpha}\,\kappa(x)\varphi(x), \qquad \text{for } x\in\Omega$$

where the functions J and κ and the constant $c_{d,\alpha}$ satisfy (with positive constants C_1, C_2 and C_3)

$$C_1\delta(x)\delta(y)\le J(x,y)\le C_2\min\left(\frac{1}{|x-y|^{d+\alpha}},\frac{\delta(x)\delta(y)}{|x-y|^{d+2+\alpha}}\right),$$

and

$$C_3^{-1}\delta^{-\alpha}(x)\le\kappa(x)\le C_3\delta^{-\alpha}(x).$$

In [22] it is proven that the two operators $(-\Delta_\Omega)^{\alpha/2}$ and $(-\Delta|_\Omega)^{\alpha/2}$ are different since, for instance, the eigenfunctions of the former are smooth up to the boundary whereas the eigenfunctions of the latter are no better than Hölder continuous up to the boundary. In recent years fractional Laplace operators have been extensively used since they seem to be more suitable for the description of phenomena such as contaminants propagating in water [5], plasma physics [12], among many others (see [24] and [21]). However, there is some literature where for the fractional Laplacian on bounded domains the definitions (1.11) and (1.13) are used interchangeably, thus leading to false results.

2 Uniform estimates and modified test functions

It is a standard result of kinetic theory that the initial-boundary value problem (1.1)–(1.3) with an equilibrium distribution M satisfying (1.4)–(1.6) and an initial position density $\rho^{in} \in L^1(dx)$ has a unique solution, which is nonnegative, if the same holds for ρ^{in} (see, e.g. [10], Chapter XXI). This will be assumed in the following, where we always denote by dx, dv, and dt the Lebesgue measures on Ω, $\mathbb{R}^d$, and, respectively, $(0,\infty)$. We start with standard estimates:

Lemma 2.1. *Let $\rho^{in} \in L^2_+(dx)$. Then the solution f_ε of (1.1)–(1.3) satisfies*

$$f_\varepsilon \in L^\infty(dt,\, L^2_+(dx\,dv/M)) \qquad \textit{uniformly as } \varepsilon \to 0\,,$$

and, with $\rho_\varepsilon := \rho_{f_\varepsilon}$,

$$f_\varepsilon - \rho_\varepsilon M = O(\varepsilon^{\alpha/2}) \qquad \textit{in } L^2(dx\,dv\,dt/M)\,, \qquad \textit{as } \varepsilon \to 0\,.$$

Proof. Multiplication of (1.1) by f_ε/M, integration with respect to x and v, the divergence theorem, and the boundary condition (1.2) yield

$$\begin{aligned} \frac{\varepsilon^\alpha}{2}\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}\frac{f_\varepsilon^2}{M}dv\,dx + \varepsilon\int_{\Gamma^+} v\cdot\nu\frac{f_\varepsilon^2}{2M}dv\,dx &= \int_\Omega\int_{\mathbb{R}^d} Q(f_\varepsilon)\frac{f_\varepsilon}{M}dv\,dx \\ &= -\|f_\varepsilon - \rho_\varepsilon M\|^2_{L^2(dx\,dv/M)}\,, \end{aligned} \tag{2.14}$$

where the second equality is a well known fact and the result of a straightforward computation (see, e.g. [11]). The nonnegativity of the second term and an integration with respect to t over $(0,T)$ give

$$\frac{\varepsilon^\alpha}{2}\|f_\varepsilon(\cdot,\cdot,T)\|^2_{L^2(dx\,dv/M)} + \int_0^T \|f_\varepsilon - \rho_\varepsilon M\|^2_{L^2(dx\,dv/M)}dt \le \frac{\varepsilon^\alpha}{2}\|\rho^{in}\|^2_{L^2(dx)}\,,$$

completing the proof. □

For the proof of Theorem 1.1 we employ the moment method introduced in [19], which relies on test functions solving a suitably chosen adjoint problem. For given $\varphi \in \mathcal{D}_\Omega$ the function $\chi_\varepsilon(x,v,t)$ is the solution of the stationary kinetic equation

$$\chi_\varepsilon - \varepsilon v\cdot\nabla_x\chi_\varepsilon = \varphi\,, \tag{2.15}$$

subject to the inflow boundary condition

$$\chi_\varepsilon = 0 \qquad \text{on } \Gamma^+\,. \tag{2.16}$$

Note that the left hand side of (2.15) is an adjoint version of a part of (1.1), where only the loss term of the collision operator and the transport operator have been kept.

We can readily solve (2.15), (2.16) via the method of characteristics, obtaining

$$\chi_\varepsilon(x,v,t) = \int_0^{r(x,v)/\varepsilon} \mathrm{e}^{-s}\varphi(x+\varepsilon sv,t)ds\,, \qquad \text{where} \quad r(x,v) = \frac{|x-x_0(x,v)|}{|v|}\,, \quad (2.17)$$

and $x_0(x,v)$ is the point closest to x in the intersection of $\partial\Omega$ and the ray starting at x with direction v. In the following a different representation will be convenient:

$$\chi_\varepsilon(x,v,t) = \varphi(x,t)\left(1-\mathrm{e}^{-r(x,v)/\varepsilon}\right) + \int_0^{r(x,v)/\varepsilon} \mathrm{e}^{-s}[\varphi(x+\varepsilon sv,t)-\varphi(x,t)]ds\,. \quad (2.18)$$

This already shows the main difference to the whole space situation [19], which is the boundary layer correction in the parenthesis on the right hand side of (2.18).

In the following we shall need a uniform boundedness result.

Lemma 2.2. *Let $\varphi \in \mathcal{D}_\Omega$ and let χ_ε be given by* (2.17). *Then*

$$\|\chi_\varepsilon\|_{L^2(M\,dx\,dv\,dt)} \le \|\varphi\|_{L^2(dx\,dt)}\,, \qquad \|\partial_t\chi_\varepsilon\|_{L^2(M\,dx\,dv\,dt)} \le \|\partial_t\varphi\|_{L^2(dx\,dt)}\,.$$

Proof. Multiplication of (2.15) by $M\chi_\varepsilon$ and integration with respect to v gives

$$\|\chi_\varepsilon\|^2_{L^2(M\,dv)} - \frac{\varepsilon}{2}\nabla_x\cdot\int_{\mathbb{R}^d} vM\chi_\varepsilon^2 dv = \varphi\int_{\mathbb{R}^d} M\chi_\varepsilon dv \le |\varphi|\,\|\chi_\varepsilon\|_{L^2(M\,dv)}\,,$$

where the Cauchy-Schwarz inequality and the normalization of M has been used. Integration with respect to x and t, the divergence theorem, and the boundary condition (2.16) for χ_ε lead to

$$\|\chi_\varepsilon\|^2_{L^2(M\,dx\,dv\,dt)} - \frac{\varepsilon}{2}\int_0^\infty\int_{\Gamma-} \nu\cdot vM\chi_\varepsilon^2 dv\,d\sigma\,dt \le \|\varphi\|_{L^2(dx\,dt)}\|\chi_\varepsilon\|_{L^2(M\,dx\,dv\,dt)}\,,$$

completing the proof of the first inequality. The proof of the second is analogous after differentiation of (2.15) with respect to t. □

3 Proof of Theorem 1.1

With $\varphi \in \mathcal{D}_\Omega$ and χ_ε defined by (2.17), multiplication of (1.1) by χ_ε and integration with respect to x, v and t gives

$$\begin{aligned} -\int_0^\infty\int_{\mathbb{R}^d}\int_\Omega & f_\varepsilon\partial_t\chi_\varepsilon dx\,dv\,dt - \int_{\mathbb{R}^d}\int_\Omega \rho^{in}M\chi_\varepsilon(t=0)dx\,dv \\ &= \varepsilon^{-\alpha}\int_0^\infty\int_{\mathbb{R}^d}\int_\Omega(\rho_\varepsilon M\chi_\varepsilon - f_\varepsilon\chi_\varepsilon + f_\varepsilon\varepsilon v\cdot\nabla_x\chi_\varepsilon)dx\,dv\,dt \\ &= \int_0^\infty\int_\Omega \rho_\varepsilon\left(\varepsilon^{-\alpha}\int_{\mathbb{R}^d} M(\chi_\varepsilon-\varphi)dv\right)dx\,dt\,. \end{aligned} \quad (3.19)$$

In the sequel we shall need the following notation: For $x, y \in \mathbb{R}^d$ we denote by $[x,y]$ the line segment connecting x and y. Furthermore, we denote by $\mathcal{S}_\Omega(x)$ the largest star shaped subdomain of Ω with center x, i.e.

$$\mathcal{S}_\Omega(x) := \{y\in\Omega : [x,y]\subset\Omega\}$$

The heart of our analysis is the asymptotics for the term in parantheses on the right hand side of (3.19).

Lemma 3.1. *Let $\varphi \in \mathcal{D}_\Omega$ and let χ_ε be given by (2.17). Then*

$$\lim_{\varepsilon\to 0} \varepsilon^{-\alpha} \int_{\mathbb{R}^d} M(\chi_\varepsilon - \varphi)dv = -h_\alpha \varphi + \mathcal{L}_\alpha(\varphi) \tag{3.20}$$

locally uniformly in x and t, where

$$h_\alpha(x) = \int_{\mathbb{R}^d} \frac{1}{|v|^{d+\alpha}} e^{-\frac{|x-x_0(x,v)|}{|v|}} dv\,,$$

$$\mathcal{L}_\alpha(\varphi)(x,t) = \Gamma(\alpha+1)\,\mathrm{P.V.}\int_{\mathcal{S}_\Omega(x)} \frac{\varphi(y,t)-\varphi(x,t)}{|y-x|^{d+\alpha}}dy\,.$$

Proof. The representation (2.18) of χ_ε induces the splitting

$$\varepsilon^{-\alpha}\int_{\mathbb{R}^d} M(\chi_\varepsilon - \varphi)dv = -h_\alpha^\varepsilon \varphi + \mathcal{L}_\alpha^\varepsilon(\varphi)\,,$$

with

$$\begin{aligned} h_\alpha^\varepsilon(x) &= -\varepsilon^{-\alpha}\int_{\mathbb{R}^d} M(v)e^{-r(x,v)/\varepsilon}dv\,, \\ \mathcal{L}_\alpha^\varepsilon(\varphi)(x,t) &= \varepsilon^{-\alpha}\int_{\mathbb{R}^d}\int_0^{r(x,v)/\varepsilon} M(v)e^{-s}[\varphi(x+\varepsilon sv,t)-\varphi(x,t)]ds\,dv\,. \end{aligned}$$

We shall consider these parts separately. In both cases we shall start by proving that the small velocities do not contribute to the limit. This splits the rest of the proof into 4 steps.

Step 1: We consider the contribution to h_α^ε coming from the small velocities. For $|v|\le 1$ we have $r(x,v)\ge \delta(x)$. Therefore

$$\varepsilon^{-\alpha}\int_{|v|\le 1} M(v)e^{-r(x,v)/\varepsilon}dv \le \varepsilon^{-\alpha}e^{-\delta(x)/\varepsilon} \le c\frac{\varepsilon^{2-\alpha}}{\delta(x)^2}\,,$$

since the map $z\mapsto z^2e^{-z}$, $z\ge 0$, is bounded.

Step 2: The previous step implies that h_α^ε is asymptotically equivalent to

$$\varepsilon^{-\alpha}\int_{|v|>1} |v|^{-d-\alpha}e^{-r(x,v)/\varepsilon}dv\,. \tag{3.21}$$

In this integral we make the coordinate transformation $w=\varepsilon v$. Observing that

$$\frac{r(x,w/\varepsilon)}{\varepsilon} = \frac{|x-x_0(x,w/\varepsilon)|}{|w|} = r(x,w)\,,$$

since $x_0(x,w/\varepsilon) = x_0(x,w)$, the expression in (3.21) is equal to

$$\int_{|w|>\varepsilon} |w|^{-d-\alpha}e^{-r(x,w)}dw\,.$$

For proving that this converges to $h_\alpha(x)$, we need to estimate

$$\begin{aligned} \int_{|w|\le\varepsilon} |w|^{-d-\alpha}e^{-r(x,w)}dw &\le \int_{|w|\le\varepsilon}|w|^{-d-\alpha}e^{-\delta(x)/|w|}dw = |S^d|\delta(x)^{-\alpha}\int_{\delta(x)/\varepsilon}^\infty s^{\alpha-1}e^{-s}ds \\ &\le |S^d|\frac{\varepsilon^{2-\alpha}}{\delta(x)^2}\sup_{\gamma\ge 0}\left(\gamma^{2-\alpha}\int_\gamma^\infty s^{\alpha-1}e^{-s}ds\right)\,. \end{aligned}$$

The supremum is finite since the integrand is bounded and decays exponentially as $s \to \infty$.

Combining this result with Step 1 shows that

$$|h_\alpha^\varepsilon(x) - h_\alpha(x)| \le c\frac{\varepsilon^{2-\alpha}}{\delta(x)^2}\,,$$

implying pointwise convergence of h_α^ε to h_α in Ω. Since $|\varphi(x,t)| \le c\,\delta(x)^2$, the convergence of $h_\alpha^\varepsilon\varphi$ to $h_\alpha\varphi$ is uniform in (x,t).

Step 3: We analyze the contributions from the small velocities to $\mathcal{L}_\alpha^\varepsilon(\varphi)$. For the test function difference, we apply the Taylor expansion:

$$\begin{aligned}
&\left|\varepsilon^{-\alpha}\int_{|v|\le 1}\int_0^{r(x,v)/\varepsilon} M(v)e^{-s}\left(\varepsilon sv\cdot\nabla_x\varphi(x,t) + \frac{\varepsilon^2 s^2}{2}v^{tr}\nabla_x^2\varphi(\hat{x},t)v\right)ds\,dv\right| \\
&\quad\le \left|\varepsilon^{1-\alpha}\nabla_x\varphi(x,t)\cdot\int_{|v|\le 1} vM(v)\int_0^{r(x,v)/\varepsilon} se^{-s}ds\,dv\right| \\
&\qquad+\varepsilon^{2-\alpha}c\int_{|v|\le 1}|v|^2M(v)dv\int_0^\infty s^2e^{-s}ds\,.
\end{aligned}$$

In the first term on the right hand side we change the order of integration:

$$\begin{aligned}
&\int_{|v|\le 1} vM(v)\int_0^{r(x,v)/\varepsilon} se^{-s}ds\,dv = \int_0^\infty se^{-s}\int_{|v|\le 1,\,\varepsilon s\le r(x,v)} vM(v)dv\,ds \\
&= \int_0^{\delta(x)/\varepsilon} se^{-s}\int_{|v|\le 1} vM(v)dv\,ds + \int_{\delta(x)/\varepsilon}^\infty se^{-s}\int_{|v|\le 1,\,\varepsilon s\le r(x,v)} vM(v)dv\,ds
\end{aligned}$$

In the first term on the right hand side, the restriction $\varepsilon s \le r(x,v)$ can be omitted, since it is automatically satisfied for $\varepsilon s \le \delta(x) \le r(x,v)$. As a consequence this term vanishes by M being even. The last term can be estimated by

$$\int_{\delta(x)/\varepsilon}^\infty se^{-s}ds\int_{|v|\le 1}|v|M(v)dv \le c\frac{\varepsilon}{\delta(x)}\sup_{\gamma\ge 0}\left(\gamma\int_\gamma^\infty se^{-s}ds\right)\,.$$

Since $\varphi \in \mathcal{D}_\Omega$ implies $|\nabla_x\varphi(x,t)| \le c\delta(x)$, we have the result

$$\varepsilon^{-\alpha}\int_{|v|\le 1}\int_0^{r(x,v)/\varepsilon} M(v)e^{-s}[\varphi(x+\varepsilon sv,t) - \varphi(x,t)]ds\,dv = O(\varepsilon^{2-\alpha})\,,$$

uniformly in (x,t).

Step 4: It remains to consider

$$\begin{aligned}
&\varepsilon^{-\alpha}\int_{|v|>1}\int_0^{r(x,v)/\varepsilon}|v|^{-d-\alpha}e^{-s}[\varphi(x+\varepsilon sv,t) - \varphi(x,t)]ds\,dv \\
&= \int_{|w|>\varepsilon}\int_0^{r(x,w)}|w|^{-d-\alpha}e^{-s}[\varphi(x+sw,t) - \varphi(x,t)]ds\,dw \\
&= \int_0^\infty s^{d+\alpha}e^{-s}\int_{|w|>\varepsilon,\,s<r(x,w)}\frac{\varphi(x+sw,t) - \varphi(x,t)}{|sw|^{d+\alpha}}dw\,ds
\end{aligned}\tag{3.22}$$

By the coordinate transformation $x + sw = y$ the condition $s < r(x,w)$ becomes s-independent:

$$|x-y| < |x - x_0(x,y-x)| \quad\Longleftrightarrow\quad y \in \mathcal{S}_\Omega(x)\,.$$

Therefore (3.22) is equal to

$$\int_0^\infty s^\alpha e^{-s} \int_{\mathcal{S}_\Omega(x)\setminus B_{\varepsilon s}(x)} \frac{\varphi(y,t)-\varphi(x,t)}{|y-x|^{d+\alpha}} dy\, ds\,,$$

where $B_r(x)$ denotes the ball with center x and radius r. In order to prove that this converges to $\mathcal{L}_\alpha(\varphi)$, we need to show that

$$\int_0^\infty s^\alpha e^{-s} \int_{\mathcal{S}_\Omega(x)\cap B_{\varepsilon s}(x)} \frac{(y-x)\cdot\nabla_x\varphi(x,t) + (y-x)^{tr}\nabla_x^2\varphi(\hat{x},t)(y-x)/2}{|y-x|^{d+\alpha}} dy\, ds$$

tends to zero. The second term involving the Hessian of the test function can be estimated by

$$c\int_0^\infty s^\alpha e^{-s} \int_{B_{\varepsilon s}(x)} |y-x|^{2-d-\alpha} dy\, ds = c\,\varepsilon^{2-\alpha} \int_0^\infty s^2 e^{-s} ds\,.$$

The estimation of the first term is more subtle. Actually, the integral with respect to y has to be understood as a principal value for $\alpha \ge 1$. Since

$$\text{P.V.} \int_{B_r(x)} \frac{y-x}{|y-x|^{d+\alpha}} dy = 0\,, \qquad \text{for } r > 0\,,$$

and $B_{\varepsilon s}(x) \subset \mathcal{S}_\Omega(x)$ for $\varepsilon s < \delta(x)$, we have

$$\begin{aligned} &\int_0^\infty s^\alpha e^{-s} \text{P.V.} \int_{\mathcal{S}_\Omega(x)\cap B_{\varepsilon s}(x)} \frac{(y-x)\cdot\nabla_x\varphi(x,t)}{|y-x|^{d+\alpha}} dy\, ds \\ &= \int_{\delta(x)/\varepsilon}^\infty s^\alpha e^{-s} \int_{(\mathcal{S}_\Omega(x)\cap B_{\varepsilon s}(x))\setminus B_{\delta(x)}} \frac{(y-x)\cdot\nabla_x\varphi(x,t)}{|y-x|^{d+\alpha}} dy\, ds\,, \end{aligned} \tag{3.23}$$

which can be estimated by

$$c\delta(x) \int_{\delta(x)/\varepsilon}^\infty s^\alpha e^{-s} \int_{B_{\varepsilon s}(x)\setminus B_{\delta(x)}} |y-x|^{1-d-\alpha} dy\, ds = c\delta(x) \int_{\delta(x)/\varepsilon}^\infty s^\alpha e^{-s} \int_{\delta(x)}^{\varepsilon s} r^{-\alpha} dr\, ds\,.$$

With

$$\int_{\delta(x)}^{\varepsilon s} r^{-\alpha} dr \le \begin{cases} c(\varepsilon s)^{1-\alpha}\,, & \alpha < 1\,, \\ \log(\varepsilon s/\delta(x))\,, & \alpha = 1\,, \\ c\delta(x)^{1-\alpha}\,, & \alpha > 1\,, \end{cases}$$

it is straightforward to obtain that (3.23) is $O(\varepsilon^{2-\alpha})$ for $\alpha \ne 1$ and $O(\varepsilon \log(1/\varepsilon))$ for $\alpha = 1$, uniformly in (x,t). This completes the proof of the uniform convergence of $\mathcal{L}_\alpha^\varepsilon(\varphi)$ to $\mathcal{L}_\alpha(\varphi)$. □

Corollary 3.2. *Let $\varphi \in \mathcal{D}_\Omega$ and let χ_ε be defined by* (2.17)*. Then*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} M(v)[\chi_\varepsilon(x,v,t) - \varphi(x,t)]dv = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} M(v)[\partial_t\chi_\varepsilon(x,v,t) - \partial_t\varphi(x,t)]dv = 0, \tag{3.24}$$

uniformly with respect to $(x,t) \in \text{supp}(\varphi)$.

Proof. The first statement is an immediate consequence of Lemma 3.1. The second statement follows, since $\varphi \in \mathcal{D}_\Omega$ implies $\partial_t\varphi \in \mathcal{D}_\Omega$ and since the map $\partial_t\varphi \mapsto \partial_t\chi_\varepsilon$ is the same as $\varphi \mapsto \chi_\varepsilon$. □

The remaining steps in the proof of Theorem 1.1 are rather standard. As a consequence of Lemma 2.1 and of the estimate

$$|\rho_\varepsilon| \le \|f_\varepsilon\|_{L^2(dv/M)} \qquad \Longrightarrow \qquad \|\rho_\varepsilon\|_{L^2(dx)} \le \|f_\varepsilon\|_{L^2(dx\,dv/M)}\,,$$

we obtain

$$\rho_\varepsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(dt;\, L^2(dx))\,, \qquad f_\varepsilon \overset{*}{\rightharpoonup} \rho M \quad \text{in } L^\infty(dt;\, L^2(dx\,dv/M))\,,$$

when restricting to subsequences. Now we are ready for passing to the limit in (3.19). We decompose the first term by using

$$\int_{\mathbb{R}^d} f_\varepsilon \partial_t \chi_\varepsilon dv = \int_{\mathbb{R}^d} (f_\varepsilon - \rho_\varepsilon M)\partial_t \chi_\varepsilon dv + \rho_\varepsilon \int_{\mathbb{R}^d} M \partial_t \chi_\varepsilon dv\,.$$

The first term on the right hand side tends to zero by

$$\left| \int_{\mathbb{R}^d} (f_\varepsilon - \rho_\varepsilon M)\partial_t \chi_\varepsilon dv \right| \le \|f_\varepsilon - \rho_\varepsilon M\|_{L^2(dv/M)} \|\partial_t \chi_\varepsilon\|_{L^2(M\,dv)}\,,$$

Lemma 2.1, and Lemma 2.2. In the second term we may pass to the limit $\rho\,\partial_t\varphi$ by the weak$*$ convergence of ρ_ε and the strong convergence of $\int_{\mathbb{R}^d} M\partial_t\chi_\varepsilon dv$ (Corollary 3.2). The limit in the second term of (3.19) is a consequence of Corollary 3.2. Finally, passing to the limit in the right hand side of (3.19) is justified by the weak$\star$ convergence of ρ_ε and by Lemma 3.1. This completes the proof of Theorem 1.1.

4 Discussion

In this section we discuss properties of the fractional diffusion operator. First we show that the function h_α defined in (1.9) is well defined and tends to infinity at the boundary of Ω.

Proposition 4.1. *Let h_α be defined by* (1.9)*, then there exists $C > 0$ such that*

$$0 < h_\alpha(x) \le C\delta(x)^{-\alpha}\,, \qquad x \in \Omega\,. \tag{4.25}$$

Proof. In order to prove (4.25) let us chose $x \in \Omega$ and note that $|x - x_0(x,w)| \ge \delta(x)$. Next, let us introduce a polar coordinates change of variables in the integral (1.9), and note the following:

$$h_\alpha(x) = \int_0^{2\pi} \int_0^\infty \frac{1}{\eta^{d+\alpha}} \mathrm{e}^{-|x-x_0(x,\sigma)|/\eta} \eta^{d-1}\, \mathrm{d}\eta\, \mathrm{d}\sigma$$

where η denotes the radial variable. Now, introducing the change of variables $r = \delta(x)\eta$ we obtain

$$\begin{aligned} h_\alpha(x) &\le \int_0^{2\pi} \int_0^\infty \frac{1}{r^{d+\alpha}} \mathrm{e}^{-\delta(x)/r} r^{d-1}\, \mathrm{d}r\, \mathrm{d}\sigma \\ &\le \int_0^{2\pi} \int_0^\infty \frac{1}{(\delta(x)\eta)^{1+\alpha}} \mathrm{e}^{-1/\eta} \delta(x)\, \mathrm{d}\eta\, \mathrm{d}\sigma \\ &= \frac{1}{\delta^\alpha(x)} \int_0^{2\pi} \int_0^\infty \frac{1}{\eta^{1+\alpha}} \mathrm{e}^{-1/\eta}\, \mathrm{d}\eta\, \mathrm{d}\sigma, \end{aligned}$$

from which (4.25) follows. In addition, we obtain that $h_\alpha(x)$ is finite for every $x \in \Omega$. □

In [13] it has been shown that the fractional heat equation

$$\begin{aligned}
\partial_t u(x,t) &= -c_{d,\alpha}\,\text{P.V.}\int_{\mathbb{R}^d} \frac{u(x,t)-u(y,t)}{|x-y|^{d+\alpha}}\,\mathrm{d}y && \text{in } \Omega, t>0,\\
u(x,t) &= 0 && \text{in } \mathbb{R}^d\setminus\Omega,\\
u(x,0) &= u^{in}(x) && \text{in } \Omega,
\end{aligned}$$

has a unique solution such that for any fixed $t_0>0$ the following estimate holds

$$\sup_{t\ge t_0}\left\|\frac{u(\cdot,t)}{\delta^{\alpha/2}(\cdot)}\right\|_{C^\alpha(\overline{\Omega})} \le C(t_0)\|u^{in}\|_{L^2(\Omega)}.$$

Therefore, for any fixed time $t>0$, $u(x,t)$ behaves like $\delta^{\alpha/2}(x)$ when $x\to\partial\Omega$.

In this work we neither prove the uniqueness of weak solutions nor any Hölder regularity results, however, formally using $\varphi(x,t)=\rho(x,t)\mathbf{1}_{[0,T]}(t)$ in (1.8) yields

$$\begin{aligned}
\frac{1}{2}\|\rho(\cdot,T)\|^2_{L^2(\Omega)} &+ \int_0^T\int_\Omega h_\alpha\rho^2 dx\,dt\\
&+\Gamma(\alpha+1)\int_0^T\int_{x,y\colon[x,y]\subset\Omega}\frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{d+\alpha}}dx\,dy\,dt = \frac{1}{2}\|\rho^{in}\|^2_{L^2(\Omega)}.
\end{aligned}$$

This implies uniqueness at least formally. Also the boundedness of the second integral together with Proposition 4.1 induces results on the behaviour of ρ close to the boundary. In particular for $\alpha>1$, as a consequence of Proposition 4.1, h_α is not integrable, implying some decay of $\rho(x,t)$ as $\delta(x)\to 0$.

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