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Abstract

The aim of this thesis is to provide a summary of different versions of Zariski's Main Theorem, or ZMT for short. In particular, the original statement on birational correspondences is stated and proven for more general schemes. For the proof we make use of Peskine's version of ZMT, which is an algebraic generalization of the Zariski's original statement. We will continue by proving the so-called power series version and discuss the Nagata property. Finally, we present an extended version of ZMT which shows how the different statements interrelate and gives a geometric characterization of normal points.

Zusammenfassung

Das Ziel dieser Arbeit ist es, einen Überblick über verschiedene Versionen von Zariski's Main Theorem zu geben. Insbesondere wird das ursprüngliche Theorem über birational Korrespondenzen für allgemeinere Schemata bewiesen. Für den Beweis verwenden wir Peskine's Variante des ZMT, welche eine algebraische Verallgemeinerung des Originaltheorems darstellt. Weiters beweisen wir die sogenannte Potenzreihenversion und behandeln die Nagata-Eigenschaft. Schließlich präsentieren wir eine erweiterte Version des ZMT, welche aufzeigt, wie die verschiedenen Versionen zusammenhängen. Zusätzlich liefert diese eine geometrische Charakterisierung normaler Punkte.

Contents

Introduction

There exist many different results in the literature called Zariski's Main Theorem (or ZMT for short) and often, there is no immediately apparent relation between them. Mumford gives a list of five statements he calls Zariski's Main Theorem in [Mum99]; but apart from that, there is surprisingly little information on how these different versions compare. In particular, the reader seldomly finds any reference to the context in which the theorem initially appeared.
Consider an example: for $n > 2$, take the blow-up $f : Y \to \mathbb{A}^n_k$ of the affine n-space at the origin. Write U for the open subset $\mathbb{A}^n_k - \{0\}$. Then f induces an isomorphism $Y - f^{-1}(U) \simeq U$, i.e., f is *birational.* Observe that 0 is a point at which f is not a local isomorphism. Such points are called *fundamental.* The preimage of 0 under f is isomorphic to $\mathbb{P}^{n-1}_k$, in particular, the fiber $f^{-1}(0)$ is positive dimensional. The original statement of Zariski's Main Theorem is that this property holds for the fundamental points of any birational morphism $X' \to X$ between k-varieties, provided that the variety X is *normal.* More precisely, it says that every irreducible component of the fiber of a fundamental point on X has positive dimension. Recall that a variety is called normal if, for every point x of X, the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.
Our aim here is two-fold: first, to provide the historical context for Zariski's original theorem and present the results in a more modern language. Second, we want to prove an extended version of ZMT, which contains the original version while at the same time showing its relation to other versions. The precise statement is as follows: let X be an integral scheme fulfilling some appropriate properties; namely, that X is of finite type over Spec(A), where A is a Nagata Dedekind domain. For any point $x \in X$, the following are equivalent:

(1) X is normal at x.
(2) The completion $\widehat{\mathcal{O}_{X,x}}$ of the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.
(3) For every every birational morphism $f : X' \to X$ such that f is not a local isomorphism at x, every irreducible component of the fiber $f^{-1}(x)$ is positive dimensional.

Note that the implication (1) $\implies$ (3) is just the statement of the original ZMT. To prove (1) $\implies$ (2) is the main objective of Chapter 3. The converse implications (2) $\implies$ (1) and (3) $\implies$ (1) are, in contrast, very easy. This gives us a rather geometric characterization of normal points.
This thesis is structured in such a way as to take into account the historical progress. We start with **Chapter 1**, where our main aim is to translate the concepts and results of Zariski's original article into the language of schemes. First, we give a precise definition of birational maps for general integral schemes. Following the approach

of Zariski, we go on to define birational correspondences using valuations. In particular, this allows us to prove a weak version of ZMT. The definitions and results of Chapter 1 are all essentially contained in [Zar43], though stated and proven here in much more generality.

In **Chapter 2**, we want to present *Peskine's version*, which is a rather natural generalization of the original ZMT. It roughly states that a finite-type morphism with finite fibers between affine schemes factors as a composition of an open immersion, followed by a finite morphism. Most of this chapter is taken up by the proof of Peskine's theorem, which is based on Zariski's original proof. At the end, we prove some easy corollaries and briefly mention Grothendieck's version of ZMT, which generalizes the statement of Peskine's version to non-affine schemes. Grothendieck proves his result using different methods from the ones covered here. We will not give any proof since doing so would go beyond the scope of this thesis.

Chapter 3 is devoted to the *power series version* of Zariski's Main Theorem. It was first proven by Zariski in 1950. The original statement is the following: let $\mathcal{O}_{X,x}$ be the local ring of a point x of a k-variety and assume that (1) $\mathcal{O}_{X,x}$ is an integrally closed domain and (2) the residue field $\kappa(x)$ of x is separable over k. Then the completion $\widehat{\mathcal{O}_{X,x}}$ is again a normal domain. Surprisingly, the original ZMT can be proven quite easily using the power series version. One of the key steps in the proof of the power series version is the finiteness of certain integral closures, which we discuss in detail. The last part of Chapter 3 is proving the power series version in a more general situation, following Nagata in [Nag62]. Nagata's proof is almost exactly the same as Zariski's, modulo a few modifications for the inseparable case.

Finally, in **Chapter 4** we state and prove the extended version of ZMT, which summarizes both the original statement and the power series version. Thus we obtain the aforementioned characterization of normal points. We finish by providing some examples illustrating the statement of the extended version.

As a final remark, there exists a result by Zariski known as the "Connectedness Theorem" (see [Zar57]), which is sometimes also called "Zariski's Main Theorem". While it is closely related and can be used to prove a version of ZMT (see Theorem 2.20), the Connectedness theorem holds under weaker assumptions. In particular, it does not characterize normal points in the sense of the extended ZMT, so we will not mention it here.

Prerequisites: As Zariski's Main Theorem is a result in algebraic geometry whose proof is almost entirely algebraic, this thesis uses quite a lot of commutative algebra. We assume basic knowledge both of commutative algebra and the theory of schemes. There is an appendix included which contains the most important results and definitions. The reader is recommended to use [AM94], [Mat89] and [Har77] as a general reference.

1. **Zariski's original version: Birational correspondences**

The original version of Zariski's Main Theorem was first stated and proven in "Foundations of a general theory of birational correspondences" (1943) by Oscar Zariski, see [Zar43, p.522]. In this paper, the theorem was stated as follows:

> MAIN THEOREM. If W is an irreducible fundamental variety on V of a birational correspondence T between V and V' and if T has no fundamental elements on V', then - under the assumption that V is locally normal at W - each irreducible component of the transform $T[W]$ is of higher dimension than W.

Zariski's objective was to systematically introduce concepts and results of the theory of birational correspondences. An important role there belongs to *normal varieties*, which were introduced as a class of varieties more general than nonsingular ones, such that many known properties of birational correspondences still hold true. Our aim in this chapter is to extend birational correspondences to general schemes. First, we give a precise definition of rational maps for schemes over an arbitrary base scheme S. Zariski used the theory of valuations in order to define birational correspondences for varieties. We will keep this approach while translating the constructions and results into the language of schemes. This will provide the original context for Zariski's Main Theorem. For the proof, we reduce the statement to a purely algebraic one, see Lemma 1.24. We will not prove Lemma 1.24 in this chapter, since the proof is essentially the same as Peskine's one, which we will carry out in Chapter 2.

1.1. Rational maps and birational transformations

In this section we will define rational maps for integral schemes X, Y as equivalence classes of morphisms defined on some open subset $U \subset X$. Let us start with the following topological result.

Lemma 1.1. *Let X be an irreducible scheme. Then every open subset of X is either empty or contains the generic point of X. In particular, every nonempty open of X is dense and the intersection of any two nonempty opens is nonempty.*

Proof. This follows from the fact that every open subset U of X is *stable under generization*: if $z \in U$ is a point and $z' \in X$ such that $z \in \overline{\{z'\}}$, then we have $z' \in U$. □

Let X, Y be integral schemes over some base scheme S. Consider the set of all pairs (U, f) where $U \subset X$ is a nonempty open and $f : U \to Y$ an S-morphism. Define an equivalence relation on this set by considering two pairs (U, f), (V, g) equivalent if there exists a nonempty open $W \subset U \cap V$ such that $f|_W = g|_W$. An *S-rational map* $f : X \dashrightarrow Y$ is given by an equivalence class $(U, \tilde{f})$ and denote the set of all such maps by $\mathrm{Rat}_S(X,Y)$. The morphism $\tilde{f}: U \to X$ is called a *representative* for f. Note that there exists a natural map $\mathrm{Hom}_S(X,Y) \to \mathrm{Rat}_S(X,Y)$. This map is not injective in general, since two different morphisms might agree on an nonempty open subset. However, when Y is separated, we have the following result:

Lemma 1.2. *Let X, Y be two schemes over a scheme S, with Y separated over S. Let $U \subset X$ be a dense open subset and $f, g : X \to Y$ two S-morphisms that agree on U. Then $f|_{X_{\mathrm{red}}} = g|_{X_{\mathrm{red}}}$. In particular, if X is reduced we have $f = g$.*

We skip the proof here and refer the reader to [GW10, Corollary 9.9].
Let $f : X \dashrightarrow Y$ be a rational S-map. Denote by $\mathrm{dom}_S(f)$ the set of points $x \in X$ such that there exists a open $U \subset X$ with $x \in U$ and an S-morphism $\tilde{f} : U \to Y$ which is a representative of f. Clearly $\mathrm{dom}_S(f)$ itself is open. We call it the *domain of definition* of f and say that $x \in X$ is *regular* for f if $x \in \mathrm{dom}_S(f)$; otherwise we call x a *fundamental point*. From the previous lemma we obtain immediately:

Proposition 1.3. *Let $f : X \dashrightarrow Y$ be a rational S-map between integral schemes and assume that Y is separated over S. Then there exists a unique representative $\tilde{f} : \mathrm{dom}_S(f) \to Y$ for f. In particular, the natural map $\mathrm{Hom}_S(X,Y) \to \mathrm{Rat}_S(X,Y)$ is injective.*

Proof. By definition, the open set $\mathrm{dom}_S(f)$ is covered by opens U_i such that there exists a representative $\tilde{f}_i : U_i \to Y$ for f. By Lemma 1.2 the $\tilde{f}_i$ agree on the intersections of the U_i, hence we can glue them to obtain a unique representative $\tilde{f} : \mathrm{dom}_S(f) \to Y$ (see [GW10, Proposition 3.5] for details). □

In general it does not make sense to compose rational maps: let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be rational maps, then $f(\mathrm{dom}_S(f))$ and $\mathrm{dom}_S(g)$ might not intersect (and hence the composition $g \circ f$ is defined nowhere). This happens if $f(\mathrm{dom}_S(f))$ is not dense anymore. In order to be able to build a category we restrict to integral schemes and rational maps between them which allow a *dominant* representative. We call a rational S-map $f : X \dashrightarrow Y$ *dominant* if there exists a dominant representative $\tilde{f} : U \to Y$ of f. Since every nonempty open subset of X contains the generic point ξ of X, by Lemma C.2, f is dominant if and only if $f(\xi)$ is the generic point of Y. In particular, every representative of a dominant S-rational map is dominant.
Now let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be rational S-maps between integral schemes and assume that f is dominant. Choose representatives $\tilde{f} : U \to Y$ and $\tilde{g} : V \to Z$ for f and g. Since f is dominant, we have that $\tilde{f}^{-1}(V)$ contains the generic point of X and hence $U \cap \tilde{f}^{-1}(V)$ is nonempty. We define the composition $g \circ f$ to be the rational map given by the representative $\tilde{g} \circ \tilde{f} : U \cap \tilde{f}^{-1}(V) \to Z$. It is easy to check that this definition is independent of the choice of representatives for f and g.
Let $f : X \dashrightarrow Y$ be a dominant rational S-map. Then f is called *birational* if f

is an isomorphism in the category of integral S-schemes together with dominant rational S-maps as morphisms. The following lemma gives some characterizations for a rational map to be birational.

Lemma 1.4. *Let $f : X \dashrightarrow Y$ be a dominant S-rational map between integral S-schemes. The following are equivalent:*

(1) *f is birational.*
(2) *For every representative $\tilde{f} : U \to Y$ there exists an S-morphism $\tilde{g} : V \to X$ with $V \subset Y$ nonempty open such that $\tilde{f} \circ \tilde{g} = \mathrm{id}$ and $\tilde{g} \circ \tilde{f} = \mathrm{id}$, where they are defined.*
(3) *There exists a representative $\tilde{f} : U \to Y$ which induces an isomorphism of U with some open $V \subset Y$.*

Proof. If f is birational, then this means that there exists a rational S-map $g : Y \dashrightarrow X$ such that $g \circ f = \mathrm{id}$ (where id denotes the rational map represented by $\mathrm{id}_X : X \to X$). In order to prove that (1) implies (2), take any representative $\tilde{g} : V \to X$. Since f is dominant we have that $U \cap \tilde{f}^{-1}(V)$ is nonempty. By assumption, there exists a nonempty open $U' \subset U \cap \tilde{f}^{-1}(V)$ such that $\tilde{g} \circ \tilde{f} \equiv \mathrm{id}$ on U'. After shrinking V we get that $\tilde{g} \circ \tilde{f} = \mathrm{id}$ where the composition is defined. Repeat the same argument to obtain $\tilde{f} \circ \tilde{g} = \mathrm{id}$.
Now assume (2). Choose a representative $\tilde{f} : U' \to Y$, then by assumptions there exists a morphism $\tilde{g} : V' \to X$ such that $\tilde{g} \circ \tilde{f} = \mathrm{id}$ where they are defined. Set $U = U' \cap \tilde{f}^{-1}(V')$ and $V = V' \cap \tilde{g}^{-1}(U')$, then it is easy to check that $\tilde{f}|_U : U \to V$ is an isomorphism and again a representative for f.
Finally, let us prove that (3) implies (1). By assumption there exists a nonempty open $V \subset Y$ and a morphism $\tilde{g} : V \to X$ such that $\tilde{g} \circ \tilde{f} = \mathrm{id}$ on U, and $\tilde{f} \circ \tilde{g} = \mathrm{id}$ on V. Hence the S-rational map g given by $\tilde{g}$ is a two-sided inverse for f and we are done. ☐

Given any S-rational $f : X \dashrightarrow Y$ map as above, denote by f' the rational map represented by the morphism $\tilde{g}$ in (2) and call it the *inverse* of f.

Remark 1.5. Let X, Y be integral separated S-schemes and $f : X \dashrightarrow Y$ a birational S-map. Denote by $\tilde{f}$ and $\tilde{f}'$ the unique representatives for f and f' from Lemma 1.3. Set

$$U := \{x \in X : x \text{ is regular for } f,\ f(x) \text{ is regular for } f'\}.$$

Then $U = \mathrm{dom}_S(f) \cap \tilde{f}^{-1}(\mathrm{dom}_S(f'))$ and U is the maximal open set such that there exists a representative for f with property (3) in Lemma 1.4.

Any birational S-map $f : X \dashrightarrow Y$ between integral S-schemes induces an isomorphism of function fields $K(X) \simeq K(Y)$. The converse does not hold true in general. A necessary condition is that the generic points of X and Y lie over the same point in S and $\mathrm{Spec}(K(X)) \to \mathrm{Spec}(K(Y))$ factors over S. However, there exist examples where such a morphism does not induce an isomorphism of open subset. It turns out that a sufficient additional condition for X and Y is to be *locally of finite presentation* over S. This result is taken from [GW10, p.257].

Proposition 1.6. *Let X, Y be S-schemes and $x \in X, y \in Y$ points lying over the same point $s \in S$.*

(1) *Suppose that Y is locally of finite type over S. Let $f, f' : X \to Y$ be S-morphisms with $f(x) = f'(x) = y$ and such that local ring maps $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ coincide. Then f and f' coincide on an open neighborhood of x.*
(2) *Suppose that Y is locally of finite presentation over S. Let $\varphi_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a local $\mathcal{O}_{S,s}$-map. Then there exists an open neighborhood U of x and an S-morphism $f : U \to Y$ with $f(x) = y$ and such that the local ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induced by f is given by φ_x.*
(3) *If in (2) in addition X is locally of finite type over S, then one can find U and f such that f is of finite type.*
(4) *If in (2) in addition X is locally of finite presentation over S and φ_x is an isomorphism, then one can find U and f such that f is an isomorphism of U onto an open neighborhood of y.*

We will now apply the proposition to the special case of integral schemes which are locally of finite presentation over an *affine* scheme S and obtain the following result:

Proposition 1.7. *Let X and Y be integral schemes locally of finite presentation over an affine base scheme $S = \mathrm{Spec}(A)$. Assume the generic points of X and Y lie over the same base point in S. Then:*

(1) *Any A-algebra map $\varphi : K(X) \to K(Y)$ induces a rational map $f : X \dashrightarrow Y$. If φ is an isomorphism, then f is birational.*
(2) *Let $f : X \dashrightarrow Y$ be a birational map and $x \in X$ a point. Then x is regular for f if and only if there exists $y \in Y$ such that the local rings $\mathcal{O}_{X,x}$ dominates $\mathcal{O}_{Y,y}$, where both are seen as local subrings of $K(X) \simeq K(Y)$.*

Proof. The first part follows directly from Proposition 1.6. For the second part, we only have to show that x and y lie over the same point $s \in S$. But this follows from the fact that, by our assumptions, the structure ring map $A \to \mathcal{O}_{X,x}$ factors as $A \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. □

We now want to extend the definition of birational maps: let X,X' be arbitrary integral schemes over some base scheme S. A *birational transformation* $T : X \dashrightarrow X'$ is defined as an S-isomorphism $\mathrm{Spec}(K(X)) \to \mathrm{Spec}(K(Y))$. We say that a point x of X is regular for T if there exists a point x' of X' and an S-morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{X',x'})$. The point x' is called the *image* of x. If there does not exist such an S-morphism we say that x is *fundamental* for T. If X, X' and S fulfill the conditions of Proposition 1.7, then any birational transformation T between X and X' induces a birational map $f : X \dashrightarrow X'$ and $x \in X$ is regular/fundamental for T if and only if x is regular/fundamental for f.

Using the above notions, it is possible to give a slightly more general definition of birational morphisms: if $f : X \to Y$ is an S-morphism, then, from now on, we call f *birational* if there exists a diagram of S-morphisms

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \uparrow & & \uparrow \\ \operatorname{Spec}(K(X)) & \xrightarrow{\simeq} & \operatorname{Spec}(K(Y)) \end{array}$$

with the vertical arrows given by the canonical maps and the lower horizontal arrow an isomorphism. In particular, a birational morphism induces a birational transformation $T : X \dashrightarrow Y$ which is regular at all $x \in X$ and such that the image of x under T is $f(x)$.

Finally, note that, if X is an integral scheme over S, then the normalization $\widetilde{X} \to X$ is always birational under the above definition.

1.2. Valuation-theoretic approach to birational correspondences

Zariski's original definition of birational correspondences was purely valuation-theoretic and many proofs in [Zar43] made use of this. In particular, a weak form of the Main theorem can be proven using only the fact that every integrally closed domain is the intersection of all valuation rings containing it. While we want to keep Zariski's approach, there is also a way to define birational correspondences avoiding valuations entirely, as we will see below.

Let us make a brief remark on the different assumptions made in this section. The constructions presented at the beginning work for schemes over an arbitrary base scheme S. However, for later results it will be convenient to assume that our schemes are of finite type over S, where S is Noetherian affine. We will repeat these assumptions at the exact place where they are needed.

We start with a definition made for technical reasons: If S is any scheme and $\operatorname{Spec}(K) \to S$ a morphism, then we call a valuation ring R an S-valuation ring if the natural morphism $\operatorname{Spec}(K) \to \operatorname{Spec}(R)$ is an S-morphism. We may sometimes omit S if there is no ambiguity. In [Zar43], Zariski originally considered the case $S = \operatorname{Spec}(k)$, with k a subfield of K. Observe that for $S = \operatorname{Spec}(k)$, any valuation ring of K is an S-valuation ring if and only if its associated valuation is trivial on k.

Let X be an integral scheme over S and denote by K the function field of X. Let x be any point of X; we say that an S-valuation ring R of K has *center* x on X if there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & S \end{array}$$

such that the morphism $\mathrm{Spec}(R) \to X$ maps the unique closed point of $\mathrm{Spec}(R)$ to x. In particular, by Lemma E.4, any such diagram implies that the valuation ring R dominates $\mathcal{O}_{X,x}$ as a subring of K.

Note that for every point x of X there exists an S-valuation ring of K which has center x; indeed, by Zorn's lemma, $\mathcal{O}_{X,x}$ is contained in some valuation ring R and $\mathrm{Spec}(R)$ is a scheme over S via the composition $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \hookrightarrow R$, where s is the image of x under $X \to S$.

Let $T : X \dashrightarrow X'$ be a birational transformation over S and denote by K their common function field; we say that $x' \in X$ *corresponds* to $x \in X$ if there exists an S-valuation ring R of K with center x' on X' and x on X. In other words, if there exists a commutative diagram of S-morphisms

$$\begin{array}{ccc} & \mathrm{Spec}(K) & \\ \swarrow & \downarrow & \searrow \\ X & & X' \\ \nwarrow & \downarrow & \nearrow \\ & \mathrm{Spec}(R) & \end{array}$$

such that the closed point of $\mathrm{Spec}(R)$ gets mapped to x' respectively x.

Assume that $x \in X$ is a regular point for T with image x'. Let R be an S-valuation ring with center $x \in X$. Note that the morphism $\mathrm{Spec}(R) \to X$ factors as

$$\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$$

and composing the first morphism with $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{X',x'})$ yields a diagram

$$\begin{array}{ccc} & \mathrm{Spec}(K) & \\ \swarrow & \downarrow & \searrow \\ \mathrm{Spec}(\mathcal{O}_{X,x}) & & \mathrm{Spec}(\mathcal{O}_{X',x'}) \\ \nwarrow & \downarrow & \nearrow \\ & \mathrm{Spec}(R) & \end{array}$$

Since the closed point of R maps to the unique closed point of $\mathrm{Spec}(\mathcal{O}_{X',x'})$ we see that R has center x' on X'; so in particular the point x' corresponds to x. Observe that for a regular point $x \in X$, there might still correspond more than one point of X' to x. This happens when the center of a valuation ring on X' is not unique.

Let us now introduce an alternative way to define birational correspondences: let $T : X \dashrightarrow X'$ be a birational transformation and consider the canonical morphisms $\mathrm{Spec}(K) \to X$, $\mathrm{Spec}(K) \to X'$ (with K the common function field). These morphisms give rise to a unique morphism $\mathrm{Spec}(K) \to X \times_S X'$, which identifies a point ξ on $X \times_S X'$ with residue field K. Define the *graph* Γ of T to be the closure of the singleton $\{\xi\}$ inside $X \times_S X'$ with its reduced scheme structure (in [Zar43] this construction was called the *join* of T). Then Γ is again an integral scheme with function field K. Restricting the projections of $X \times_S X'$ to Γ gives rise to birational morphisms $p : \Gamma \to X$ and $p' : \Gamma \to X'$. Note that Γ obtains many nice properties from X and X'.

Lemma 1.8. *Let $T : X \dashrightarrow X'$ be a birational transformation and Γ the graph of T. If X and X' are of finite type over S then so is Γ. If furthermore both X and X' are separated (resp. proper), then Γ is separated (resp. proper).*

Proof. Note that Γ becomes a scheme over S via the composition

$$\Gamma \hookrightarrow X \times_S X' \longrightarrow S \ .$$

Since any closed immersion is proper, it suffices to prove all assertions for $X \times_S X'$. Observe that the morphism $X \times_S X' \to S$ is again obtained by a composition, namely

$$X \times_S X' \longrightarrow X' \longrightarrow S \ ,$$

where the first arrow is just the base change of $X \to S$ by $X' \to S$. The statement follows since all properties - finite type, separated, proper - are stable under base change and composition. □

We now want to connect the notion of birational correspondences to the graph Γ.

Lemma 1.9. *Let $T : X \dashrightarrow X'$ be a birational transformation and Γ,p,p' defined as above. Then for each $x \in X$ the set of all points of X' corresponding to x' equals $p'(p^{-1}(x))$.*

Proof. Let $x' \in X'$ correspond to x and let R be an S-valuation ring with centers x,x'. By definition this induces a diagram

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & X \times X' \\ \uparrow & \nearrow & \\ \mathrm{Spec}(K) & & \end{array}$$

Call z the image of the closed point of $\mathrm{Spec}(R)$ in $X \times X'$. By Lemma E.4 the element z lies on Γ and it is clear that z gets mapped to x respectively x'.
Conversely, assume $z \in \Gamma$ with $p(z) = x$ and $p'(z) = x'$. Let R be an S-valuation ring with center z on Γ. Composing the morphism $\mathrm{Spec}(R) \to \Gamma$ with p, p' we see that R has center x on X and x' on X'. Hence x corresponds to x'. □

For $T : X \dashrightarrow X'$ we call $p'(p^{-1}(x))$ the *transform* of x and denote it by $T[x]$.

Remark 1.10. In [Zar43] the transform $T[Z]$ of an irreducible closed subset Z was defined as the Zariski closure of $p'(p^{-1}(\eta_z))$, where η_z denotes the generic point of Z. The original statement of Zariski's Main Theorem concerned the dimension of the transform for fundamental points. Note that $T[Z]$ can be a proper subset of the *total transform* $\overline{p'(p^{-1}(Z))}$: consider the blowup $X \to \mathbb{A}^2_k$ of $\mathbb{A}^2_k$ at the origin (see [Har77, I.4] for a definition). Let Y be the plane nodal curve defined by $y = x^2(x+1)$ in $\mathbb{A}^2_k$. Then the transform $T[Y]$ does not contain the exceptional divisor $E = \mathbb{P}^1_k$, since the generic point of E gets mapped to $0 \in Y$.

Lemma 1.11. *Let $T : X \dashrightarrow X'$ be a birational transformation and assume that x is regular for T. Then x is regular for $X \dashrightarrow \Gamma$.*

Proof. Choose an affine open neighborhoods $U = \operatorname{Spec}(B)$, $U' = \operatorname{Spec}(B')$ of x and the unique point x' corresponding to x; denote by $\mathfrak{q}, \mathfrak{q}'$ the prime ideals associated to them. Then $U \times_S U' = \operatorname{Spec}(B \otimes_A B')$ is an open affine of $X \times_S X'$. Hence we get that $\Gamma \cap U \times U'$ is an affine open of Γ with

$$\Gamma \cap U \times U' = \operatorname{Spec}((B \otimes B')/J),$$

where J is the kernel of the map $B \otimes B' \to K$, which sends $b \otimes b'$ to $bb' \in K$. By construction $\Gamma \cap U \times U'$ contains all $z \in \Gamma$ corresponding to both x and x'. Write $C = (B \otimes B')/J$ and $\mathfrak{n} \subset C$ for a prime associated to such a point z. Since $\mathfrak{n}$ lies over $\mathfrak{q}$ we get a map $B_{\mathfrak{q}} \to C_{\mathfrak{n}}$. It is injective since $\Gamma \to X$ is dominant. We are done if we show that this map is surjective. Note that, by assumption, $B'_{\mathfrak{q}'} \subset B_{\mathfrak{q}}$, and hence, when considered as subrings of K, the ring C is contained in $B_{\mathfrak{q}}$. This implies that $B_{\mathfrak{q}} = C_{\mathfrak{n}}$. □

Example 1.12. We will now illustrate the constructions above using an example: Take the quadratic Cremona transformation $T : X \dashrightarrow X'$, with $X = X' = \mathbb{P}^2_k$, given by $(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1)$ in homogeneous coordinates. It has three fundamental points $(0 : 0 : 1)$,$(0 : 1 : 0)$ and $(1 : 0 : 0)$. Denote by x the point given by $(1 : 0 : 0)$. Its local ring embeds into the function field as

$$\mathcal{O}_{X,x} \simeq k[X, Y]_{(X,Y)}.$$

Now consider the two points z,z' given by $(0 : 1 : 0)$ and $(0 : 0 : 1)$. Via the isomorphism their local rings embed into the function field as follows:

$$\mathcal{O}_{X',z} \simeq k[X, X/Y]_{(X,X/Y)},\ \mathcal{O}_{X',z'} \simeq k[Y, Y/X]_{(Y,Y/X)}.$$

Note that $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X',z}$ are dominated by the local ring $k[X, Y, X/Y]_{(X,Y,X/Y)} \simeq k[X, Y, Z]_{(X,Y,Z)}/(ZY - X)$ inside $k(X, Y)$; so in particular there exists a k-valuation ring of $k(X, Y)$ with center x on X and z on X'. The same argument works for x and z', so both z and z' correspond to x.

We will now use the graph Γ of T to find all points of X' corresponding to x. Let U_0 be the open affine of X where $x_0 \neq 0$ and V_1 the open affine of X' where $y_1 \neq 0$. Then it is easy to check that $U = \Gamma \cap (U_0 \times V_1)$ is an open affine of Γ with

$$\mathcal{O}_U = (k[X,Y] \otimes_k k[S,T])/(S - X, TY - X).$$

Now via the map $k[X,Y] \to \mathcal{O}_U$ the ideal (X, Y, S) of $\mathcal{O}_U$ lies over (X, Y). Translating back to geometry this implies that the set of all points of X' corresponding to x is the closed set defined by $y_0 = 0$ (in homogeneous coordinates).

From now on, for the sake of convenience, we will tacitly assume that all schemes are of finite type over $S = \mathrm{Spec}(A)$ affine, with A Noetherian. In particular, by Proposition 1.7, every birational transformation $T : X \dashrightarrow X'$ induces a rational S-map $f : X \dashrightarrow Y$. For example, in the next lemma we need X to be Noetherian in order to be able to apply Theorem E.5.

Lemma 1.13. *Let X be separated over S, then any S-valuation ring R of K has at most one center on X. If X is proper over S then R has a unique center on X.*

Proof. Note that X is Noetherian since it is of finite type over S, which is Noetherian affine. Then the lemma is a direct application of the valuative criterion of separatedness resp. properness, see Theorem E.5. Namely, an S-valuation ring R of K gives rise to a unique diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & S \end{array} .$$

and a diagonal arrow corresponds to a center of R on X. ☐

Corollary 1.14. *X*

(1) *Let $T : X \dashrightarrow X'$ be a birational transformation and suppose that x is regular for T, with image $x' \in X'$. If X' is separated then x' is the only point corresponding to x.*
(2) *Let $f : X' \to X$ be a birational morphism. If both X and X' are separated, then for each $x \in X$ the set of all points of X' corresponding to x equals $f^{-1}(x)$.*

Proof. Let us prove (1). By assumption, any S-valuation ring R with center x on X has center x' on X' and using Lemma 1.13 it follows that this x' is unique.
From (1) it follows that each element of $f^{-1}(x)$ corresponds to x. Now suppose that x' corresponds to x and let R be an S-valuation ring with centers x,x'. Then R also has center $f(x')$ on X and since X is separated we have $x = f(x')$. ☐

The following lemma says that for a normal scheme X, all fundamental points of X have codimension at least 2. The statement follows for the same reason that normal varieties are nonsingular in codimension 1.

Lemma 1.15. *Let $T : X \dashrightarrow X'$ be a birational transformation and assume X is normal and X' is proper. Then any fundamental point of X for T has codimension ≥ 2.*

Proof. Let $x \in X$ be a point of codimension 1, then $\mathcal{O}_{X,x}$ is an integrally closed domain of dimension 1 and hence a (discrete) valuation ring. Since X' is proper, by Lemma 1.13, there exists a local ring map $\mathcal{O}_{X',x'} \to \mathcal{O}_{X,x}$ for some (unique) x'. But this implies that x is regular for T. □

Corollary 1.14 states that, if x was a regular point for a birational transformation $T : X \dashrightarrow X'$ and under the assumption that X' is separated, there corresponds only one point on X' to x. The converse only holds under additional assumptions; in particular, that x is a normal point of X. We first need the following lemma, which is the scheme-theoretic version of Corollary E.3.

Lemma 1.16. *Let B be a normal local domain with quotient field K such that* $\operatorname{Spec}(B)$ *is a scheme over S. Let C be a ring such that for all S-valuation rings R of K dominating B there exists a diagram*

$$\begin{array}{ccc} \operatorname{Spec}(R) & \longrightarrow & \operatorname{Spec}(C) \\ & \nwarrow \quad \nearrow & \\ & \operatorname{Spec}(K) & \end{array}$$

of S-morphisms. Then there exists a unique S-morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(C)$ *making all the diagrams commute.*

Proof. Since B is normal, by Corollary E.3 it is equal to the intersection of all valuation rings of K dominating it. Note that every valuation ring R dominating B is an S-valuation ring of K and for each such R we have, by assumption, $C \subset R \subset K$. Hence $C \subset B \subset K$ and this induces an S-morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(C)$ with the desired properties. □

Lemma 1.17. *Let $T : X \dashrightarrow X'$ be a birational transformation and assume that X' is proper and X is normal. If to $x \in X$ there corresponds only one $x' \in X'$ then x is regular for T.*

Proof. Let R be a valuation ring with center x. Since X' is proper R has center x' on X', which means we can apply Lemma 1.16 with $B = \mathcal{O}_{X',x'}$ and we are done. □

Remark 1.18. Note that the lemma is false if we drop the assumption that x is a normal point: Let X be the projectivized cusp in $\mathbb{P}^2_k$ defined by $x_0 x_2^2 - x_1^3 = 0$.

Consider the birational morphism $\mathbb{P}^1_k \to X$ induced by the usual parametrization of the cusp; in homogeneous coordinates, it can be written as

$$\mathbb{P}^1_k \to X;\ (t_0 : t_1) \mapsto (t_0^3 : t_0 t_1^2 : t_1^3).$$

By considering its inverse we get a birational transformation $T : X \dashrightarrow \mathbb{P}^1_k$ with one fundamental point $z = (1 : 0 : 0) \in X$. It is easy to check that to z there corresponds only $z' = (1 : 0) \in \mathbb{P}^1_k$, but z is not regular for T since $\mathcal{O}_{X',z'} \not\subset \mathcal{O}_{X,z}$.

In the case where X' is not only proper but *projective*, we can prove the stronger statement that a point $x \in X$ is regular for $X \dashrightarrow X'$ if and only if there correspond only finitely many points of X' to x. Let us briefly recall the definition of a projective scheme: for any base scheme S, define projective n-space over S as $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$, where $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[X_0, \ldots, X_n])$. Note that if $S = \operatorname{Spec}(A)$ is affine, then $\mathbb{P}^n_S$ is just $\operatorname{Proj}(A[X_0, \ldots, X_n])$. An S-scheme X is said to be *projective* if the structure morphism $X \to S$ factors through a closed immersion $X \to \mathbb{P}^n_S$ for some n. Note that any projective scheme is proper, see for example [Har77, Theorem II.4.9]. The only property that we need from projective schemes is the following result.

Lemma 1.19. *Let X be a projective scheme over S and let $z_1, \ldots, z_s$ be a finite number of points in X. Then there exists an affine open $U = \operatorname{Spec}(R)$ of X containing all the z_i's.*

Proof. It is sufficient to prove the statement for $X = \operatorname{Proj}(A)$, where A is any graded ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the homogeneous primes of S corresponding to $z_1, \ldots, z_s$ and denote by S_+ the ideal $\bigoplus_{d>0} S_d$, where S_d are the homogeneous elements of S of degree d. Since $S_+ \not\subset \mathfrak{p}_i$ for all i and by the Prime avoidance lemma (see A.3) there exists $f \in S_+$ such that $f \notin \mathfrak{p}_i$ for all i. We can assume f to be homogeneous; then $D_+(f) = \{\mathfrak{q} \in \operatorname{Proj}(S) : f \notin \mathfrak{q}\}$ is an affine open of X containing all the z_i (for details see [Har77, II.2.5]). ☐

Compare the next theorem with [Zar43, Theorem 10].

Theorem 1.20. *Let $T : X \dashrightarrow X'$ be a birational transformation and assume that X' is projective (over S). Let $x \in X$ be a normal point such that there correspond only finitely many $x' \in X'$ to x. Then x is regular for T (in particular, to x there corresponds a unique $x' \in X'$).*

Proof. By Lemma 1.19 we find an affine open $U' = \operatorname{Spec}(B)$ containing all points $x' \in X'$ corresponding to x. Since X' is proper, the center (on X') of each S-valuation ring R with center x is contained in U', so for each such R we get a diagram

$$\begin{array}{ccc} \operatorname{Spec}(R) & \longrightarrow & \operatorname{Spec}(B) \\ \downarrow & & \uparrow \\ \operatorname{Spec}(\mathcal{O}_{X,x}) & \longleftarrow & \operatorname{Spec}(K) \end{array} .$$

Using Lemma 1.16 we get an S-morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(B)$, which implies that x is regular for the birational transformation. □

The following corollary is a weaker form of Zariski's Main Theorem. It states roughly that the transform of a fundamental point has to have at least one positive dimensional component.

Corollary 1.21. *Let $f : X' \to X$ be a birational morphism and assume that X' is projective and X is separated. If $x \in X$ is a normal point and x fundamental for f' then $\dim f^{-1}(x) \geq 1$, i.e. there exists at least one component of $f^{-1}(x)$ of positive dimension.*

Proof. Note that, since f is of finite type, the scheme-theoretic fiber $X' \times_X \kappa(x)$ is of finite type over the field $\kappa(x)$; so $f^{-1}(x)$ is zero-dimensional if and only if it is finite as a set. So the statement follows by applying Theorem 1.20 to the birational transformation $f' : X \dashrightarrow X'$. □

If we assume that the normalization $\widetilde{X} \to X$ is finite - which is fulfilled if X is a *Japanese* scheme, see Chapter 3 - then we can prove the next result. Compare this with [Gro66, (8.11.1)], which says that any proper, quasi-finite morphism locally of finite presentation is finite.

Corollary 1.22. *Let $f : X' \to X$ be a birational morphism with X' projective and X separated. Assume that the normalization $\widetilde{X} \to X$ is a finite morphism and that, for every $x \in X$, the fiber $f^{-1}(x)$ is finite. Then the morphism f is finite.*

Proof. Consider the normalization $\widetilde{X} \to X$, which induces a birational transformation $\widetilde{X} \dashrightarrow X'$. Let $\widetilde{x} \in \widetilde{X}$ and x the image of $\widetilde{x}$. Any valuation ring with center $\widetilde{x}$ on $\widetilde{X}$ has center x on X. Hence, by assumption, there correspond only finitely many $x' \in X'$ to $\widetilde{x}$. Now apply Theorem 1.20 to get that $\widetilde{x}$ is regular for $\widetilde{X} \dashrightarrow X'$. This holds for any $\widetilde{x} \in \widetilde{X}$, hence we get a factorization $\widetilde{X} \to X' \to X$. But this implies that $X' \to X$ is finite. □

Now we come to the original formulation of Zariski's Main Theorem, which first appeared in [Zar43, p.522]. It says that, for any birational morphism $X' \to X$ and under the assumption that X is normal, *every* component of the transform of a fundamental point $x \in X$ is positive-dimensional.

Theorem 1.23 (Main theorem)**.** *Let $f : X' \to X$ be a birational morphism and assume that X is normal. Then, if $x \in X$ is fundamental for f', each irreducible component of $f^{-1}(x)$ has positive dimension.*

Assume the contrary, i.e. that $f^{-1}(x)$ has an irreducible component Z of dimension 0. Then Z consists of a single point z and we identify z with the corresponding point $x' \in X'$ mapping to x. Note that, by assumption, Z is both open and closed in $f^{-1}(x)$. Thus we have arrived quite naturally at the notion of *isolated points*, which are defined as points of a topological space whose corresponding singleton is open as a topological subspace. We will provide algebraic characterizations for this topological definition in the next chapter. What we have to prove is that x is regular for f'. Choose affine open subsets $U' = \mathrm{Spec}(B')$ of x' and $U = \mathrm{Spec}(B)$ of x such that $f(U') \subset U$. Since X and X' are integral the rings B and B' are domains, which are of finite type over a Noetherian ring A. The morphism f induces an extension $B \subset B'$ such that $\mathrm{Quot}(B) = \mathrm{Quot}(B')$, and since X is normal, we have that B is integrally closed. Let $\mathfrak{q}$ denote the prime of B' corresponding to x', and similarly for $\mathfrak{p} \subset B$ and x. Our assumptions say that $\mathfrak{q} \cap B = \mathfrak{p}$ and that the image of $\mathfrak{q}$ in the fiber ring $B' \otimes_B B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is isolated; we say that $\mathfrak{q}$ is *isolated in its fiber*. Hence it is sufficient to prove the following lemma (compare to [Zar43, Theorem 14]):

Lemma 1.24. *Let $B \subset B'$ be an extension of domains which are of finite type over a Noetherian ring A and such that* $\mathrm{Quot}(B) = \mathrm{Quot}(B')$. *Assume that B is integrally closed. Let $\mathfrak{q}$ be a prime of B' with $\mathfrak{p} = \mathfrak{q} \cap B$ and such that $\mathfrak{q}$ is isolated in its fiber. Then $B_\mathfrak{p} = B'_\mathfrak{q}$.*

We will not prove this result here, but show in the next chapter that it is a direct consequence of Peskine's theorem (see Theorem 2.8). In fact, the proof of the lemma is more or less exactly the same as the proof given by Peskine. The main observation is that, along with the assumption that $\mathfrak{q}$ is isolated in its fiber, the crucial conditions here are:

(1) The extension $B \subset B'$ is of finite type since both B, B' are of finite type over A.
(2) B is integrally closed in B', since B is integrally closed in $\mathrm{Quot}(B)$ and $\mathrm{Quot}(B) = \mathrm{Quot}(B')$.

Instead of assuming that B is integrally closed, Peskine considers the *relative* integral closure of B in B'. Furthermore, by assuming that $B \to B'$ is of finite type, it is possible to prove the result while omitting all Noetherianity conditions on B and B'.

As a final comment, if we want to extend Corollary 1.21 and Theorem 1.23 to the more general case of a birational transformation, we obtain results on the dimension of the components of $p^{-1}(x)$, where $p : \Gamma \to X$ is the projection of the graph of the birational transformation $X \dashrightarrow X'$.

2. Peskine's version: A structure theorem for quasi-finite morphisms

In this chapter, we will formulate and prove a variant of Zariski's main theorem, originally introduced by Christian Peskine in [Pes66]. His main observation was that the original proof of Zariski could be generalized to arbitrary rings by adding a few minor arguments. Peskine's statement and proof can be found, with slight differences, in various books, see for example [Pes96, Chapter 13], [Ive73, IV.2], [Ray70, IV, Proposition 1] or [GW10, Proposition 12.76].
Since it avoids any Noetherianity assumption, the proof of Peskine lends itself to a constructive version. See [ACL14] for a proof in the case where a whole fiber of a ring map is finite.
First, we will define the notion of isolated primes and its geometric counterpart, quasi-finite morphisms. Then, we will state and prove Peskine's theorem. The proof itself is completely algebraic in nature and uses only elementary commutative algebra. Finally, we will briefly discuss the *Grothendieck version* of Zariski's Main Theorem, which can be seen as a natural generalization of Peskine's result. It appeared first in its weakest form in [Gro61, (4.4.3)]. We will not give a proof here since this would go beyond the scope of this thesis.

2.1. Isolated primes and quasi-finite morphisms

Fix some notation first: let $R \to S$ be any ring map and $\mathfrak{p}$ a prime of R. Then we denote by $S_{[\mathfrak{p}]}$ the ring $S \otimes_R \kappa(\mathfrak{p})$ and call it the *fiber ring* of $\mathfrak{p}$. Note that, if $R \to S$ is of finite type, then so is $\kappa(\mathfrak{p}) \to S_{[\mathfrak{p}]}$. In particular, $S_{[\mathfrak{p}]}$ is Noetherian, even if R and S are not.
We start with the definition of an isolated point: if X is any topological space, a point $x \in X$ is called *isolated* if the singleton $\{x\}$ is open as a topological subspace of X. If R is a ring and $\mathfrak{p}$ a prime of R, then $\mathfrak{p}$ is called isolated if the corresponding point in $\operatorname{Spec}(R)$ is isolated. Now let $R \to S$ be any ring map, $\mathfrak{q}$ a prime of S and write $\mathfrak{p} = \mathfrak{q} \cap R$. Then we say that $\mathfrak{q}$ is *isolated in its fiber* (or alternatively: $\mathfrak{q}$ is isolated over $\mathfrak{p}$) if the image of $\mathfrak{q}$ in $S_{[\mathfrak{p}]}$ is an isolated point. With the next lemma we obtain an algebraic characterization of this definition:

Lemma 2.1. *Let $R \to S$ be a ring map of finite type and $\mathfrak{q} \subset S$ a prime lying over $\mathfrak{p} \subset R$. Write $\overline{\mathfrak{q}}$ for the image of $\mathfrak{q}$ in $S_{[\mathfrak{p}]}$. Then the following are equivalent:*

1. *$\mathfrak{q}$ is isolated over $\mathfrak{p}$.*
2. *$\mathfrak{q}$ is maximal and minimal among all primes of S lying over $\mathfrak{p}$, or equivalently, $\overline{\mathfrak{q}}$ is maximal and minimal in the fiber ring $S_{[\mathfrak{p}]}$.*
3. *$(S_{[\mathfrak{p}]})_{\overline{\mathfrak{q}}}$ is a finite $\kappa(\mathfrak{p})$-algebra.*
4. *The extension of residue fields $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite and* $\dim(S_{[\mathfrak{p}]})_{\overline{\mathfrak{q}}} = 0$.

Proof. Write $F = S_{[\mathfrak{p}]}$ and observe that $F_{\overline{\mathfrak{q}}} = S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$. Let us first prove that (2) implies (1), so assume $\overline{\mathfrak{q}}$ is maximal and minimal in F. Note that F is Noetherian and denote by $\overline{\mathfrak{q}} = \mathfrak{q}_1, \ldots, \mathfrak{q}_r$ the minimal primes of F. It follows that $\mathrm{Spec}(F) - \{\overline{\mathfrak{q}}\} = \overline{V(\mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r)}$, hence $\{\overline{\mathfrak{q}}\}$ is an open subset of $\mathrm{Spec}(F)$.

Assume (1). Since the principal opens form a basis for the Zariski topology there exists a nonzero $g \in F$ with $\mathrm{Spec}(F_g) = \{\overline{\mathfrak{q}}\}$. Observe that F_g is local with maximal ideal $\overline{\mathfrak{q}}F_g$, hence $F_g = F_{\overline{\mathfrak{q}}}$. Since F_g is of finite type over $\kappa(\mathfrak{p})$ we obtain (3) by Lemma A.2.

The implication (3) $\Longrightarrow$ (4) is clear since $\kappa(\mathfrak{q}) = \kappa(\overline{\mathfrak{q}})$. Let us finish by proving that (4) implies (2). That $\overline{\mathfrak{q}}$ is minimal in F follows immediately from $\dim F_{\overline{\mathfrak{q}}} = 0$. Note that we have a chain of inclusions $\kappa(\mathfrak{p}) \subset F/\overline{\mathfrak{q}} \subset \kappa(\mathfrak{q})$. But any intermediate ring of an algebraic field extension is a field itself (see Lemma B.6), so in particular $\overline{\mathfrak{q}}$ is maximal in F. □

Lemma 2.2. *Let $R \to S$ be a ring map of finite type such that every prime $\mathfrak{q}$ of S is isolated in its fiber. Then for each prime $\mathfrak{p} \subset R$ the fiber ring $S_{[\mathfrak{p}]}$ is zero-dimensional, or equivalently,* $\mathrm{Spec}(S_{[\mathfrak{p}]})$ *is finite and discrete.*

Proof. By Lemma 2.1 each prime $\overline{\mathfrak{q}}$ of $S_{[\mathfrak{p}]}$ is maximal and minimal, hence $\dim S_{[\mathfrak{p}]} = 0$. The other assertions follow from Lemma A.2. □

Let us now define the analogue of isolated primes for schemes. Recall that for a morphism $f : X \to Y$ the scheme-theoretic fiber of a point $y \in Y$ is defined as $X_y = X \times_Y \kappa(y)$. Its underlying topological space is homeomorphic to $f^{-1}(y)$. If $X = \mathrm{Spec}(S)$ and $Y = \mathrm{Spec}(R)$ are affine and $\mathfrak{p}$ is the prime of R corresponding to y, then $X_y = \mathrm{Spec}(S_{[\mathfrak{p}]})$.

Lemma 2.3. *Let $f : X \to Y$ be a morphism locally of finite type and $x \in X$ a point. Write $y = f(x)$. Then the following are equivalent:*

(1) *x is an isolated point of $f^{-1}(y)$.*
(2) *The image $\overline{x}$ of x in X_y is a closed point and no point $x' \in X_y$ specializes to $\overline{x}$.*
(3) *For all affine opens $U = \mathrm{Spec}(S) \subset X$, $V = \mathrm{Spec}(R) \subset Y$ with $f(U) \subset V$ and $x \in U$ the prime $\mathfrak{q}$ of S corresponding to x is isolated in its fiber with respect to the ring map $R \to S$.*
(4) *There exist affine opens $U = \mathrm{Spec}(S) \subset X$, $V = \mathrm{Spec}(R) \subset Y$ with $f(U) \subset V$ and $x \in U$ such that the prime $\mathfrak{q} \subset S$ corresponding to x is isolated in its fiber with respect to the ring map $R \to S$.*

Proof. By the above remark (1) is equivalent to say that $\{\overline{x}\}$ is open in X_y. We want to prove that (1) implies (2). Any open subset is closed under generization, so no point $x' \in X_y$ can specialize to $\overline{x}$. For the second assertion, note that $X_y \to \mathrm{Spec}(\kappa(y))$ is a base change of $X \to Y$ and hence X_y is locally of finite type over $\kappa(y)$. But by C.1 every open subset of X_y contains a closed point, so we are done.

Now assume (2) and let $U = \mathrm{Spec}(S) \subset X$, $V = \mathrm{Spec}(R) \subset Y$ be affine opens with $f(U) \subset V$ and $x \in U$. By assumption the ring map $R \to S$ is of finite type. Denote by $\mathfrak{q} \subset S$ the prime corresponding to x and by $\mathfrak{p}$ the contraction of $\mathfrak{q}$ to R. Note that $\mathfrak{p}$ corresponds to the point $y \in Y$ and we have $\kappa(y) = \kappa(\mathfrak{p})$. Then $U \times_V \kappa(y) = \mathrm{Spec}(S_{[\mathfrak{p}]})$ is an affine open of X_y containing the point $\overline{x}$. In particular, $\overline{x}$ is an isolated point of $U \otimes_V \kappa(y)$, hence $\mathfrak{q}$ is isolated over $\mathfrak{p}$.

Condition (3) trivially implies (4), so to finish the proof all that is left to show is that (4) implies (1). Assume (4), then we have to prove that $\{\overline{x}\}$ is open in X_y. Note that it is sufficient to prove that there exists an open subset $\overline{U}$ of X_y containing $\overline{x}$ such that $\{\overline{x}\}$ is open in $\overline{U}$. But this follows again from the fact that $U \times_V \kappa(y) = \mathrm{Spec}(S_{[\mathfrak{p}]})$ is affine open in X_y. □

If any of the equivalent conditions in Lemma 2.3 is satisfied we say that f is *quasi-finite at* x. A morphism $f : X \to Y$ is called *locally quasi-finite* if it is locally of finite type and quasi-finite at all $x \in X$. If in addition f is quasi-compact, we say that f is *quasi-finite.*

Corollary 2.4. *Let $f : X \to Y$ be a morphism locally of finite type.*

(1) *f is locally quasi-finite if and only if, for all $y \in Y$, the fiber X_y is discrete (as a topological space).*
(2) *Assume that f is in addition quasi-compact. Then f is quasi-finite if and only if, for all $y \in Y$, the fiber X_y is finite (as a set).*

Proof. The first part follows immediately from the definitions. For (2), note that, since f is quasi-compact, each fiber X_y is quasi-compact. Hence X_y is discrete if and only if it is finite. □

A prime $\mathfrak{q} \subset S$ being isolated over $\mathfrak{p}$ does not mean that the fiber of $\mathfrak{p}$ is finite, as the following example shows:

Example 2.5. Consider the algebraic set $X = V(y, z) \cup V(y - 1) \subset \mathbb{A}^3_{\mathbb{R}}$, which corresponds to the union of a hyperplane and a line, and the morphism $X \to \mathbb{A}^1_{\mathbb{R}}$ induced by $(x, y, z) \mapsto x$. Clearly the fiber of each point $a \in \mathbb{A}^1_{\mathbb{R}}$ consists of the union of a line and a point. Algebraically, we have a ring extension $R = k[X] \subset S = k[X, Y, Z]/(Y, Z) \cap (Y - 1)$ and a prime $\mathfrak{q} = (X, Y, Z) \subset S$ which is isolated over $\mathfrak{p} = \mathfrak{q} \cap k[X] = (X)$. However, the fiber ring $S_{[\mathfrak{p}]}$ is not zero-dimensional (i.e. $\mathrm{Spec}(S_{[\mathfrak{p}]})$ is not finite) since $(X, Y - 1)$ is not isolated over $\mathfrak{p}$. The same argument also works for the (localized) extension $U^{-1}R \subset U^{-1}S$ with $U := k[X] - (X)$. In this case $\mathrm{Spec}(U^{-1}S)$ can be interpreted as the set of all affine varieties contained in X which have nonempty intersection with the hyperplane $x = 0$.

Any composition of locally quasi-finite morphisms is again locally quasi-finite.

Lemma 2.6. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms locally of finite type and assume that f is quasi-finite at $x \in X$ and g is quasi-finite at $y = f(x)$. Then $g \circ f$ is quasi-finite at x.*

Proof. Write $z = g(y)$. Since g is quasi-finite at y, the subset $f^{-1}(y)$ is open as a subspace of $(g \circ f)^{-1}(z)$. By assumption, $\{x\}$ is open in $f^{-1}(y)$, therefore $\{x\}$ is open in $(g \circ f)^{-1}(z)$. □

Conversely, we have the following result:

Remark 2.7. Assume we have the following situation:

$$\begin{array}{ccccc} R & \longrightarrow & S & \longrightarrow & T \\ | & & | & & | \\ \mathfrak{p} & \text{——} & \mathfrak{q} & \text{——} & \mathfrak{n} \end{array}$$

with $\mathfrak{n}$ isolated over $\mathfrak{p}$. Then $\mathfrak{n}$ is isolated over $\mathfrak{q}$; however, in general, $\mathfrak{q}$ will not be isolated over $\mathfrak{p}$.
For the geometric analogue, assume we have

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \qquad x \longmapsto y \longmapsto z$$

with $g \circ f$ quasi-finite at x. Then f is quasi-finite at x, but, in general, g need not be quasi-finite at y.

Let us give some examples for quasi-finite morphisms:

- An open immersion is locally quasi-finite, since it is locally an isomorphism. A closed immersion is clearly quasi-finite. Hence, any immersion is locally quasi-finite.
- A finite morphism is quasi-finite. This follows directly from Corollary B.4.

These two examples are in some sense exhaustive: Grothendieck's version of ZMT says that, under some reasonable assumptions on Y and f, any quasi-finite morphism factors as a composition of an open immersion followed by a finite morphism, see Theorem 2.21.

2.2. Peskine's proof

Let us now go back to algebra. The following is the orginal formulation of Peskine's theorem, see [Pes66, p.1].

Theorem 2.8 (Zariski's Main Theorem, Peskine's version)**.** *Let $\varphi : R \to S$ be a ring map of finite type and $\mathfrak{q}$ be a prime in S isolated over $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Denote by R' the integral closure of R in S. Then there exists an element $t \in R'$, $t \notin \mathfrak{q}$ such that $R'_t = S_t$.*

Remark 2.9. Note that under the assumptions of the theorem, the statement is equivalent to finding a finite R-subalgebra $C \subset S$ and $t \in C$, $t \notin \mathfrak{q}$ with $C_t = S_t$. For assume we have such a C, then, since $C \subset R' \subset S$, it follows immediately that $R'_t = S_t$. Conversely, write $S = R[s_1, \dots, s_m]$. Since $R'_t = S_t$ there exists $d > 0$ such that $t^d s_i \in R'$. Set $C = R[t^d s_1, \dots, t^d s_m]$, then C is finite over R and $C_t = S_t$.

Let us start with the proof. As a first observation, note that we can replace the ring R by its image under φ and assume R is a subring of S. For $\varphi(R) \subset S$ is of finite type and $\mathfrak{q}$ is isolated over $\varphi(\mathfrak{p})$, which is a prime ideal of $\varphi(R) \simeq R/\ker(\varphi)$. Following an idea first introduced in [EGE70], we actually prove a slightly stronger statement:

Theorem 2.10. *Let $R \subset S$ be a ring extension and assume there exist $x_1, \dots, x_n \in S$ such that S is integral over $R[x_1, \dots, x_n]$. Denote by R' the integral closure of R in S. If $\mathfrak{q}$ is a prime of S isolated over $\mathfrak{p} = \mathfrak{q} \cap R$, then there $t \in R'$, $t \notin \mathfrak{q}$ such that $R'_t = S_t$.*

The proof of Theorem 2.10 will be done by induction on the number of elements $x_1, \dots, x_n$. Let us start with the induction step and finish by proving the case $n = 1$.

Induction step: Assume the statement is true for $k < n$ and let $R \subset S$ be a ring extension with S finite over $R[x_1, \dots, x_n]$. Let $\mathfrak{q}$ be a prime of S which is isolated over $\mathfrak{p} = \mathfrak{q} \cap S$. Consider the ring extensions

$$R[x_1, \dots, x_{n-1}] \subset R[x_1, \dots, x_{n-1}][x_n] \subset S$$

and observe that, by Remark 2.7, the prime $\mathfrak{q}$ is isolated over $\mathfrak{q} \cap R[x_1, \dots, x_{n-1}]$. Denote by R'' the integral closure of $R[x_1, \dots, x_{n-1}]$ in S. Then apply the case $n = 1$ to obtain $t_1 \in R''$, $t_1 \notin \mathfrak{q}$ with $R''_{t_1} = S_{t_1}$. We want to use induction on

$$R \subset R[x_1, \dots, x_{n-1}] \subset R'',$$

but first we have to prove that $\mathfrak{q}'' = \mathfrak{q} \cap R''$ is isolated over $\mathfrak{p}$. We have the following situation:

$$\begin{array}{ccccc} R & \longrightarrow & R'' & \longrightarrow & S \\ | & & | & & | \\ \mathfrak{p} & \text{———} & \mathfrak{q}'' & \text{———} & \mathfrak{q} \end{array}$$

Any prime of R'' contained in $\mathfrak{q}''$ appears in the localization $R''_{t_1} = S_{t_1}$. Since $\mathfrak{q}$ is isolated over $\mathfrak{p}$, we get that $\mathfrak{q}''$ is minimal among all primes of R'' lying over $\mathfrak{p}$.
By localizing R, R'' and S with regard to $U = R - \mathfrak{p}$, we may assume that $\mathfrak{p}$, $\mathfrak{q}$ are maximal. What is left to prove is that $\mathfrak{q}''$ is maximal too. Set $K = R/\mathfrak{p}$, $A = R''/\mathfrak{q}''$ and $L = S/\mathfrak{q}$. Then A is a subring of the field extension $K \subset L$. If we prove that L is algebraic over K, then A is a field by Lemma B.6 and we are done. For this last step we want to use Theorem A.1, which is a variant of Hilbert's Nullstellensatz.

Observe that it cannot be directly applied to L, since it is not of finite type over K. However, L is integral over $K[x_1, \ldots, x_n]$ and the latter is a field, for example by the Lying-Over property. Then Theorem A.1 says that $K[x_1, \ldots, x_n]$ is finite over K and hence L is algebraic over K.
Going back, note that R' is the integral closure of R inside R''. Hence we can apply our induction hypothesis to get $t_2 \in R'$, $t_2 \notin \mathfrak{q}$ with $R'_{t_2} = R''_{t_2}$. Now all that remains to prove is the following lemma.

Lemma 2.11. *Given ring extensions $R \subset S \subset T$, a prime $\mathfrak{q}$ of T and elements $t_1 \in R$, $t_2 \in S$; $t_1, t_2 \notin \mathfrak{q}$ with $R_{t_1} = S_{t_1}$ and $S_{t_2} = T_{t_2}$. Then there exists a $t \in R$, $t \notin \mathfrak{q}$ with $R_t = T_t$.*

Proof. Since $R_{t_1} = S_{t_1}$ we can write $t_2 = \frac{\xi}{t_1^d}$ for some $\xi \in R$. Since $\xi = t_2 t_1^d$ it follows that $\xi \notin \mathfrak{q}$. Setting $t = \xi t_1 \notin \mathfrak{q}$ we get that $R_t = T_t$. □

The case n=1: This is the difficult part. Suppose we have

$$\begin{array}{ccccc} R & \longrightarrow & R[x] & \longrightarrow & S \\ | & & | & & | \\ \mathfrak{p} & \text{———} & \mathfrak{p}' & \text{———} & \mathfrak{q} \end{array}$$

with S finite over $R[x]$ and $\mathfrak{q}$ isolated over $\mathfrak{p}$. The idea now is to define the *conductor ideal* $J = (R[x] : S) = \{t \in R[x], t \cdot S \subset R[x]\}$. Observe that J is both an ideal of $R[x]$ and S; in fact, J is maximal with this property. We now consider two cases, namely whether J is contained in $\mathfrak{q}$ or not.
Let us first assume that $J \not\subset \mathfrak{q}$. Then there exists $t \in R[x]$, $t \notin \mathfrak{q}$ such that $tS \subset R[x]$. In particular, $R[x]_t = S_t$. Using Lemma 2.11 and a similar argument to the one we used in the induction step, we reduce to the case $S = R[x]$.

Lemma 2.12. *Let $R \subset S = R[x]$ and $\mathfrak{q}$ a prime in S isolated over $\mathfrak{p} = \mathfrak{q} \cap R$. Then there exists an element $s \in S - \mathfrak{q}$ such that s, sx are integral over R.*

Proof. Consider the fiber ring $S_{[\mathfrak{p}]}$, which is finitely generated as an $\kappa(\mathfrak{p})$-algebra by the image of x. Denote by $\overline{\mathfrak{q}}$ the image of $\mathfrak{q}$ in $S_{[\mathfrak{p}]}$. Since $\overline{\mathfrak{q}}$ is isolated, the ring $S_{[\mathfrak{p}]}$ cannot be a polynomial ring. Hence we have a monic equation for $x \in S_{[\mathfrak{p}]}$ with coefficients in $\kappa(\mathfrak{p})$. By multiplying with denominators we get an expression $a_r x^r + \ldots + a_0 \in \mathfrak{p}R[x]$ with $a_r \notin \mathfrak{p}$. Since $\mathfrak{p}R[x] = \mathfrak{p}[x]$, we obtain an equation $b_d x^d + \ldots + b_0 = 0$ with coefficients in R and $b_i \notin \mathfrak{p}$ for some i. Note that $b_d x$ is integral over R. If $b_d \notin \mathfrak{q}$ then the statement follows. Otherwise, set $b'_{d-1} = b_d x + b_{d-1}$, which is integral over R. Furthermore, we have

$$b'_{d-1} x^{d-1} + b_{d-2} x^{d-2} + \ldots + b_1 x + b_0 = 0$$

and hence $b'_{d-1} x$ is integral over R. If $b_{d-1} \notin \mathfrak{p}$ then $b'_{d-1} \notin \mathfrak{q}$ and we are done. Otherwise, set $b'_{d-2} = b'_{d-1} x + b_{d-2}$ and proceed inductively. □

Finally, assume that $\mathfrak{q}$ contains J. The goal is to obtain a contradiction to the assumption that $\mathfrak{q}$ isolated over $\mathfrak{p}$. Choose a prime $\mathfrak{n}$ with $J \subset \mathfrak{n} \subset \mathfrak{q}$ and such that $\mathfrak{n}$ is minimal with this property. Write $\mathfrak{m} = \mathfrak{n} \cap R$. We proceed now in two steps: first, we show that the element x of $S/\mathfrak{n}$ must be transcendental over $R/\mathfrak{m}$. Second, we prove that x being transcendental implies that no prime of $S/\mathfrak{n}$ can be isolated over its image in $R/\mathfrak{n} \cap R$. Since $\mathfrak{n} \subset \mathfrak{q}$, this finishes the proof.

For technical reasons, we may replace R by R', the integral closure of R in S. This follows from Remark 2.7, as any prime of $S/\mathfrak{n}$ isolated over its image in $R/\mathfrak{m}$ is isolated over its image in $R'/\mathfrak{n} \cap R'$. Therefore we can assume that R is integrally closed in S.

For the first part, write $U = R - \mathfrak{m}$. Observe that it is sufficient to prove that x (as an element of $U^{-1}S/\mathfrak{n}U^{-1}S$) is transcendental over $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. Furthermore, we have

$$U^{-1}J = U^{-1}(R[x] : S) = (U^{-1}R[x] : U^{-1}S)$$

and $U^{-1}\mathfrak{n}$ is minimal over $U^{-1}J$. By Lemma B.1, $U^{-1}R$ is integrally closed in $U^{-1}S$. Hence we can assume that R is local with maximal ideal $\mathfrak{m}$.

Suppose that $x \in S/\mathfrak{n}$ is algebraic over $R/\mathfrak{m}$. Let $f(X) \in R[X]$ be a monic polynomial with $f(x) \in \mathfrak{n}$. By Lemma A.4, there exists $u \in S$, $u \notin \mathfrak{n}$ and $d > 0$ with $uf(x)^d \in J$. Applying the following lemma finishes the first step (this is also the reason why we needed the additional condition that R is integrally closed).

Lemma 2.13. *Let $R \subset S$ be a ring extension such that S is integral over $R[x]$ for some x. Set $J = (R[x] : S)$. Assume that there exists a monic polynomial $f(X) \in R[X]$ and an element $u \in S$ such that $uf(x) \in J$. Then $u \in J$.*

Proof. We prove the statement by induction on $\deg(f)$. If $\deg(f) = 0$, then $f = 1$ and we are done. So suppose $\deg(f) > 0$. Let s be any element of S. By assumption, there exists a polynomial $g(X) \in R[X]$ such that $suf(x) = g(x)$. Since $f(X)$ is monic, we can divide $g(X)$ by $f(X)$ and obtain $g(X) = h(X)f(X) - g'(X)$ with $\deg(g') < \deg(f)$. Write $t = su - h(x)$, then t is an element of S with $tf(x) = g'(x)$. This gives us an integral equation for x as an element of S_t over $R[t^{-1}]$. Since t is integral over $R[x]$ we obtain an integral equation for t over $R[t^{-1}]$. Multiplying by a suitable power of t yields an integral equation for t over R. But then, since R is integrally closed in S, we have $t \in R$ and hence $su \in R[x]$. Therefore $u \in J$. □

As for the second and last step, it is sufficient to prove the next lemma.

Lemma 2.14. *Let $A \subset B$ be a extension of domains such that B is integral over $A[x]$, with x transcendental over A. Then no prime of B is isolated in its fiber.*

Proof. By assumption the ring $A[x]$ is a polynomial ring over A. Hence, if $\mathfrak{p}$ is a prime of A, we have $\mathfrak{p}A[x]$ and $(\mathfrak{p}, x)A[x]$ which are both primes of $A[x]$ lying over $\mathfrak{p}$. Now assume that A is normal, which implies that $A[x]$ is normal as well. Since

$A[x] \subset B$ is an integral extension it satisfies both the Going-Up and the Going-Down property. Therefore, a chain of primes in $A[x]$ can be extended to a chain of primes in B in any direction. So if $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $\mathfrak{q}$ lies over $\mathfrak{p}A[x]$ or $(\mathfrak{p}, x)A[x]$. Either way, we see that $\mathfrak{q}$ is not isolated over $\mathfrak{p}$.
In the general case where A is not normal, replace A and B by their respective integral closure A' and B'. Then x is still transcendental over A' and $A'[x] \subset B'$ finite. Now take a prime $\mathfrak{q}$ of B and write $\mathfrak{p} = \mathfrak{q} \cap A$. Since $B \subset B'$ is integral we can find a prime $\mathfrak{q}'$ of B' lying over $\mathfrak{q}$. By the above, $\mathfrak{q}'$ is not isolated over $\mathfrak{p}' = \mathfrak{q}' \cap A'$ and hence not isolated over $\mathfrak{p}$ as well. Without loss of generality, assume $\mathfrak{n}' \subset \mathfrak{q}'$ with $\mathfrak{n}'$ lying over $\mathfrak{p}$. Setting $\mathfrak{n} = \mathfrak{n}' \cap B$ we get $\mathfrak{n} \subset \mathfrak{q}$ and, again since $B \subset B'$ is integral, the inclusion is strict. So $\mathfrak{q}$ is not isolated over $\mathfrak{p}$. □

This finishes the proof of Theorem 2.10.

Remark 2.15. We want to give a geometric interpretation of the conductor ideal J appearing in the proof. Namely, let $X = \operatorname{Spec}(R)$ be a integral affine scheme and suppose the integral closure R' of R is finite over R. Let Z be the closed subset defined by the conductor $(R : R')$. Then Z consists of those points which are not normal, or equivalently, all points of X fundamental for $X \dashrightarrow \widetilde{X} = \operatorname{Spec}(R')$. Compare with [Zar43, Corollary 4, p.512].

2.3. Consequences and Grothendieck's version of Zariski's Main Theorem

First, let us see that Peskine's theorem immediately implies Lemma 1.24. Let $R \subset S$ be a ring extension of finite type with R normal and $\operatorname{Quot}(R) = \operatorname{Quot}(S)$ and assume that $\mathfrak{q}$ is a prime of S isolated over $\mathfrak{p} = \mathfrak{q} \cap S$. Since R is integrally closed in S, there exists a $t \in R$, $t \notin \mathfrak{q}$ with $R_t = S_t$. But then, in particular $R_{\mathfrak{p}} = S_{\mathfrak{q}}$ and we are done.
The following two results are consequences of Theorem 2.8 and more geometric in nature (see [Ray70, p.42]):

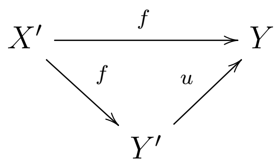
Corollary 2.16. *Let $R \to S$ be a ring map of finite type. Then the set of all primes of S isolated in their fiber form an open subset of* $\operatorname{Spec}(S)$.

Proof. Let $\mathfrak{q}$ be a prime of S isolated over $\mathfrak{p} = \mathfrak{q} \cap R$. By Theorem 2.8 and Remark 2.9 there exists a finite R-algebra C and an element $t \in C$, $t \notin \mathfrak{q}$ such that $C_t = S_t$. Since every prime of C is isolated in its fiber over R, we obtain the same statement for C_t and hence for S_t as well. Hence the principal open $D(t) = \operatorname{Spec}(S_t)$ is a neighborhood of $\mathfrak{q}$ and contains only primes which are isolated in their fiber. This proves the claim. □

Corollary 2.17. *Let $R \to S$ be a ring map of finite type such that every prime of S is isolated in its fiber. Then there exists a factorization $R \to C \to S$ with C finite over R and such that the induced morphism* $\operatorname{Spec}(S) \to \operatorname{Spec}(C)$ *is an open immersion.*

Proof. By assumption, for every prime $\mathfrak{q}$ of S we can find find a pair (C, t) with C a finite R-subalgebra of S, t an element of C with $t \notin \mathfrak{q}$ and such that $C_t = S_t$. In particular, we obtain an open covering of $\operatorname{Spec}(S)$ by the sets $\operatorname{Spec}(S_t)$ and since every affine scheme is quasi-compact there exists a finite subcovering given by $(C_1, t_1), \ldots, (C_r, t_r)$. Write C for the R-subalgebra generated by the C_i. Note that C is finitely generated as an R-subalgebra and hence finite over R. Furthermore, we have that $C_{t_i} = S_{t_i}$ for all i. Therefore $\operatorname{Spec}(S) \to \operatorname{Spec}(C)$ is an open immersion. □

Remark 2.18. Let $f : X \to Y$ be a quasi-finite morphism between *affine* schemes. Then Corollary 2.17 says that there exists an affine scheme Y' and a diagram



with f' an open immersion and u a finite morphism. In other words, every quasi-finite morphism between affine schemes is obtained by restricting a finite morphism to an open subset.

As the next step we want to generalize Corollary 2.16 to arbitrary, not necessarily affine, schemes:

Corollary 2.19. *Let $f : X \to Y$ be a morphism locally of finite type between schemes X, Y. Then the set of all $x \in X$ such that f is quasi-finite at x is open in X.*

Proof. Let $x \in X$ be an isolated point of $f^{-1}(f(x))$. We have to show that there exists an open neighborhood U of x such that f is quasi-finite at every $x' \in U$. By Lemma 2.3 there exist $U = \operatorname{Spec}(S) \subset X$, $V = \operatorname{Spec}(R) \subset Y$ open affines with $f(U) \subset V$ and $x \in U$ such that the prime $\mathfrak{q} \subset S$ corresponding to x is isolated in its fibre with respect to $R \to S$. Now use Corollary 2.16 to see that the set of all points $x' \in U$ such that f is quasi-finite at x' is open in U and hence in X as well. □

It turns out that it is much harder to extend the statement of Corollary 2.17 in the same way. In fact, additional techniques are needed in order to deduce such a statement from Theorem 2.8. The desired result was proven by Grothendieck and is stated in various levels of generality in EGAIII and EGAIV. The first version can be found in [Gro61, (4.4.3)] and says the following:

Theorem 2.20. *Let Y be a Noetherian scheme and $f : X \to Y$ a quasi-projective morphism. Denote by X' the set of all $x \in X$ such that f is quasi-finite at x. Then X' is open and isomorphic to an open subset of a scheme Y' which is finite over Y.*

Grothendieck's proof uses completely different methods from those used in the original proof of Zariski. The main ingredient is the *Theorem on Formal Functions*, see [Har77, III.11] or [Gro61, §4], which is a result on the cohomologies of sheaves. In [Gro66, (8.12.6)] Grothendieck gave the following, more general version:

Theorem 2.21. *Let Y be a quasi-separated, quasi-compact scheme and $f : X \to Y$ a quasi-finite, separated morphism locally of finite presentation. Then there exists a factorization*

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & Y \\ & {\searrow}^{f} \quad {\nearrow}_{u} & \\ & Y' & \end{array}$$

with f' an open immersion and u a finite morphism.

The proof is different from both Theorem 2.8 and Theorem 2.20, avoiding the machinery of cohomology entirely. Instead, it uses similar arguments to the proof of Theorem 3.16.

Finally, the statement of Theorem 2.21 was generalized in [Gro67, (18.12.13)] to the case where f need not be locally of finite presentation. This last step uses étale localization, see [Gro67, (18.12.1)].

3. Zariski's power series version: Formal normality

In this chapter we shall present the power series version of Zariski's Main Theorem, which roughly states that, for a normal point x of an algebraic variety X over a perfect field k, the completion of the local ring $\mathcal{O}_{X,x}$ is a normal local ring (so in particular a domain). This theorem was proven by Oscar Zariski in the course of various papers: First, he proved that the completion of $\mathcal{O}_{X,x}$ is a domain in [Zar48]. In [Zar49] he used this result to prove the original Main theorem. Finally, the theorem in the above form was first proven in [Zar50]. A summary of all results and arguments can be found in [ZS60, Chapter VIII, §13].
Zariski also asked the question whether it is true that, for every normal local Noetherian ring R, the completion $\widehat{R}$ is normal. This question was answered in the negative by Nagata: he published two counterexamples, the first in [Nag55] and the second in [Nag58]. It turns out that the problem is closely related to the question if, for a domain A with quotient field K and a finite field extension L/K, the integral closure of A in L is finite over A. By turning this property into an assumption, Nagata then generalized the Power series version to a bigger class of rings in [Nag62, Chapter V.37]. The proof there is more or less exactly the same as in [ZS60] and we will reproduce it here.
In this chapter we will tacitly assume from the beginning that all rings which appear are Noetherian. This was the original situation in all the cited sources and ensures that taking completions is well-behaved.
Note that for any regular local ring R, its completion $\widehat{R}$ is easily seen to be regular again: choose a minimal generating set for the maximal ideal $\mathfrak{m}$ of R, then the chosen elements will generate $\mathfrak{m}\widehat{R}$ which is the maximal ideal of $\widehat{R}$. Since $\dim(R) = \dim(\widehat{R})$ our assertion follows. In particular, the completion of a regular local ring is a normal domain. The main idea behind the proof of Theorem 3.16 is reducing to the case of a regular ring via a strong normalization-type result.

3.1. Formally unramified local rings and the Nagata property

In some results and proofs we will consider finite extensions of local rings, which in general will not be local anymore. However, by the Going-Up theorem, any such extension will only have finitely many maximal ideals. Rings with finitely many maximal ideals are called *semi-local*. Recall that the Jacobson radical is defined to be the intersection of all maximal ideals of a ring. We will write $(R, \mathfrak{m})$ for a semi-local ring R with Jacobson radical $\mathfrak{m}$. Note that this notation is consistent with that of local rings.
For a semi-local ring $(R, \mathfrak{m})$ consider the completion $\widehat{R}$ with respect to the ideal $\mathfrak{m}$. The next result tells us that $\widehat{R}$ is just a direct sum of completions of local rings.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a semi-local ring and let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Denote by $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ the maximal ideals of R. Then $\widehat{R} = \prod_{i=1}^{r} \widehat{R}_i$, where $\widehat{R}_i$ is the completion of the local ring $(R_{\mathfrak{m}_i}, \mathfrak{m}_i)$.*

Proof. See [Nag62, Theorem 17.7]. □

Finite extensions behave very well under completion, as the following lemma shows.

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a semi-local ring and $R \subset S$ a finite ring extension. Then S is again semi-local. If $\widehat{S}$ is the completion of S with respect to its Jacobson radical then we have $\widehat{S} = S \otimes_R \widehat{R}$ and $\widehat{R}$ is a topological subspace of $\widehat{S}$.*

Proof. Each maximal ideal of S necessarily lies over a maximal ideal of R and since S is finite over R, there can only be finitely many primes of S lying over a particular prime of R. Hence S is semi-local. For the second assertion see [Nag62, Theorem 17.8]. □

Corollary 3.3. *Let $(R, \mathfrak{m})$ be a semi-local domain and $R \subset S$ a finite extension of domains such that* $\mathrm{Quot}(R) = \mathrm{Quot}(S)$. *Assume $\widehat{R}$ is a subring of $\widehat{S}$. Then* $\mathrm{Frac}(\widehat{R}) = \mathrm{Frac}(\widehat{S})$. *In particular, each minimal prime of $\widehat{S}$ is of the form $\mathfrak{q}\widehat{S}$, where $\mathfrak{q}$ is a minimal prime of $\widehat{R}$.*

Proof. Let $s_1, \ldots, s_t \in \mathrm{Quot}(R)$ be generators for S as an R-module and denote by $r \in R$ the product of the denominators of the s_i. By Lemma 3.2 we have $\widehat{S} = \sum_i \widehat{R} s_i$. Let us prove that each $x \in \widehat{R}$ which is not a zero divisor in $\widehat{R}$ is also not one in $\widehat{S}$. Assume that there exists $y \in \widehat{S}$ such that $xy = 0$. Then, since $ry \in \widehat{R}$, we have that $ry = 0$. But, since $\widehat{S}$ is faithfully flat over S , the element r is not a zero divisor in $\widehat{S}$, so $y = 0$. Therefore, there exists an injective ring map $\mathrm{Frac}(\widehat{R}) \to \mathrm{Frac}(\widehat{S})$ which is also surjective since each $s_i \in \mathrm{Frac}(\widehat{R})$.
If $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ denote the minimal primes of $\widehat{R}$, then $\mathrm{Frac}(\widehat{R}) = R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_r}$ and each minimal prime of $\widehat{S}$ is the preimage of $R_{\mathfrak{q}_1} \times \ldots \times \mathfrak{q}_i \times \ldots \times R_{\mathfrak{q}_r}$ for some i. □

Lemma 3.4. *Let $(R, \mathfrak{m}), (S, \mathfrak{n})$ be semi-local rings and let $R \to S$ be a local ring map. If $S/\mathfrak{m}S$ is finite over $R/\mathfrak{m}$ then $\widehat{S}$ is finite over $\widehat{R}$.*

Proof. Let $s_1, \ldots, s_r$ be elements of S such that their residue classes generate $S/\mathfrak{m}S$. We prove the following: let $n \geq 0$, then for every $x \in \mathfrak{m}^n S$ we can find elements $a_1, \ldots, a_r \in \mathfrak{m}^n$ such that $x - \sum_i a_i s_i \in \mathfrak{m}^{n+1} S$. The case $n = 0$ follows by assumption. Now assume the statement has been proven for $n - 1$ and let $x \in \mathfrak{m}^n S$. Then we can write $x = \sum_j a'_j x'_j$ with $a_j \in \mathfrak{m}$ and $x'_j \in \mathfrak{m}^{n-1} S$. By induction, we find a''_{ij} such that $x'_j - \sum_i a''_{ij} s_i \in \mathfrak{m}^n S$. Hence $x - \sum_i \sum_j a'_j a''_{ij} s_i \in \mathfrak{m}^{n+1} S$ and we are done. We have just shown that the associated graded ring $\mathrm{gr}(S)$ is finitely generated over $\mathrm{gr}(R)$. Note that $\mathrm{gr}(R) \simeq \mathrm{gr}(\widehat{R})$ and $\mathrm{gr}(S) \simeq \mathrm{gr}(\widehat{S})$ (see [AM94, Proposition 10.22]).

Since $\widehat{S}$ is Noetherian and the induced ring map $\widehat{R} \to \widehat{S}$ is local, Krull's intersection theorem yields

$$\bigcap \mathfrak{m}^n \widehat{S} \subset \bigcap (\mathfrak{m}\widehat{S})^n \subset \bigcap \mathfrak{n}^n \widehat{S} = (0).$$

Hence we can apply [AM94, Proposition 10.24] to get the statement. □

We say that a domain R is *Japanese* if, for every finite extension L of $\mathrm{Quot}(R)$, the integral closure of R in L is a finite R-module. Note that, since taking integral closure and localization commutes (see Lemma B.1), any localization of a Japanese domain is again Japanese.

A ring R is called *Nagata* if it is Noetherian and $R/\mathfrak{p}$ is Japanese for every prime $\mathfrak{p}$ of R. If R is a Nagata ring, then the following rings are again Nagata:

- Any $\phi(R)$, where $\phi : R \to S$ is a ring homomorphism. Note that $\phi(R) \cong R/I$ for some ideal $I \subset R$.
- Any localization $U^{-1}R$, where $U \subset R$ is a multiplicative subset. This follows from the fact that every localization of a Japanese domain is Japanese..
- Any R-algebra S which is finite over R. If $\mathfrak{q}$ is any prime of S and $\mathfrak{p} = \mathfrak{q} \cap R$, then $R' = R/\mathfrak{p} \to S/\mathfrak{q} = S'$ is finite with $\mathrm{Quot}(R') \subset \mathrm{Quot}(S')$ a finite field extension. Hence the integral closure of S' in some L is the same as the integral closure of R' in L.

Now apply these notions to schemes: We say that an integral scheme X is Japanese if for every point $x \in X$ there exists an affine open neighborhood $U = \mathrm{Spec}(R)$ of x with R a Japanese domain. A scheme X is called Nagata if for every point x we can find an affine open neighborhood $U = \mathrm{Spec}(R)$ of x with R a Nagata ring. By definition, a Nagata scheme is locally Noetherian. For the normalization $\widetilde{X} \to X$ we get the following result:

Lemma 3.5. *Let X be a scheme.*

1. *If X is integral and Japanese then the normalization $\widetilde{X} \to X$ is a finite morphism.*
2. *If X is Nagata (not necessarily integral) then $\widetilde{X} \to X$ is finite.*

Let $(R, \mathfrak{m})$ be any semi-local ring, then we say that R is *formally unramified* if the completion $\widehat{R}$ is a reduced ring. If $\mathfrak{p}$ is any prime of R, we say that $\mathfrak{p}$ is formally unramified if the quotient $R/\mathfrak{p}$ is formally unramified. Note that, if R is formally unramified, then in particular R itself is reduced. Moreover, applying Lemma 3.1 to a semi-normal ring gives us the following easy corollary.

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a semi-local ring and denote by $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ its maximal ideals. Then R is formally unramified if and only if each local ring $R_{\mathfrak{m}_i}$ is formally unramified.*

For the next theorem we need the following short lemma.

Lemma 3.7. *Let R be a domain and let $\mathfrak{p}$ be a prime of R with $R_\mathfrak{p}$ a discrete valuation ring. Let $R \subset S$ be a flat ring extension and assume that $\mathfrak{p}S$ is radical. Then, for every minimal prime $\mathfrak{q}$ of $\mathfrak{p}S$, the local ring $S_\mathfrak{q}$ is a discrete valuation ring.*

Proof. Take $t \in \mathfrak{p}$ a generator for $\mathfrak{p}R_\mathfrak{p}$. We have $tS_\mathfrak{q} = \mathfrak{p}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$, where the last equality holds because $\mathfrak{q}$ appears in the primary decomposition of $\mathfrak{p}S$. Since R is a domain and S is flat over R, we have that t is not a zero divisor in S and so not in $S_\mathfrak{q}$ as well. Hence $\dim(S_\mathfrak{q}) > 0$ and by Proposition E.1 we are done. ☐

The following result was, in the case of R being a local ring of an algebraic variety, originally conjectured by Zariski and first proven by Chevalley in [Che45]. The more general version presented here was taken from [Nag62].

Theorem 3.8 (Chevalley)**.** *Let $(R, \mathfrak{m})$ be a semi-local Nagata domain. Then R is formally unramified.*

Proof. Denote by R' the integral closure of R, then R' is finite over R by assumption. Using Lemma 3.2 we see that R' is a semi-local Nagata domain whose completion contains $\widehat{R}$, so it is sufficient to prove that R' is formally unramified. Hence we can assume R is normal.
We will prove the statement by induction on $n = \dim R$. If $n = 0$, then R is a field and the assertion is trivial. Assume that $n \geq 1$. Take any element $x \neq 0$ of $\mathfrak{m}$ and denote by $\mathfrak{p}_i$ the associated primes of xR. Since R is normal we have $\mathrm{ht}(\mathfrak{p}_i) = 1$ and $R_{\mathfrak{p}_i}$ is a DVR. Using induction we see that $\mathfrak{p}_i\widehat{R}$ is radical. Then, if we denote by $\mathfrak{p}_{ij}$ the associated primes of $\mathfrak{p}_i\widehat{R}$, Lemma 3.7 says that $\widehat{R}_{\mathfrak{p}_{ij}}$ is a DVR as well. Note that flatness of the completion implies that $x\widehat{R} = \bigcap \mathfrak{p}_{ij}$. Denote by $\mathfrak{q}_{ij}$ the kernel of the canonical map $\widehat{R} \to \widehat{R}_{\mathfrak{p}_{ij}}$; then $\mathfrak{q}_{ij}$ is a minimal prime of $\widehat{R}$ which is contained in $\mathfrak{p}_{ij}$ and with $x \notin \mathfrak{q}_{ij}$. Write $\mathfrak{q}$ for the intersection of all $\mathfrak{q}_{ij}$; it is clear that $\mathfrak{q}$ contains the nilradical of $\widehat{R}$. Since $\mathfrak{q}$ is contained in any $\mathfrak{p}_{ij}$-primary ideal it is is contained in $x\widehat{R}$. Furthermore, we have

$$(\mathfrak{q} : x) = (\bigcap \mathfrak{q}_{ij} : x) = \bigcap (\mathfrak{q}_{ij} : x) = \bigcap \mathfrak{q}_{ij} = \mathfrak{q},$$

so we get $x\mathfrak{q} = \mathfrak{q}$. Applying Nakayama's lemma yields $\mathfrak{q} = 0$ and hence $\widehat{R}$ has no nilpotent elements. ☐

Note that there is a kind of converse to Chevalley's theorem: if R is semi-local and formally unramified, then the integral closure of R in $\mathrm{Frac}(R)$ is finite over R. This follows from the fact that a complete local ring is Nagata, which itself can be proven by using Cohen's structure theorem to reduce to the case of a regular complete local ring.
Chevalley's theorem is used in the proof of the following result, which we will provide for completeness' sake. We omit the proof here since it is rather involved and not needed in order to prove the main theorem of this chapter. See [Nag62, Theorem (36.5)] for more details.

Theorem 3.9. *Let R be a Nagata ring, then every ring S of finite type over R is Nagata again.*

3.2. Formal irreducibility and normality

We call a local ring $(R, \mathfrak{m})$ *formally irreducible* if its completion $\widehat{R}$ is a domain and *formally normal* if $\widehat{R}$ is in addition normal. As before, we say that a prime $\mathfrak{p}$ of R is formally irreducible resp. formally normal if this holds for the local ring $R/\mathfrak{p}$. The first result of this section is one of the key ingredients in the proof of Theorem 3.16. The element of d in the statement is sometimes called an *universal denominator*. Compare with Zariski's "Condition D"

Lemma 3.10. *Let R be an integrally closed domain and $S = R[X]/(f(X))$, with $f(X) \in R[X]$ monic. Let $d \in R$ be the discriminant of $f(X)$. If S' is the integral closure of S in $Q = \mathrm{Frac}(S)$ then $dS' \subset S$.*

Proof. If $d = 0$ then we are done, so assume $d \neq 0$. Denote by x the residue class of X in S. Write $K = \mathrm{Quot}(R)$, then clearly K is contained in Q and we have $Q = K[x]$. Let L be a splitting field of $f(X)$ over K and denote by $a_1, \ldots, a_m$ the roots of $f(X)$ in L. For each i we get an inclusion $K[X]/f(X) \hookrightarrow L$ via $x \mapsto a_i$, which extends to an inclusion $\phi_i : Q \hookrightarrow L$. Let $b \in S'$, then there exist $y_0, \ldots, y_{n-1} \in K$ such that $b = \sum_{j=0}^{n-1} y_j x^j$, hence $\phi_i(b) = \sum_j y_j a_i^j$. Written differently, we get

$$\begin{pmatrix} 1 & a_1 & \ldots & a_1^{n-1} \\ 1 & a_2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & a_m & \ldots & a_m^{n-1} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \phi_1(b) \\ \phi_2(b) \\ \vdots \\ \phi_m(b) \end{pmatrix}$$

with the matrix $A = (a_i^j)$ a Vandermonde matrix. Hence the determinant of A is given by $D = \prod_{i<k}(a_k - a_i)$ and we have $D^2 = d$. Note that all $a_i, \phi_i(b)$ are integral over R. By multiplying the equation with the adjoint matrix we see that dy_j is integral over R. But dy_j is in K, so, since R is integrally closed, we have $dy_j \in R$ and hence $db \in S$. □

Corollary 3.11. *Let R be an integrally closed domain and L be a finite separable extension of $K = \mathrm{Quot}(R)$. Then the integral closure R' of R in L is finite over R.*

Proof. By the primitive element theorem write $L = K(a')$ for some $a' \in L$. The element a' satisfies a monic equation over K, hence there exists a $b \in R$ such that $a = ba'$ satisfies a monic equation over R. Then $L = K(a)$ and R' is the integral closure of $R[a]$. Let $d \in R$ be the discriminant of the minimal polynomial of a over R. By assumption $d \neq 0$. Using Lemma 3.10 we have $dR' \subset R[a]$, so R' is a submodule of the finite module $\sum_j^{n-1} \frac{a^j}{d} R$, where n is the degree of L over K. Since R is Noetherian, this implies that R' is finite over R. □

Let us now consider an application of these results to the case of polynomial rings over Japanese domains.

Theorem 3.12. *Let R be a Japanese domain and let $x_1, \ldots, x_m$ be elements which are algebraically independent over R. Then $R[x] = R[x_1, \ldots, x_m]$ is again Japanese.*

Proof. Write $k = \mathrm{Quot}(R)$, $K = \mathrm{Quot}(R[x])$ and let L be a finite field extension of K. We prove the result in three steps:
Step 1: By assumption, the integral closure R' of R (in k) is finite over R. Hence $R'[x]$ is finite over $R[x]$ and the integral closure of $R'[x]$ in L coincides with that of $R[x]$. Therefore we may assume that R is integrally closed. In particular, by Corollary 3.11, we obtain the result in the case where L is separable over K.
Step 2: Now assume that L is inseparable over K. We prove that we can take L to be *purely inseparable* over K. First, note that we may always replace L by a finite extension L': if the integral closure of $R[x]$ in L' is finite, then the integral closure of $R[x]$ in L is a submodule of the former and hence finite as well, since $R[x]$ is Noetherian. Thus we replace L by the normal closure L' of the extension L/K. Consider the subfield F of L which consists of all elements fixed by the K-automorphisms of L. Then F/K is purely inseparable and L/F is Galois. By the transitivity of integral extension, it is enough to prove that the integral closure of R in F is finite.
Step 3: Finally, assume that L is finite purely inseparable over K. Then there exists a power q of the characteristic p of K such that L is obtained by adjoining finitely many q-th roots of elements $f_i \in R[x]$ to K. Let $a_1, \ldots, a_r \in R$ be all the coefficients of the elements f_i and set $k' = k(a_1^{1/q}, \ldots, a_r^{1/q})$, which is finite over k. Replace L with the finite field extension $L' = k'(x_1^{1/q}, \ldots, x_n^{1/q})$. Write R' for the integral closure of $R[a_1^{1/q}, \ldots, a_r^{1/q}]$ (inside its quotient field k'). By assumption R' is finite over R, hence $R'[x_1^{1/q}, \ldots, x_n^{1/q}]$ is finite over $R[x]$. Observe that the elements $x_i^{1/q}$ are algebraically independent over R'. Hence $R'[x_1^{1/q}, \ldots, x_n^{1/q}]$ is normal; in particular, it is the integral closure of $R[x]$ in L' and we are done. $\square$

Remark 3.13. Note that Theorem 3.12 in particular implies that any domain A finitely generated over a field k is Japanese. Namely, by Noether normalization, A is finite over a polynomial ring $k[x_1, \ldots, x_n]$, which is Japanese by the above.

Before we come to the proof of the power series version, we need the following two results.

Lemma 3.14. *Let $(R, \mathfrak{m})$ be a normal semi-local domain and denote by $\widehat{R}$ its completion. Let $t \in R$ be nonzero and not a unit and assume that, for every associated prime $\mathfrak{p}$ of tR, the extension $\mathfrak{p}\widehat{R}$ is radical. If $z \in \mathrm{Frac}(\widehat{R})$ is integral over $\widehat{R}$ and such that $tz \in \widehat{R}$ then $z \in \widehat{R}$.*

Proof. Denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the associated primes of tR. Since R is normal, each $\mathfrak{p}_i$ is of height 1 and $R_{\mathfrak{p}_i}$ is a DVR. Set $S = \bigcap_i (R - \mathfrak{p}_i)$, then $R_S = S^{-1}R$ is a semi-local

Dedekind domain with maximal ideals given by the images of the $\mathfrak{p}_i$. Note that a semi-local Dedekind domain is already a PID and hence all $\mathfrak{p}_i R_S$ are principal; choose for example for each i an element $t_i \in \mathfrak{p}_i$ with $t_i \notin \bigcap_{j \neq i} \mathfrak{p}_j$, then t_i generates $\mathfrak{p}_i R_S$. In particular, each $\mathfrak{p}_i$-primary ideal is generated by a power of t_i, so write $tR_S = t_1^{\alpha_1} \dots t_r^{\alpha_r} R_S$. Note that $t_1^{\alpha_1} \dots t_r^{\alpha_r} \in tR$, since $tR = \bigcap_i \mathfrak{p}_i^{(\alpha_i)}$, where $\mathfrak{p}_i^{(\alpha_i)}$ denotes the α_i-th symbolic power of $\mathfrak{p}_i$.
Let $z \in \mathrm{Frac}(\widehat{R})$ be integral over $\widehat{R}$, then we claim it is enough to show that there exists $s \in S$ such that $stz \in t_1^{\alpha_1} \dots t_r^{\alpha_r} \widehat{R}$. Then $stz \in t\widehat{R}$. By the definition of S the element s is not a zero divisor in R/tR, so by flatness also not in $\widehat{R}/t\widehat{R}$. Hence tz must be zero modulo $t\widehat{R}$, so we can write $tz = tz'$ for some $z \in \widehat{R}$. Since t is not a zero divisor in $\widehat{R}$ we have $z = z'$ in $\mathrm{Frac}(\widehat{R})$ and we are done.
Consider the following set

$$\{(\xi_1, \dots, \xi_r) \in \mathbb{Z}_{\geq 0}^r \mid \exists s \in S : stz \in t_1^{\xi_1} \dots t_r^{\xi_r} \widehat{R}\}$$

and take a maximal element $(\beta_1, \dots, \beta_r)$ with respect to the degree partial ordering on $\mathbb{Z}_{\geq 0}^r$. It is sufficient to prove that $\beta_i \geq \alpha_i$ for all i. Assume without loss of generality that $\beta_1 < \alpha_1$. Let $s \in S, y \in \widehat{R}$ with $stz = t_1^{\beta_1} \dots t_r^{\beta_r} y$. Denote by $\mathfrak{q}_1, \dots, \mathfrak{q}_m$ the associated primes of $\mathfrak{p}_1 \widehat{R}$. Then, by Lemma 3.7, each $\widehat{R}_{\mathfrak{q}_j}$ is a DVR; let w_j be its associated valuation with $w_j(t_1) = 1$ and write ϕ_j for the canonical ring map $\mathrm{Frac}(\widehat{R}) \to \mathrm{Quot}(\widehat{R}_{\mathfrak{q}_j})$. Since z is integral over $\widehat{R}$ the element $\phi_j(z)$ is integral over $\widehat{R}_{\mathfrak{q}_j}$. But the latter is a DVR, hence in particular integrally closed, so $\phi_j(z) \in \widehat{R}_{\mathfrak{q}_j}$ and $w_j(\phi_j(z)) \geq 0$. Therefore we obtain the inequality

$$\beta_1 + w_j(\phi_j(y)) = w_j(\phi_j(t_1^{\beta_1} \dots t_r^{\beta_r} y)) = w_j(\phi_j(stz)) \geq w_j(\phi_j(t)) = w_j(t_1^{\alpha_1}) = \alpha_1$$

and, by using our assumption $\beta_1 < \alpha_1$, we see that $w_j(\phi_j(y)) \geq 1$ and hence $\phi_j(y) \in \mathfrak{q}_j \widehat{R}_{\mathfrak{q}_j}$. But this implies that $y \in \mathfrak{q}_j$ for all j. Since $\mathfrak{p}_1 \widehat{R}$ is radical it is the intersection of its associated primes and we see that $y \in \mathfrak{p}_1 \widehat{R}$. Localizing $\widehat{R}$ with respect to S gives us $y \in \mathfrak{p}_1 \widehat{R}_S = t_1 \widehat{R}_S$, so there exists $s' \in S$ such that $s'y \in t_1 \widehat{R}$. Set $s'' = ss' \in S$, then $s''tz \in t_1^{\beta_1+1} \dots t_r^{\beta_r} \widehat{R}$, which gives a contradiction to the maximality of $(\beta_1, \dots, \beta_r)$. □

Lemma 3.15. *Let $(R, \mathfrak{m})$ be a normal local domain and assume that R is formally normal. Let L be a finite separable extension of $K = \mathrm{Quot}(R)$ and denote by S the integral closure of R in L. Assume furthermore that every prime $\mathfrak{q} \subset S$ with $\mathrm{ht}(\mathfrak{q}) = 1$ is formally unramified. Then the completion $\widehat{S}$ of S is reduced and integrally closed (in its total ring of fractions).*

Proof. Using the same arguments as in the proof of Corollary 3.11, we may assume that there exists an element $a \in L$ such that $L = K(a)$, S is the integral closure of $R[a]$ (in $L = \mathrm{Quot}(R[a])$) and S is finite over R. Now apply Lemma 3.2 and 3.3 to get that $\widehat{R}[a]$ is a subring of $\widehat{S}$ such that $\mathrm{Frac}(\widehat{R}[a]) = \mathrm{Frac}(\widehat{S})$. Let $(\widehat{S})'$ denote the integral closure of $\widehat{S}$ inside its total ring of fractions and d be the discriminant of the monic irreducible polynomial over R which has a as a root. Then by Lemma 3.10 we have $d(\widehat{S})' \subset \widehat{R}[a]$. Apply Lemma 3.14 to see that $\widehat{S} = (\widehat{S})'$. In order to see that $\widehat{S}$ is reduced, let $s \in \widehat{S}$ be a nilpotent element. By faithful flatness, every nonzero $x \in S$ is not a zero divisor in $\widehat{S}$, so $\frac{s}{x}$ is an element of $\mathrm{Frac}(\widehat{S})$ which is integral over $\widehat{S}$. Hence $s \in x\widehat{S}$ for every $x \in S, x \neq 0$. In particular, s is contained in arbitrarily high powers of the Jacobson radical of $\widehat{S}$. By Krull's intersection theorem, we have $s = 0$. □

Now we come to the proof of the power series version of Zariski's Main Theorem, which we will prove for local rings essentially of finite type either over field or over a Nagata Dedekind domain. Note that for any domain of dimension 1, being Nagata is equivalent to Japanese. Hence, by Corollary 3.11, every Dedekind domain of characteristic 0 is Nagata, for example $\mathbb{Z}$.

Theorem 3.16 (Power series version)**.** *Let A be either a field or a Nagata Dedekind domain and $(R, \mathfrak{m})$ a local domain containing A such that R is essentially of finite type over A. If R is normal then R is formally normal.*

Proof. We will only prove the theorem in the case where $\mathrm{Quot}(A) \subset \mathrm{Quot}(R)$ is separable. For the inseparable case, we would need an additional argument at the end of the proof, see [Nag62, p.139].
For the proof of the theorem we need to show that the hypothesis of Lemma 3.15 holds, namely that any local ring essentially of finite type over A is formally unramified. For this we could use Theorem 3.8 in combination with Theorem 3.9; however, since we skipped the proof of the latter, we provide another proof in our special case. In fact, we will prove the following slightly stronger statement:
If $(R, \mathfrak{m})$ is a local domain essentially of finite type over A, then R is formally unramified and the integral closure R' of R is finite over R. If in addition R is normal, then R is formally normal.
As a first step we show that we can assume that A is a field or a DVR with maximal ideal $\mathfrak{n}$ such that the extension of residue fields $A/\mathfrak{n} \subset R/\mathfrak{m}$ is algebraic. Write $\mathfrak{n} = \mathfrak{m} \cap A$, then $A_{\mathfrak{n}}$ is either a field or a DVR and Nagata again. Hence we can assume $A = A_{\mathfrak{n}}$. Let $a_1, \ldots, a_r$ be elements of R such that their images in $R/\mathfrak{m}$ form a transcendence basis over $A/\mathfrak{n}$. We want to prove that the x_i are algebraically independent over A. Let $P(a_1, \ldots, a_r) = 0$ be a polynomial relation with coefficients in A. Modulo $\mathfrak{m}$ all the coefficients of P have to be zero, so they must be elements of $\mathfrak{n}$. If A is a field then we are done, so consider the case where A is a DVR. Then $\mathfrak{n}$ is principal, for example $\mathfrak{n} = (t)$, so we can write every coefficient as a product of a unit and a power of t. If one of the coefficients of P is nonzero, then divide $P = 0$ by suitable power of t to obtain a polynomial relation $\widetilde{P}(a_1, \ldots, a_r) = 0$ with one coefficient not in $\mathfrak{n}$, which gives a contradiction. Hence $a_1, \ldots, a_r$ are algebraic over A. Consider the ring $B = A(a_1, \ldots, a_r)$. If A was a field, then B is again a field and

$R/\mathfrak{m}$ is algebraic over B. If A is a DVR, then B is again a DVR and what is left to prove is that B is Nagata. Note that it is sufficient to show that B is Japanese, which follows from Corollary 3.12. Finally, since $\mathrm{Quot}(B)$ is an intermediate field of $\mathrm{Quot}(A) \subset \mathrm{Quot}(R)$, the field extension $\mathrm{Quot}(B) \subset \mathrm{Quot}(R)$ is separable and we can assume $A = B$.

Let us prove the statement by induction on the dimension of R. If $\dim(R) = 0$, then R is a field and the assertion is trivial. Otherwise let $x_1, \ldots, x_m$ be a *system of parameters* of R, i.e. $x_1, \ldots, x_m$ generate an $\mathfrak{m}$-primary ideal of R and they are minimal with this property. By the definition of the x_i we have that $m = \dim(R)$. If A is a DVR, we can assume without loss of generality that $x_1 \in \mathfrak{n}$. Note that there exists a chain of primes

$$(0) \subset \mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \ldots \mathfrak{q}_m = \mathfrak{m}$$

such that $x_i \in \mathfrak{q}_j$ only if $j \leq i$. Write $A[x] = A[x_1, \ldots, x_m]$ and set $A' = A[x]_{(x)}$, where $(x) = (x_1, \ldots, x_m)$ is a maximal ideal of $A[x]$; note that if A is a DVR, the ideal $\mathfrak{n}$ is contained in (x). Intersecting the above chain with A' yields that A' has at least dimension m. But the maximal ideal of A' is generated by $x_1, \ldots, x_m$, so A' is a regular local ring of dimension m. Since A is either a field or a DVR and hence universally catenary we have that A' is universally catenary again. Hence, we can use the dimension formula for $A' \subset R$

$$\underbrace{\dim(R)}_{=m} + \underbrace{\mathrm{trdeg}_{A/\mathfrak{n}} R/\mathfrak{m}}_{=0} = \underbrace{\dim(A[x])}_{=m} + \mathrm{trdeg}_{\mathrm{Quot}(A')} \mathrm{Quot}(R)$$

and we obtain that $\mathrm{Quot}(A') \subset \mathrm{Quot}(R)$ is a finite extension. Note that, by Corollary 3.12 again, the ring A' is Japanese. Write $L = \mathrm{Quot}(R)$ and consider the integral closure C of A' in L which is semi-local and finite over A'. Since A' is regular local, its completion is regular again and hence in particular normal. By our induction assumption and Corollary 3.6, every quotient $C/\mathfrak{q}$, with $\mathfrak{q}$ a nonzero prime of C, is formally unramified. Hence the hypotheses of Lemma 3.15 are fulfilled and we get that $\widehat{C}$ is reduced and integrally closed in its total ring of fractions. Write R' for the subring of L generated by R and C. Since C is finite over A' we have that R' is finite over R and hence semi-local. Let $\mathfrak{m}'$ be any maximal ideal of R', then $\mathfrak{q} = \mathfrak{m}' \cap C$ is a maximal ideal of C and $C_\mathfrak{q}$ is formally normal. If we show that $R'_{\mathfrak{m}'} = C_\mathfrak{q}$ then it follows that R' is normal and formally normal. Hence R is formally unramified and R' is the integral closure of R (and finite over R). If R is normal then $R = R'$ and we have proven our assertion.

So for the final step, let us prove that $R'_{\mathfrak{m}'} = C_\mathfrak{q}$. Since $C_\mathfrak{q} \to R'_{\mathfrak{m}'}$ is local we get a local ring map between the completions $\phi : \widehat{C_\mathfrak{q}} \to \widehat{R'_{\mathfrak{m}'}}$. Moreover, since $R/(x)$ is finite over $A/\mathfrak{n}$ the same holds for $R'/\mathfrak{q}R'$ over $C/\mathfrak{q}$. So by Lemma 3.4 we get that

$\widehat{R'_{\mathfrak{m}'}}$ is finite over $\phi(\widehat{C_{\mathfrak{q}}})$. Hence we have the equality

$$\dim(\widehat{C_{\mathfrak{q}}}) = \dim(C_{\mathfrak{q}}) = \dim(R'_{\mathfrak{m}'}) = \dim(\widehat{R'_{\mathfrak{m}'}}) = \dim(\phi(\widehat{C_{\mathfrak{q}}})).$$

Since $\widehat{C_{\mathfrak{q}}}$ is a domain, it follows from the above equality that ϕ is injective. Hence $\widehat{R'_{\mathfrak{m}'}}$ and in particular $R'_{\mathfrak{m}'}$ are integral over $\widehat{C_{\mathfrak{q}}}$. Observe that, since L is finite separable over $\mathrm{Quot}(A')$, we have that $\mathrm{Quot}(C) = L$. To finish the proof, it is sufficient to prove the next lemma. □

Lemma 3.17. *Let R be a domain and S a faithfully flat extension of R. Then $x \in \mathrm{Quot}(R)$ is integral over R if and only if $x \otimes 1 \in \mathrm{Quot}(R) \otimes_R S$ is integral over S.*

Proof. Let $x \otimes 1$ be integral over S, which means that there exist $s_0, \ldots, s_{d-1} \in S$ such that

$$x^d \otimes 1 + x^{d-1} \otimes s_{d-1} + \ldots + 1 \otimes s_0 = 0$$

Write $x = \frac{a}{b}$ with $a, b \in R$. By multiplying the equation with b^d, and since $S \to \mathrm{Quot}(R) \otimes S$ is injective, we get

$$a^d + a^{d-1} b s_{d-1} + \ldots + b^d s_0 = 0$$

in S. Hence a^d lies in the ideal $(a^{d-1}b, a^{d-2}b^2, \ldots, b^d)S$. Since S is faithfully flat over R we have $(a^{d-1}b, \ldots, b^d)S \cap R = (a^{d-1}b, \ldots, b^d)R$ (see [Mat89, Theorem 7.5] for example) and hence we obtain an integral equation for a over R. □

The following is an immediate consequence of the theorem. It states roughly that "normalization separates formal branches".

Corollary 3.18. *Let $(R, \mathfrak{m})$ be a local domain essentially of finite type over A, where A is either a field or a Nagata Dedekind domain. Let R' be the integral closure of R. Then the set of maximal ideals of R' is in bijection to the set of minimal primes of $\widehat{R}$.*

Proof. From the proof of Theorem 3.16 we see that R' is finite over R. So R' is in particular semi-local; denote its maximal ideals by $\mathfrak{m}'_1, \ldots, \mathfrak{m}'_r$. Lemma 3.1 and 3.2 show that $\widehat{R}$ is a subring of $\widehat{R'} \simeq \widehat{R'_{\mathfrak{m}'_1}} \times \ldots \times \widehat{R'_{\mathfrak{m}'_r}}$. By Theorem 3.16 each $\widehat{R'_{\mathfrak{m}_i}}$ is a normal domain. Using Corollary 3.3 we see that

$$\mathfrak{m}'_i \mapsto (\widehat{R'_{\mathfrak{m}'_1}} \times \ldots \times \widehat{R'_{\mathfrak{m}'_{i-1}}} \times (0) \times \widehat{R'_{\mathfrak{m}'_{i+1}}} \times \ldots \times \widehat{R'_{\mathfrak{m}'_r}}) \cap \widehat{R}$$

yields the desired bijection. □

4. **Conclusion: An extended version of ZMT**

In this last chapter we want to put together previous results in order to prove a geometric characterization of normal points. The main ideas and arguments are taken from unpublished lectures notes by David Mumford and Tadao Oda, which can be found at `http://www.dam.brown.edu/people/mumford/alg_geom/papers/AGII.pdf`.

4.1. A characterization of normal points

First, fix the category for which we will prove the statement: let A be either a field or a Nagata Dedekind domain and set $S = \mathrm{Spec}(A)$. Every scheme that will be considered is assumed to be integral and of finite type over S. In particular, every scheme will be Noetherian. These are the minimum assumptions such that we can use both Theorem 1.23 and 3.16.

Theorem 4.1 (Extended ZMT). *Let X be a scheme as above and let $x \in X$ be a point. The following are equivalent:*

1. *X is normal at x.*
2. *X is formally normal at x.*
3. *For every scheme X' and every birational morphism $f : X' \to X$ such that x is fundamental for $X \dashrightarrow X'$, every component of the fiber $f^{-1}(x)$ is positive dimensional.*

Proof. Let $\mathfrak{m}_x$,$\mathfrak{m}_{x'}$ be the maximal ideals of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X',x'}$.
(1) $\Longrightarrow$ (2): This is the power series version, Theorem 3.16.
(2) $\Longrightarrow$ (1): Suppose $\widehat{\mathcal{O}_{X,x}}$ is a normal domain. The local ring $\mathcal{O}_{X,x}$ is a subring of $\widehat{\mathcal{O}_{X,x}}$ and observe that, by faithful flatness, we have $\widehat{\mathcal{O}_{X,x}} \cap \mathrm{Quot}(\mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$. Hence $\mathcal{O}_{X,x}$ is a normal domain.
(1) $\Longrightarrow$ (3): This is the original version, Theorem 1.23.
(3) $\Longrightarrow$ (1): Consider the normalization $\widetilde{X} \to X$. By our assumptions, this is a finite and birational morphism, in particular it has finite fibers. Hence x is regular for f' with image $\tilde{x} \in \widetilde{X}$, which implies that $\mathcal{O}_{X,x} \simeq \mathcal{O}_{\widetilde{X},\tilde{x}}$.
(2) $\Longrightarrow$ (3): We will give here an additional proof without using Theorem 1.23, following an idea in [Mum99, p.212]. Let $f : X' \to X$ be birational, x' a point of X' with $f(x') = x$ and assume that x' is isolated in its fiber. We have to show that the local ring map $\varphi : \mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is an isomorphism. Denote by $\widehat{\varphi}$ be the induced map on the completions. Note that by assumption $\mathcal{O}_{X',x'}/\mathfrak{m}_x\mathcal{O}_{X',x'}$ is finite over $\kappa(x)$. We can repeat the argument at the end of the proof of Theorem 3.16: Lemma 3.4 yields that $\widehat{\mathcal{O}_{X',x'}}$ is finite over $\widehat{\varphi}(\widehat{\mathcal{O}_{X,x}})$. Note that $\mathcal{O}_{X,x}$ is universally

catenary and hence the dimension formula yields

$$\dim(\mathcal{O}_{X',x'}) + \underbrace{\operatorname{trdeg}_{\kappa(x)} \kappa(x')}_{=0} = \dim(\mathcal{O}_{X,x}) + \underbrace{\operatorname{trdeg}_{K(X)} K(X')}_{=0}.$$

Therefore we have $\dim(\widehat{\mathcal{O}_{X,x}}) = \dim(\widehat{\varphi}(\widehat{\mathcal{O}_{X,x}}))$. But $\widehat{\mathcal{O}_{X,x}}$ is a domain, so $\widehat{\varphi}$ is injective. Then using Lemma 3.17 finishes the proof. □

A straightforward application of Corollary 3.18 yields the following result.

Theorem 4.2. *Let X be scheme as above and let $x \in X$ be a point. The following are equivalent:*

(1) *X is formally irreducible at x.*
(2) *X is unibranch at x, i.e., the preimage of x under the normalization $\pi : \widetilde{X} \to X$ is a single point.*

Moreover, any of the equivalent conditions of Theorem 4.1 implies the above properties.

Proof. Choose an affine open $U = \operatorname{Spec}(R)$ of X which contains x. Denote by U' the preimage of U under π; then $U' = \operatorname{Spec}(R')$ with R' the integral closure of R. Write $\mathfrak{p}$ for the prime of R corresponding to x and $T = R - \mathfrak{p}$. Lemma B.1 says that $T^{-1}R'$ is the integral closure of $R_{\mathfrak{p}}$. Note that the maximal ideals of $T^{-1}R'$ are just the primes of R' lying over R. Hence, applying Lemma 3.18 yields a bijection between the set of irreducible components of $\operatorname{Spec}(\widehat{\mathcal{O}_{X,x}})$ and $\pi^{-1}(x)$.
For the last part, a formally normal domain is by definition formally irreducible. □

4.2. Examples

Let us finish this chapter by presenting three examples which will illustrate the statements of Theorem 4.1 and 4.2. The first two are singular curves which serve as (partial) counterexamples to the theorem in the non-normal case. The third one is an isolated surface singularity which is normal.

Example 4.3. Let $X = V(y^2 - x^3)$ be the cubic cusp in $\mathbb{A}^3_k$. Then $0 \in X$ is a singular point of X and hence not normal. For example, the element $y/x \in K(X)$ is integral over the local ring $\mathcal{O}_{X,0} = k[x,y]_{(x,y)}/(y^2 - x^3)$. Consider the parametrization $\mathbb{A}^1_k \to X$ given by $t \mapsto (t^2, t^3)$. By Remark 1.18 we see that statement (3) in Theorem 4.1 does not hold for $\mathbb{A}^1_k \to X$. Consider the completion $\widehat{\mathcal{O}_{X,0}}$, which is a domain. This can be checked by a calculation. Alternatively, note that our parametrization $\mathbb{A}^1_k \to X$ is the normalization of X and the preimage of $0 \in X$ is a single point. Hence X is unibranch at 0 and we can apply Theorem 4.2 to see that X is formally irreducible at 0. On the other hand, $\widehat{\mathcal{O}_{X,0}}$ is not normal; for example, it does not contain the element y/x.

Example 4.4. Consider the cubic node $X = V(y^2 - x^2(x+1))$ in $\mathbb{A}^3_k$. Then X is not normal at 0; again, the element y/x is integral over $\mathcal{O}_{X,0}$. As before, we consider a parametrization $f : \mathbb{A}^1_k \to X$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$. The morphism f is birational and 0 is fundamental for the rational inverse of f. But $f^{-1}(0)$ consists of the two points $-1, +1 \in \mathbb{A}^1_k$. In particular, statement (3) of Theorem 4.1 fails. It is easy to check that f is the normalization of X. Hence X is not unibranch at x.
We want to see explicitly that the normalization separates the formal branches of X at x. Consider first the local ring map $\mathcal{O}_{X,0} \to \mathcal{O}_{\mathbb{A}^1_k,-1}$. The induced ring map on the completions is given by

$$\varphi : k[[X,Y]]/(Y^2 - X^2(X+1)) \longrightarrow k[[T]],$$
$$X \mapsto (T-1)^2 - 1,\ Y \mapsto (T-1)((T-1)^2 - 1).$$

We want to compute the kernel of φ. Consider the power series $F(X)$ in X given by the Taylor expansion of $\sqrt{1+X}$. Then, a computation shows that

$$\varphi(F(X)) = \varphi(F(T^2 - 2T)) = 1 - T.$$

Hence, we have

$$\varphi(Y + XF(X)) = (T-1)(T^2 - 2T) + (T^2 - 2T)(1-T) = 0,$$

and since the ideal $(Y + XF(X))$ is a minimal prime of $\widehat{\mathcal{O}_{X,0}}$ we get $\ker(\varphi) = (Y + XF(X))$. Observe that the other minimal prime of $\widehat{\mathcal{O}_{X,0}}$ is given by $(Y - XF(X))$, and, by the same argument as above, it is the kernel of the local ring map

$$\widehat{\mathcal{O}_{X,0}} \longrightarrow \widehat{\mathcal{O}_{\mathbb{A}^1_k,+1}}.$$

Hence we see that, on the level of completed local rings, the normalization of X identifies the formal components of X at 0.

Example 4.5. Let $X = V(x^2 + y^2 - z^2)$ be the double cone in $\mathbb{A}^3_k$. It is regular outside the origin $0 \in X$. The inverse of the usual stereographic projection induces a morphism $f : \mathbb{A}^2_k \to X$, given by

$$(s,t) \mapsto (2st, s(1-t^2), s(1+t^2)).$$

Algebraically, this embeds $\mathcal{O}_X$ into $k[S,T]$ as the subring $k[S, ST, ST^2]$. Hence f is birational and an isomorphism outside of the closed subset $Z = \{s = 0\}$. Furthermore, $0 \in X$ is the only point fundamental for $X \dashrightarrow \mathbb{A}^2_k$ and $f^{-1}(0) = Z$, which is closed irreducible of dimension 1. So the statement of the original version of ZMT holds for f. In fact, one can prove that X is normal at 0.

Appendix A. General results of commutative algebra

In this section we want to recall some results from commutative algebra. Let us start with the following version of the Hilbert's Nullstellensatz.

Theorem A.1. *Let k be a field and $K \supset k$ a field extension which is finitely generated as a k-algebra. Then K is finite over k.*

The following gives a characterization of Artinian rings.

Lemma A.2. *Let R be a Noetherian ring. Then the following are equivalent:*

(1) *R is Artinian.*
(2) $\dim(R) = 0$.
(3) $\mathrm{Spec}(R)$ *is finite and discrete.*
(4) $\mathrm{Spec}(R)$ *is finite.*

If R is finitely generated over a field k, then any of the conditions above is equivalent to

(5) *R is finite over k.*

Lemma A.3 (Prime avoidance)**.** *Let R be a ring and $I_1, \dots, I_s$ be ideals of R such that I_k are prime for $k > 2$. Let $J \subset R$ be any subset; if J is not contained in any of the I_k, then J is not contained in $\bigcup I_k$.*

Proof. Prove the statement by induction on s. The case $s = 1$ is trivial; use the induction hypothesis to choose for each j an element z_j in $J - \bigcup_{k \neq j} I_k$. If $z_j \notin I_j$ then we are done, so assume $z_j \in I_j$. Set $z = \prod_{j=1}^{s-1} z_j + z_s \in J$. If $z \in I_j$ for some $j < s$ then z_s is in I_j, a contradiction. Assume that $z \in I_s$, then $\prod_{j=1}^{s-1} z_j$ is in I_s. If $s = 2$, then this gives a contradiction to the choice of z_1. If $s > 2$ then, since I_s is prime, $z_j \in I_s$ for some $j < s$, which is again a contradiction. □

In rings which are not Noetherian, there might not exist a primary decomposition any more. Nonetheless, we have the following result.

Lemma A.4. *Let R be a ring, I an ideal of R and $\mathfrak{n}$ a prime containing I which is minimal with this property. Then, for every element $x \in \mathfrak{n}$, there exists $t \in R$, $t \notin \mathfrak{n}$ and $d > 0$ such that $tx^d \in I$.*

Proof. We pass to the ring R/I and assume that $I = 0$. Consider the localization $R_{\mathfrak{n}}$. By assumption, the nilradical of this ring equals $\mathfrak{n}R_{\mathfrak{n}}$. Therefore, the image of x in $R_{\mathfrak{n}}$ is nilpotent. This in turn implies that there exist $t \notin \mathfrak{n}$ and $d > 0$ such that $tx^d = 0$. □

Finally, we want to introduce universally catenary rings: Let R be a ring and $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_r$ a chain of primes of R, then this chain is said to have *length* r. A chain of primes of R is called *saturated* if there does not exist a prime of R

contained inbetween two consecutive terms. The ring R is called *catenary* if for any two primes $\mathfrak{p}, \mathfrak{p}'$ of R with $\mathfrak{p} \subset \mathfrak{p}'$, there exists a saturated chain starting with $\mathfrak{p}$ and ending with $\mathfrak{p}'$ and all such chains have equal length. Notice that any localization and any quotient of a catenary ring is again catenary.
We say that a ring R is *universally catenary* if R is Noetherian and all rings which are finitely generated over R are catenary. In particular, for any finite type ring map $R \to S$ with R universally catenary we have that S is universally catenary as well. Almost all Noetherian rings encountered in algebraic geometry are universally catenary; for example, by [Mat89, Theorem 17.9], we have that a Cohen-Macaulay ring (and any quotient of it) is universally catenary. Note that a regular local ring is Cohen-Macaulay local. In particular, a Dedekind domain and any algebra of finite type over it are universally catenary. We will use this fact in Chapter 3 and 4.
The main property we need from universally catenary rings is the following: Let R be a Noetherian domain and $R \subset S$ an extension of domains; we say that the *dimension formula* holds between R and S if, for every prime $\mathfrak{q}$ of S with $\mathfrak{p} = \mathfrak{q} \cap R$, we have that

$$\mathrm{ht}(\mathfrak{q}) + \mathrm{trdeg}_{\kappa(\mathfrak{p})}\, \kappa(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + \mathrm{trdeg}_{\mathrm{Quot}(R)}\, \mathrm{Quot}(S).$$

The dimension formula relates to universally catenary rings as follows.

Theorem A.5. *Let R be a Noetherian ring. Then R is universally catenary if and only if for every prime $\mathfrak{p}$ of R and every finite type ring map $R/\mathfrak{p} \to S$ with S a domain the dimension formula holds between $R/\mathfrak{p}$ and S.*

See [Mat89, Theorem 15.6].

Appendix B. Integral closure and normal rings

We assume that the reader is familiar with the definition of integral extensions and integral closure. Let us just fix the terminology: If R is a domain, then by $\mathrm{Quot}(R)$ we denote the quotient field of R. We say that R is *integrally closed* if R is integrally closed in $\mathrm{Quot}(R)$. The first result is that integral closure behaves well under localization.

Lemma B.1. *Let $R \to S$ be a ring map and U any multiplicatively closed subset of R. Denote by R' the integral closure of R in S; then $U^{-1}R'$ is the integral closure of $U^{-1}R$ in $U^{-1}S$.*

Recall that a ring R is called *normal* if all localizations $R_\mathfrak{p}$ for $\mathfrak{p} \subset R$ a prime are integrally closed domains. Note that a normal ring is by definition reduced.
If R is any ring, we consider the multiplicatively closed set U consisting of all elements of R which are not zero divisors. We call $U^{-1}R$ the *total ring of fractions*

of R and denote it by $\mathrm{Frac}(R)$. Note the natural map $R \to \mathrm{Frac}(R)$ is injective. If R is a domain, then clearly $\mathrm{Frac}(R) = \mathrm{Quot}(R)$.

Lemma B.2. *Let R be a reduced ring. Then the following are equivalent:*

(1) *R is normal.*
(2) *R is integrally closed in* $\mathrm{Frac}(R)$*.*

With the next result we want to recall the Going-Up resp. Going-Down properties:

Theorem B.3. *Let $R \subset S$ be an integral extension of rings. Then $R \subset S$ satisfies the following properties:*

> **Lying Over:** *For every prime $\mathfrak{p}$ of R there exists a prime $\mathfrak{q}$ of S with $\mathfrak{q} \cap R = \mathfrak{p}$.*
> **Incomparability:** *If $\mathfrak{q}$, $\mathfrak{q}'$ are two primes of S lying over the same prime $\mathfrak{p}$ of R, then $\mathfrak{q} \not\subset \mathfrak{q}'$ and $\mathfrak{q}' \not\subset \mathfrak{q}$.*
> **Going-Up:** *Let $\mathfrak{p} \subset \mathfrak{p}'$ be two primes of R and $\mathfrak{q}$ a prime of S with $\mathfrak{q} \cap R = \mathfrak{p}$. Then there exists a prime $\mathfrak{q}'$ of S containing $\mathfrak{q}$ and such that $\mathfrak{q}' \cap R = \mathfrak{p}'$.*

If in addition S is a domain (which implies that R is domain) and R is integrally closed, then the following holds:

> **Going-Down:** *Let $\mathfrak{p} \subset \mathfrak{p}'$ be two two primes of R and $\mathfrak{q}'$ a prime of S with $\mathfrak{q}' \cap R = \mathfrak{p}'$. Then there exists a prime $\mathfrak{q}$ of S, contained in $\mathfrak{q}'$, and such that $\mathfrak{q} \cap R = \mathfrak{p}$.*

Corollary B.4. *Let $R \subset S$ be an extension of rings with S finite over R. Then over each prime $\mathfrak{p}$ of R there exist only finitely many primes of S lying over $\mathfrak{p}$.*

All the above statements are proven in [Mat89, Ch.9].
There exist several characterizations of normal rings, for example Serre's criterion, which uses the ring-theoretic notion of depth. See [Mat89, Theorem 23.8] for a reference. Another result characterizes integrally closed domains which are Noetherian:

Theorem B.5. *Let R be a Noetherian domain. Then R is integrally closed iff R is a* Krull ring*, i.e. if the following properties hold:*

(1) $R = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} R_{\mathfrak{p}}$.
(2) *Each $R_{\mathfrak{p}}$ with* $\mathrm{ht}(\mathfrak{p}) = 1$ *is a discrete valuation ring.*

A complete proof is contained in [Mat89, Ch. 11 and 12].
We want to add a useful lemma here. It says that in particular, every ring R which is an intermediate ring of an algebraic field extension $k \subset K$ is a field itself.

Lemma B.6. *Let $R \subset S$ be an extension of domains. Let $x \in R$ and assume that $x^{-1} \in S$ and x^{-1} is integral over R. Then $x^{-1} \in R$.*

Proof. By assumption we have an integral equation

$$x^{-d} + a_{d-1}x^{-d+1} + \ldots + a_1 x^{-1} + a_0 = 0$$

with $a_i \in R$. Multiplying with x^{d-1} yields

$$x^{-1} = -a_{d-1} - \ldots - a_1 x^{d-2} - a_0 x^{d-1} \in R.$$

□

Appendix C. Schemes

We assume that the reader is familiar with the definition of an abstract scheme and proceed by recalling the basic definitions and results needed here. Let us first fix some notation: if X is a scheme, write $\mathcal{O}_X$ for the structure sheaf on X. If U is an open subset of X, we write $\Gamma(U, \mathcal{O}_X)$ for the ring of sections associated to U. For any point x of X denote by $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at x and by $\kappa(x)$ the residue field of $\mathcal{O}_{X,x}$.

Let X be a scheme. We say that X is *reduced* if for every affine open U of X the ring $\Gamma(U, \mathcal{O}_X)$ is reduced. X is called *integral* if it is reduced and irreducible (more precisely, if its underlying topological space is irreducible). It is easy to show that if X is integral, then $\Gamma(U, \mathcal{O}_X)$ is a domain for every open affine U of X. Note that any integral scheme X has a unique generic point $\xi \in X$ and the stalk $\mathcal{O}_{X,\xi}$ is a field. We write $K(X) = \mathcal{O}_{X,\xi}$ and call it the *function field* of X.

A scheme X is called *locally Noetherian* if, for every affine open U of X, the ring $\Gamma(U, \mathcal{O}_X)$ is Noetherian. If in addition X is quasi-compact (i.e. every open covering of X admits a finite subcovering) we say that X is Noetherian. Note that for any Noetherian scheme X its underlying topological space is Noetherian as well, in particular X has only finitely many irreducible components.

Let $f : X \to Y$ be a morphism of schemes. We say that f is *locally of finite type* if for every affine open U of X and V of Y with $f(U) \subset V$, the ring map $\Gamma(V, \mathcal{O}_Y) \to \Gamma(U, \mathcal{O}_X)$ is of finite type. Again, if f is in addition quasi-compact then f is said to be *of finite type*. We can define *locally of finite presentation* and *of finite presentation* in a completely analogous way. Note that if S is locally Noetherian, then for any morphism $X \to S$ being locally of finite type is equivalent to being locally of finite presentation. Furthermore, if a scheme X is of finite type over a Noetherian base scheme S, then X is Noetherian itself.

We say that a scheme X is of finite type over a field k if there exists a morphism $X \to \mathrm{Spec}(k)$ which is of finite type. The next result is just a variant of Hilbert's Nullstellensatz:

Theorem C.1. *Let X be of finite type over a field k, then the underlying topological space of X is* Jacobson*, i.e. every nonempty locally closed subset contains a closed point of X.*

A morphism $f : X \to Y$ is called *affine* if the preimage of any affine open of Y under f is an affine open of X. We say that f is *integral* if f is affine and for every affine open $V = \mathrm{Spec}(R)$ of Y with preimage $f^{-1}(V) = \mathrm{Spec}(S)$ such that the ring map $R \to S$ is integral. The morphism f is called *finite* if it is affine and for every $V = \mathrm{Spec}(R)$, $f^{-1}(V) = \mathrm{Spec}(S)$ as above the ring map $R \to S$ is finite. By the definitions it is clear that f finite is equivalent to f integral and locally of finite type.

Let $f : X \to Y$ be a morphism and consider the diagonal morphism $\Delta : X \to X \times_Y X$ given by the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

Then f is called *separated* if Δ is a closed immersion.

Recall that a map between topological spaces is called closed if the image of any closed subset is closed. If $f : X \to Y$ is a morphism, then we say that f is *universally closed* if f is closed (as a map between the underlying topological spaces of X and Y) and for every morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ of f is closed again.

Finally, a morphism $f : X \to Y$ is called *proper* if it is separated, of finite type and universally closed. Both separatedness and properness are properties closely related to topological notions, see more in [Har77, II.4]. The main result on separated/proper morphisms we will need is the valuative criterion, see Chapter E.

Let X be a locally Noetherian scheme. Then X is called *universally catenary* if for every affine open $U = \mathrm{Spec}(R)$ of X the ring R is universally catenary, which is equivalent to $\mathcal{O}_{X,x}$ universally catenary for all $x \in X$. From the results of Chapter A we get that every scheme X of finite type over $\mathrm{Spec}(A)$, where A is a Dedekind domain, is universally catenary. Furthermore, if $X \to Y$ is a morphism between integral and universally catenary schemes, the dimension formula says that

$$\dim(\mathcal{O}_{X,x}) + \mathrm{trdeg}_{\kappa(f(x))}\, \kappa(x) = \dim(\mathcal{O}_{Y,f(x)}) + \mathrm{trdeg}_{K(Y)}\, K(X).$$

Finally, recall that a morphism $f : X \to Y$ is said to be *dominant* if $f(X)$ is dense in Y. In the case where X has only finitely many irreducible components (for example, if X is Noetherian) there is the following result:

Lemma C.2. *Let $f : X \to Y$ be a morphism of schemes and assume X has only finitely many irreducible components. Then f is dominant if and only if for every component of Y its generic point is contained in $f(X)$. In this case Y has only finitely many irreducible components as well.*

Proof. Every point z of Y lies in an irreducible component Z. Denote by ξ_Z the generic point of Z; if $\xi_Z \in f(X)$ then $y \in Z \subset \overline{f(X)}$.

Conversely, assume f is dominant. Decompose $X = X_1 \cup \ldots \cup X_r$ into its components and denote by η_i the generic point of X_i. We have $Y = \bigcup \overline{f(X_i)}$ with each $\overline{f(X_i)}$ irreducible and such that its generic point $f(\eta_i)$ is contained in $f(X)$. □

Let us finish by showing that, for a dominant morphism between integral schemes, all ring maps between the rings associated to affine opens are injective.

Lemma C.3. *Let $\varphi : R \to S$ be a ring map with corresponding morphism f : $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$. Then $\overline{f(\operatorname{Spec}(S))} = \{\mathfrak{p} \in \operatorname{Spec}(R) : \ker(\varphi) \subset \mathfrak{p}\}$. In particular, f is dominant if and only if φ is injective.*

Corollary C.4. *Let $f : X \to Y$ be a dominant morphism between integral schemes. Then, for affine opens $U \subset X, V \subset Y$ such that $f(U) \subset V$ the induced ring homomorphism $\Gamma(V, \mathcal{O}_Y) \to \Gamma(U, \mathcal{O}_X)$ is injective. In particular, for every point $x \in X$ the local homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective.*

Proof. Denote by η the generic point of X and by ξ the generic point of Y. Since f is dominant it maps η to ξ. Furthermore, since X is integral, we get that every affine open $U \subset X$ must contain η. So taking an affine open $V \subset Y$ with $f(U) \subset V$ we get that the morphism $\operatorname{res}(f) : U \to V$ maps the generic point of U to the generic point of V. Hence it is dominant and the corresponding map of rings is injective. □

Appendix D. Normal schemes and normalization

Let X be any scheme and $x \in X$ a point. We say that x is a *normal point* of X (or alternatively, that X is normal at x) if the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain. The scheme X is called *normal* if X is normal at all points $x \in X$. By definition, a normal scheme is necessarily reduced.
Let us now define the *normalization* of an integral scheme X. First assume that $X = \operatorname{Spec}(R)$ is affine. Consider the integral closure R' of R inside $\operatorname{Quot}(R)$. Set $\widetilde{X} = \operatorname{Spec}(R')$. Then the inclusion $R \subset R'$ induces a morphism $\widetilde{X} \to X$ and we will call this the normalization of X. In the case where X is not affine, we can globalize this construction. This process is rather technical and makes use of a few results and definitions not covered here. We will only need the properties listed below, so any proofs are omitted. As a general reference, see for example [GW10, (12.10)].
Let X be an integral scheme with function field $K = K(X)$. Then K induces a quasi-coherent sheaf K_X on X via $\mathit{\Gamma}(U, K_X) = K$ for all opens U of X. We define a pre-sheaf $\mathcal{A}$ on X by

$$\mathit{\Gamma}(U, \mathcal{A}) = \{z \in K : \ z \text{ is integral over } \Gamma(U, \mathcal{O}_X)\}.$$

One can prove that $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_X$-algebra. Hence $\mathcal{A}$ defines an X-scheme $\widetilde{X} = \operatorname{Spec}(\mathcal{A})$. We call the natural morphism $\pi : \widetilde{X} \to X$ the normalization of X.

By definition π is affine. Furthermore, from the affine case we obtain the following properties:

- The scheme $\widetilde{X}$ is integral and normal.
- The morphism π is integral, surjective and we have that $\dim(\widetilde{X}) = \dim(X)$.
- The induced map between function fields $K(X) \to K(\widetilde{X})$ is an isomorphism.

It is possible to extend the definition of the normalization to more general schemes, for example to reduced schemes with finitely many irreducible components. However, we will only need to consider the normalization of integral schemes here.

Appendix E. Valuations

We start with the definition of a valuation. Let Γ be a abelian group, written additively, and $\geq$ a total order on Γ which is compatible with the group operation. Introduce a new symbol ∞ and extend both operation and order to $\Gamma \cup \{\infty\}$ via the rules $a + \infty = \infty + a = \infty$ and $\infty \geq a$ for all $a \in \Gamma$. Let K be a field. A *valuation* of K is a map $v : K \to \Gamma \cup \{\infty\}$ satisfying the following properties for all $x, y \in K$:

(1) $v(x) = \infty$ if and only if $x = 0$.
(2) $v(xy) = v(x) + v(y)$.
(3) $v(x + y) \geq \min(v(x), v(y))$.

The subgroup $v(K^*)$ of Γ is called the *valuation group* of v. Set $R_v = \{x \in K : v(x) \geq 0\}$ and $\mathfrak{m}_v = \{x \in K : v(x) > 0\}$. It is easy to check that R_v is a local ring with maximal ideal $\mathfrak{m}_v$, we say that R_v is the *valuation ring* of v.
Conversely, assume that R is a subring of a field K. We have that the following properties are equivalent:

(1) For every $x \in K$, we have that $x \in R$ or $x^{-1} \in R$.
(2) R is local, the quotient field of R is K and R is maximal among the set of local subrings of K together with the partial order given by domination, i.e. $(R, \mathfrak{m}) \leq (R', \mathfrak{m})$ if $R \subset R'$ and $\mathfrak{m}' \cap R = \mathfrak{m}$.
(3) There exists an abelian group Γ and a valuation $v : K \to \Gamma \cup \{\infty\}$ such that $R = R_v$.

If R satisfies any of these properties we say that R is a *valuation ring* and call v of (3) its *associated valuation*.
We say that a subring R of K is a *discrete valuation ring*, or DVR for short, if there exists a valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ such that $R_v = R$. There are many equivalent definition for a DVR and we want to list those that will be used here.

Proposition E.1. *Let $(R, \mathfrak{m})$ be a local domain. Then the following are equivalent:*

(1) *R is a DVR.*
(2) *R is a Noetherian valuation ring.*
(3) *R is Noetherian, integrally closed and* $\dim(R) = 1$.

(4) *R is Noetherian, $\mathfrak{m}$ is principal and* $\dim(R) > 0$.

For a proof see Theorem 11.1 and 11.2 in [Mat89].
Let us now proceed with an important lemma connecting valuation rings to normal local rings. In fact, Corollary E.3 is the main result behind Theorem 1.20.

Lemma E.2. *Let A be a normal domain with quotient field K. For every $x \in K$, $x \notin A$ there exists a valuation ring $A \subset R \subset K$ of K such that $x \notin R$.*

Proof. Suppose $x \in K$ not contained in A. Set $B = A[x^{-1}]$, then $x \notin B$; since otherwise $x = x^{-d} + \ldots + a_0$, $a_i \in A$, and multiplying this equation with x^d yields an integral equation for x over A. So x^{-1} is not a unit in B and hence is contained in a maximal ideal $\mathfrak{m}$ of B. Consider $B_{\mathfrak{m}}$ and take any valuation ring R dominating it. Then $x \notin R$ since x^{-1} is contained in the maximal ideal of R. □

Note that if in the above statement we suppose that A is local then for each x the chosen valuation ring R will dominate A. Hence we get the following easy consequence:

Corollary E.3. *Let A be a normal local domain. Then A is the intersection of all valuation rings of $K = \mathrm{Quot}(A)$ which dominate A inside K.*

Lemma E.4. *Let R be a valuation ring of a field K. Let X be any scheme. To give a morphism $\mathrm{Spec}(K) \to X$ is equivalent to giving a point $x_1 \in X$ and a field extension $\kappa(x_1) \subset K$. To give a morphism $\mathrm{Spec}(R) \to X$ is equivalent to giving two points $x_0, x_1 \in X$ with $x_0 \in \overline{\{x_1\}}$ and a field extension $\kappa(x_1) \subset K$ such that R dominates the local ring $\mathcal{O}_{Z,x_0}$ on the subscheme $Z = \overline{\{x_1\}} \subset X$ with its reduced scheme structure.*

For a proof see [Har77, Lemma II.4.4].

Theorem E.5 (Valuative criterion of separatedness/properness)**.** *Let $f : X \to Y$ be a morphism of schemes and assume that X is Noetherian.*

(1) *The morphism f is separated if and only if, for every valuation ring R with quotient field K and every commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & Y \end{array}$$

with $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ induced by the inclusion $R \subset K$, there exists at most one diagonal morphism $\mathrm{Spec}(R) \to X$ making the diagram commute.

(2) *Assume furthermore that f is of finite type. Then f is proper if and only if for every valuation ring R with $K = \mathrm{Quot}(R)$ and every diagram as above, there exists exactly one diagonal morphism $\mathrm{Spec}(R) \to X$ such that the diagram commutes.*

This is Theorem II.4.3 and Theorem II.4.7 in [Har77].

Appendix F. Completion and flatness

We refer the reader to [AM94, Chapter 10] and [Mat89, Chapter 8] for the definition of the completion of a ring with respect to one of its ideals. Our aim here is to repeat the properties of the completion which we will make use of in Chapter 3. From now on, let R be any ring, I an ideal of R and denote by $\widehat{R}$ the I-adic completion of R. Assume that R is Noetherian, then we have the following:

- The completion $\widehat{R}$ is flat over R (see [Mat89, Theorem 8.8]). Furthermore, if I is contained in the Jacobson radical of R, then $\widehat{R}$ is faithfully flat over R.
- The associated graded rings of R and $\widehat{R}$ are isomorphic. In particular, $\widehat{R}$ is Noetherian again (see [AM94, Theorem 10.26]).
- If $(R, \mathfrak{m})$ is local, then $\widehat{R}$ is local again with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ and we have $\dim(R) = \dim(\widehat{R})$ (see [AM94, Corollary 11.19]).

The following version of Krull's intersection theorem can be found in [AM94, Corollary 10.19].

Theorem F.1 (Krull's intersection theorem)**.** *If A is Noetherian and I is contained in the Jacobson radical of R, then $\bigcap_n I^n = (0)$. In particular, the canonical map $A \to \widehat{A}$ is injective.*

To finish, let us recall some basic properties of flat resp. faithfully flat extensions.

Lemma F.2. *Let $R \to S$ be a flat ring map.*

1. *If $t \in R$ is not a zero divisor in R then it is not one in S as well.*
2. *If I,J are ideals of R, then $IS \cap JS = (I \cap J)S$.*
3. *Assume that $R \subset S$ is faithfully flat. Then, for every ideal I of R, we have $IS \cap R = I$.*

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