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DISSERTATION

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“On the Localization Properties of Quantum Fields with Zero Mass and Infinite Spin”

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Abstract

Infinite spin at zero mass occurs alongside with the well-known spin- and helicity representations in Wigner's classification of irreducible representations of the Poincaré group, but unlike the latter these are known to be incompatible with point-like localization in the sense of Wightman fields. However, the construction of string-localized fields by Mund, Schroer and Yngvason, which is applicable in particular in the case of infinite spin, has also lead to two-particle wave functions transforming as scalars under the action of these representations. The main focus of the present thesis is to show that despite their resulting compact localization properties in the sense of modular localization, these wave functions cannot be the result of applying an operator with compact localization to the vacuum. In addition, various methods aiming to extend this result to cover the nonexistence of local observable algebras with infinite spin in general and to unify the construction of massive and infinite spin representations in order to sharpen the understanding of their relation to locality are presented.

Zusammenfassung

Unendlicher Spin bei verschwindender Masse tritt zusammen mit den bekannten Spin- und Helizitätsdarstellungen in Wigners Klassifikation der irreduziblen Darstellungen der Poincaré-Gruppe auf; im Gegensatz zu Letzteren sind diese jedoch bekannt dafür mit punktartiger Lokalisierung im Sinne von Wightmanfeldern inkompatibel zu sein. Andererseits hat die Konstruktion stringlokalisierter Felder nach Mund, Schroer und Yngvason, welche insbesondere auf den Fall unendlichen Spins angewendet werden kann, ebenfalls zu Zweiteilchen-Wellenfunktionen geführt, welche sich unter diesen Darstellungen als Skalare transformieren. Das Hauptaugenmerk der vorliegenden Arbeit besteht darin zu zeigen, dass diese Wellenfunktionen trotz ihrer folglich kompakten Lokalisierungseigenschaften im Sinne der modularen Lokalisierung nicht das Ergebnis der Anwendung eines Operators mit kompakter Lokalisierung auf das Vakuum sein können. Weiterhin werden verschiedene Methoden mit dem Ziel dieses Ergebnis zu erweitern um die allgemeine Nichtexistenz lokaler Observablenalgebren mit unendlichem Spin zu beinhalten und um die Konstruktion von massiven Darstellungen und solchen mit unendlichem Spin zu vereinheitlichen um das Verständnis ihres Bezugs zur Lokalisierung zu verbessern vorgestellt.

Contents

List of Symbols

notation	definition	reference
$\mathbb{N}$	set of natural numbers, including zero	
$\mathbb{R}$	set of real numbers	
$\mathbb{R}^+$	set of strictly positive real numbers	
$\mathbb{R}^n$	n-dimensional Euclidean space	
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space of test functions on $\mathbb{R}^n$	
$\mathcal{D}(A)$	space of compactly supported smooth functions on the set A	
$\hat{f}$	Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$	
$\|\| \cdot \|\|_{\alpha,\beta}$	Schwartz space seminorm, $\alpha, \beta \in \mathbb{N}$	
$\mathbb{C}$	set of complex numbers	
H^+	set of complex numbers with strictly positive imaginary part	
$\underline{H}$	Hilbert space	
$\langle \cdot \| \cdot \rangle$	Hilbert space scalar product, linear in the right entry	
$\|\| \cdot \|\|$	Hilbert space norm	
$\mathcal{P}(\underline{H})$	projective Hilbert space	
$\mathcal{B}(\underline{H})$	bounded operators on $\underline{H}$	
$L^2(A)$	Hilbert space of complex-valued square-integrable functions on the set A	
$L^2_{\text{loc}}(A)$	locally square-integrable functions on A	
$\Re$, $\Im$	real and imaginary part	
η	Minkowski metric	(2.1.4)
$\mathbb{M}$	four-dimensional Minkowski space	(2.1.5)
xy	Minkowski product for $x, y \in \mathbb{M}$	(2.1.6)
$\underset{\smile}{x}, \overset{\frown}{x}$	contra- and covariant matrix representation of $x \in \mathbb{M}$	(2.1.9)
σ_i	Pauli matrices, $i = 1, 2, 3$	
$\mathcal{L}_+^\uparrow$	proper orthochronous Lorentz group	
$\mathrm{SL}(2, \mathbb{C})$	group of complex special 2×2 matrices, the twofold covering group of $\mathcal{L}_+^\uparrow$	
$\mathcal{P}_+^\uparrow$	proper orthochronous Poincaré group	(2.1.7)
$\mathcal{P}^c$	twofold covering of $\mathcal{P}_+^\uparrow$	(2.1.8)
Λ	covering homomorphism	(2.1.9)
V^+	forward light cone	
H_m^+	upper mass shell for mass m	(2.1.12)

∂V^+	boundary of the forward light cone	(2.1.13)
∂V^-	boundary of the backward light cone	(3.2.8)
p_+, p_-	light cone coordinates of the momentum four-vector p	(2.1.9)
p	complex representation of the transverse components of p	(2.1.9)
$\mathrm{x} \cdot \mathrm{y}$	Euclidean scalar product for $x, y \in \mathbb{C}$	(2.1.25)
$\widetilde{\mathrm{d}p}$	Lorentz-invariant measure on ∂V^+	(2.1.14)
U	representation of the group $\mathcal{P}^c$ on $\mathcal{H}$	
$\mathcal{H}_1$	one-particle Hilbert space	(2.1.19)
$\mathcal{H}_n$	n-particle Hilbert space	(2.2.4)
$\mathcal{H}$	bosonic Fock space over $\mathcal{H}_1$	(2.2.5)
U_n	representation of $\mathcal{P}^c$ on $\mathcal{H}_n$	
m	particle mass, $m \geq 0$	
B_p	covering group element for the Wigner boost	(2.1.18)
G_q	little group for the reference momentum q	(2.1.15)
$\mathrm{SO}(3)$	group of special orthogonal 3×3 matrices	
$\mathrm{SU}(2)$	group of special unitary 2×2 matrices, the twofold covering group of $\mathrm{SO}(3)$	
D^l	spin-l representation of $\mathrm{SU}(2)$	
Y^l_n	spherical harmonic function, $l \in \mathbb{N}$, $\lvert n\rvert \leq l$	
$E(2)$	two-dimensional Euclidean group	
$\widetilde{E(2)}$	twofold covering of $E(2)$	(2.1.22)
λ	covering homomorphism	
D	representation of the little group	(2.1.24)
J_n	Bessel function, $n \in \mathbb{Z}$	
$\mathcal{H}_q$	little Hilbert space for the reference momentum q	
$R(A,p)$	Wigner rotation for $A \in \mathrm{SL}(2,\mathbb{C})$ and momentum p	(2.1.21)
κ	Pauli-Lubanski parameter, $\kappa \geq 0$	
H	hyperboloid of normalized spacelike directions	(2.2.9)
$\mathrm{d}\sigma(e)$	Lorentz-invariant measure on H	
u_1, u_{1c}	one-particle intertwiner and conjugate intertwiner	(2.2.10)
u_2	two-particle intertwiner	(3.1.3)
$\Phi(x,e)$	unsmeared string-localized field with endpoint $x \in \mathbb{M}$ and spacelike direction $e \in H$	(2.2.16)
$\Phi(f,h)$	smeared string-localized field for $f \in \mathcal{S}(\mathbb{M}), h \in \mathcal{D}(H)$	(2.2.17)
$B(g)$	two-particle observable for $g \in \mathcal{S}(\mathbb{M}^{\times 2})$	(3.1.1)

Chapter 1

Introduction

1.1 A brief History of Infinite Spin Representations in Quantum Field Theory

1.1.1 Wigner classification of irreducible representations of the Poincaré group

The application of representation theory to Quantum Mechanics originates from the description of symmetries in a physical system. Wigner's Theorem [Wig31] states that transformations on the space of states which correspond to a symmetry of the system can be implemented by an either unitary or anti-unitary operator acting on its Hilbert space. It is also discussed in [Bar64] and more recently an alternative proof has been given by [SA90].

In Quantum Field Theory the most important symmetries occur due to the fact that its main use is to describe the behaviour of elementary particles at high energies, where the finiteness and observer-independence of the speed of light become relevant [BLOT90] [Wei95][IZ05][Sre07]. These phenomena are described by the theory of Special Relativity, which is based on the concept of Minkowski spacetime. Consequently, the relevant symmetry operations are elements of the Poincaré group which is generated by translations, spatial rotations and boosts, i.e. transformations relating inertial observers with different velocities [Rin06].

Since these symmetries form a group structure the question whether this property is reflected in the corresponding operators naturally arises. Building on previous applications of the representations of the Lorentz group Wigner proceeded to answer this question by classifying the irreducible representations of the Poincaré group [Wig39].

The representations associated to different types of particles are required to have positive energy, i.e. the timelike component of the generators of the translation subgroup, which are interpreted as the components of the relativistic four-momentum, has positive spectrum. They fall into three classes which are identified by the spectrum of certain Casimir operators, namely particles with finite rest mass and spin, massless particles with discrete helicity and finally massless particles with infinite spin and a positive Pauli-Lubanski parameter.[1] In [BW75] this classification is discussed in terms of the wave equation-based

[1] There is also a recent application of massless particles in a theory of observables in the forward light cone which is suitable for the description of long-range forces [Buc13].

approach to the treatment of elementary particles. One key element is the reduction of the problem to the classification of the little group which is defined as the stabilizer group of a conveniently chosen reference momentum. For massive representations this is a four-vector with vanishing spatial components, such that the stabilizer is the group of spatial rotations, while for massless representations the reference momentum is lightlike, such that the stabilizer is isomorphic to the two-dimensional Euclidean group.

The observation that the composition of operators implementing the elements of a symmetry group will form a representation only up to a phase factor in general, while it is possible to obtain a true representation by passing to the covering group, has been discussed further by Bargmann for the Lorentz group [Bar47].

Due to the simultaneous application of Quantum Mechanics and Special Relativity the concept of localization becomes important, since no measurements taking place in regions of Minkowski space which are spacelike separated, i.e. which cannot be connected by a beam of light, should have an influence on each other within the theory. This is a fundamental concept in the algebraic approach to Quantum Field Theory [Haa96]. In view of the classification of particle types in terms of wave equations one approach to define the localization of a particle would be to consider the spatial support of the solutions to the corresponding equation at any given time, similar to the way the nonrelativistic Schrödinger equation is usually discussed when interpreting its position-space solutions as the probability amplitude for the position of the particle. However, this construction is incompatible with the basic principles of Special Relativity and it will be advantageous to introduce the concept of modular localization [FS02] which builds on the Wigner representations themselves and is constructed in a manifestly covariant way [BGL02].

1.1.2 Previous attempts to localize infinite spin observables

The infinite spin representations of the Poincaré group have repeatedly been considered in works on Quantum Field Theory, but a recurrent phenomenon is the fact that the resulting models are nonlocal: Field operators which transform covariantly under these representations are generally found to display some deficiency from the behaviour that is expected from a local theory, that is, a theory where field operators localized in spacelike separated regions of Minkowski space should commute.

G. J. Iverson and G. Mack have constructed free quantum fields with infinite spin which are covariant in the sense that they depend on a position-space variable transforming under the Lorentz transformations given by elements of the covering group $\mathrm{SL}(2,\mathbb{C})$ as well as a complex two-dimensional variable transforming by pullback under the action of this group and can therefore be regarded as infinite component fields [IM71]. Although all fields considered in that context are nonlocal, some cases satisfying the Spin-Statistics Theorem lead to a vanishing commutator for a proper subset of the spacelike separated configurations. The authors also investigate an infinite spin theory with interactions and observe that for the Pauli-Lubanski parameter approaching zero, where the free fields become finite-component fields again, this theory of “infinite spin neutrinos” reduces to the conventional theory of weak interactions. They have contributed to an understanding of the representations of $\mathrm{SL}(2,\mathbb{C})$ in terms of its subgroup $\widetilde{E(2)}$, the double covering of the two-dimensional Euclidean group, as well [IM70].

A construction by L. F. Abbott starts directly from the representation space obtained from Wigner's classification [Abb76]. A four-dimensional spinor is introduced to ensure the covariant transformation behaviour of the free quantum field, but it appears in an extra exponential factor in the matrix element between the vacuum and a one-particle state whenever the Pauli-Lubanski parameter is different from zero. This non-polynomial factor reappears in the two-point function which is rewritten as an integral over the spatial momentum components for these fields, where the exponent turns out to contain a momentum dependency as well. However, since the position-space requirement of locality translates to a polynomial dependency on the momentum, which has thus been shown to be violated, the result is again that the infinite spin case is not compatible with locality.[2]

On the other hand, the construction of a free quantum field with infinite spin due to K. Hirata [Hir77] is again based on the relativistic wave equations for both single and double valued representations. Quantizing the solutions of these equations using canonical (anti)commutation relations leads to a covariant local Hamiltonian density for fermionic quantization in the case of single valued representations and for bosonic quantization in the case of double valued representations while causal commutation relations for the constructed field operators result only in accordance with the Spin-Statistics Theorem, i.e. the inverse connection between single-/double-valued representations and the chosen commutation relations.

Another instance of the peculiar properties of infinite component fields regarding the TCP-Theorem is discussed in [OT68].

Within the axiomatic framework of Wightman theory, a result indicating the incompatibility of the infinite spin representations with local commutativity has been proved by J. Yngvason [Yng69][Yng70]. In addition to the mentioned requirements of positive energy and local commutativity the generalized Wightman axioms used here state in particular that the fields are operator-valued distributions in the smearing functions, that they are covariant in the sense that apart from the Poincaré transformation acting on the corresponding position-space variable the field depends on a vector in the representation space of an arbitrary representation of the Lorentz group and that repeatedly applying the field operators to the vacuum creates a dense subspace of the Hilbert space. Under these assumptions it is shown that the one-particle states which the Wightman fields create from the vacuum are orthogonal to any irreducible representation subspace of the Hilbert space for zero mass and infinite spin.

In a recent series of papers P. Schuster and N. Toro have presented a theory of interacting infinite spin particles in the path integral approach to Quantum Field Theory. The central object of consideration is an action integral for fields which depend on a four-dimensional

[2]The author also mentions an argument by Wigner against the physical existence of infinite spin particles, which is based on the idea that this would result in an infinite heat capacity of the vacuum. The effects that a locally infinite number of degrees of freedom can have is discussed in [BJ89] for a model theory of free massive particles. See also [Buc74][BL04] and [BP90][BPS91].

position-space variable and an additional four-vector describing the infinite spin degrees of freedom [ST13a]. This setting allows for the introduction of interactions by adding potential terms to the action, similar to Quantum Electrodynamics, Yang-Mills Theory and the Standard Model. However, unlike the previously cited works the fields in this construction do not transform in a covariant way but pick up additional gauge phases under the action of the Lorentz group. In a subsequent discussion of the scattering amplitudes in the interacting theory it is further shown that they are approximated by a theory of helicity particles whenever the energy is large compared to the Pauli-Lubanski parameter which is also referred to as the "spin-scale" in this context [ST13b]. The authors also discuss the possibility of applying the infinite spin theory at small spin-scale to describe long-range forces [ST13c]. It is pointed out in [ST15] that the existence of a "local matter sector" for a nonzero spin-scale is still unknown.

1.2 Modular Localization of Infinite Spin States and Observables

1.2.1 Introduction of modular localization

Introduced by R. Brunetti, D. Guido and R. Longo in [BGL02], the idea of modular localization is to associate to a given wedge-shaped region W of Minkowski space a certain closed and real linear subspace $\mathcal{K}(W)$ of the one-particle Hilbert space $\mathcal{H}_1$ from the Wigner classification. This subspace is defined as the $+1$-eigenspace of the so-called Tomita operator which can be constructed in terms of the representation of the subgroup of boosts leaving the wedge W invariant and the reflection into its causal complement. Consequently, the construction is covariant with respect to the Wigner representation, i.e. the Tomita operator and thus the corresponding real subspace for a different wedge can be obtained by applying the representation of a Poincaré transformation relating the wedges.

One application of these subspaces is then to construct an algebra $\mathcal{R}(W)$ of operators localized in any W for an interaction-free theory as the double commutant of the set of Weyl operators on the bosonic Fock space over $\mathcal{H}_1$ which correspond to $\mathcal{K}(W)$. Subspaces $\mathcal{K}(\mathcal{O})$ for regions $\mathcal{O}$ not necessarily of wedge-form are defined as the intersection of all real subspaces for wedges containing $\mathcal{O}$. Passing to operator algebras analogously leads to a covariant net of algebras $\mathcal{A}(\mathcal{O})$ of operators localized in $\mathcal{O}$. In general the Tomita operator acts by mapping any vector obtained by applying a local operator to the vacuum to the result of applying the adjoint operator to the vacuum. However, in the case of massive theories the Bisognano-Wichmann Theorem provides the required connection between the Tomita operator and the positive-energy representations [Mun01] which serves as a motivation to define this operator for zero mass theories as well.

If C is a spacelike cone the real subspaces $\mathcal{K}(C)$ are shown to be standard for all positive energy representations, i.e. while being disjoint from the corresponding -1-eigenspaces the sum of each pair of ± 1-eigenspaces is dense in $\mathcal{H}_1$. In the case of massive spin representations and massless finite helicity representations this is true even for compact regions. In terms of the resulting operator algebra this is an instance of the Reeh-Schlieder theorem [SW64] which basically says that by performing measurements localized in a given compact region of Minkowski space, any vector in the Hilbert space can be approximated.

The deformation techniques used by G. Lechner to obtain theories of interacting wedge-localized fields also make use of the concept of modular localization [Lec12]. It is shown in particular that the relation between the Tomita operator and the positive energy representations, which is provided by the Bisognano-Wichmann Theorem, is preserved when passing to the operator algebra generated by the deformed field operators which are localized in the standard wedge.

1.2.2 Construction of string-localized states and field operators

In contrast to the various presented attempts to define pointlike localized fields for massless infinite spin representations, J. Mund, B. Schroer and J. Yngvason have constructed free quantum fields, which are localized in semi-infinite spacelike strings, for all positive energy-representations [MSY04], [Mun07], [MSY06]. This is achieved by defining one-particle vectors in the representation Hilbert space obtained from Wigner's classification which are products of a Fourier transformed smearing function on Minkowski space and a so-called intertwiner, that is, a momentum-dependent distribution on the hyperboloid of normalized spacelike direction vectors taking values in the little Hilbert space which is chosen such that the action of the little group on the intertwiner translates to a covariant transformation behaviour of the direction vector. This property is encoded by the so-called intertwiner equation. In addition, the intertwiners are assumed to fulfill certain boundedness properties, such that the constructed vectors are normalizable, as well as analyticity requirements regarding their dependence on the spacelike direction vectors.

Under these assumptions it is then proved that the resulting one-particle vectors are localized in the spacelike cones defined by the supports of the smearing functions. Applying second quantization using canonical commutation relations to these vectors yields the corresponding free quantum fields as operator-valued distributions.

One method of obtaining stringlike intertwiners is to consider pullback representations on orbits of the little group in the forward light cone and map them to the Wigner representation space in an isometric way. In the case of massive representations this orbit can be chosen as the sphere of normalized four-momenta with fixed energy while for massless representations the orbit becomes a two-dimensional paraboloid embedded into the boundary of the forward light cone. The difficulties encountered when trying to obtain fields with sharper localization properties can be understood as consequences of the paraboloid being non-compact. Instead of using this "recipe-like" construction to define the string-localized intertwiners, they are obtained by solving the intertwiner equation directly in the present thesis, providing a complementary route to the uniqueness statements made in [MSY06].

String-localized intertwiners are useful not only to show the existence of string-localized fields with infinite spin, but also for the construction of spinor fields with finite helicity which have less strict relations between the spinor indices and the helicity than in the case of pointlike localized fields [PY12]. In view of the results in [BF82] regarding the non-locality of the unobservable fields involved in the description of massive charged particles it is already pointed out in [Mun07] that string-localization might be useful in the construction of interacting models for charged particles where the one-particle states as well as the asymptotic free fields are string-localized. It is indicated in the same work and further elaborated in [MSY06] that the ultraviolet behaviour, which is encoded in the distributional character of the free fields, is less singular than in the case of pointlike localization, which facilitates the perturbative construction of interacting theories.

1.2.3 Compactly localized n-particle states

In addition to the stringlike intertwiners used in the definition of string-localized fields also two-particle intertwiners are considered in [MSY06]. The corresponding vectors in the two-particle Hilbert space transform as scalars under the action of the Poincaré group. One consequence is that these vectors can be localized in compact regions in the sense of second-quantized standard subspaces. Again, the construction is based on the orbits of the little group and therefore singles out a particular class of solutions of the intertwiner equation. An analogous discussion as in the case of string-intertwiners will yield the general solution to the two-particle intertwiner equation as well.

It has already been pointed out in [Sch08] that the exponential prefactors, which are characteristic for such an intertwiner, are bound to lead to problems regarding the possibility of constructing the corresponding compactly localized two-particle observables.

1.3 A No-Go Theorem for a Class of Infinite Spin Constructions on Fock Space and its Scope

1.3.1 Heuristic attempts to obtain compactly localized operators from known states

Candidate two-particle observables have been defined in [Kö11] by rewriting the two-particle wave functions in terms of a product of two bosonic Fock space creation operators and complementing this term by a conjugate annihilation term as well as a term which leaves the particle number invariant. These terms can be chosen in such a way that when trying to show that the commutator between two such observables as well as the relative commutator between one such observable and a string-localized field vanishes if the relevant smearing function supports are spacelike separated by means of analytic continuation of an appropriately chosen one-parameter group of Lorentz transformations - which is a standard technique to show local commutativity - the boundary terms of the corresponding analytic function on the standard strip $\mathbb{R} + \mathrm{i}]0, \pi[$ can be chosen to match satisfactorily, but the analyticity in the interior cannot be established due to the formation of singularities.

Several modifications to the precise form of the two-particle intertwiners turned out to always lead to this problem in various forms and ultimately turned to the idea that these singularities are a generic feature of the intertwiners.

1.3.2 Restriction to a class of two-particle observables

The strategy to search for compactly localized infinite spin observables by building upon the known modular localized two-particle vectors has been proposed in [MSY06].

In general, nontrivial n-particle wave functions which are modular localized in a compact region are necessary for the existence of local operators which create n-particle states from the vacuum. A relaxation of the intertwiner assumption seems therefore possible if the standard subspaces for infinite spin are sufficiently understood. First results regarding the characterization of the standard subspaces are relatively recent [LL14] and apply to theories on a one-dimensional light ray and to two-dimensional massive theories. Possible generalizations of these results to the case of infinite spin representations will be discussed in section 3.3.2.

In conclusion, while the possibility of the existence of compactly modular localized n-particle wave functions which are conceptually different from those related to solutions of an intertwiner equation remains, the focus on these candidates remains presently.

1.3.3 Summary of the main theorem

The main result of the present thesis is Theorem 4, which states that although the two-particle wave functions constructed in [MSY06] can be modular localized in compact regions, there are no nontrivial operators which create these two-particle wave functions from the vacuum and are relatively local to the string-fields.

The proof strategy can be summarized as follows:

Examining the two-particle wave functions shows that they include prefactors with essential singularities in their momentum dependence. As is expected from modular localization, when considering the corresponding field operators, the vacuum expectation value of the commutator of two such operators is local, but the matrix elements of the relative commutator with a string-field are sensitive to these singularities. The central object of consideration is the matrix element of such a relative commutator between the vacuum state and an arbitrary one-particle state as a function of the parameter of a one-parameter group of lightlike translations acting on the string-field. By relative locality, the commutator vanishes for all values of the parameter in a half-space which in turn implies that the Fourier transform of the matrix element is analytic in a half plane.

At this point, it seems plausible that the only possibility to reconcile the contradicting behaviour of this function and the singular factors of the operator may be to choose the zero operator which is the statement of the theorem. However, the stated analyticity is a property of the matrix element itself, while the non-analytic prefactors occur in the integrand that appears when expanding the matrix element using the canonical commutation relations. This involves an integral over the boundary of the forward light cone and a circle that corresponds to the irreducible representation of the little group for each nonvanishing commutator between creation and annihilation operators.

In order to actually prove the theorem it is therefore necessary to exploit the possible choices in defining the matrix element of the commutator, namely the smearing functions for the string-field and the one-particle state mentioned previously, such that the remaining part of the integrand, which is essentially determined by the operator, can be probed at sufficiently many points. Of course, the smearing functions have to be chosen in such a way that the support properties of the matrix element which have been used are still valid. Once the singular factors in the integrand have been related to the integral itself in this way, it is a task in complex analysis to show that the integrand, and consequently the operator itself, has to vanish.

Chapter 2

Mathematical Preliminaries

Here and in the following, natural units of measurement are chosen such that the speed of light c and Planck's constant $\hbar$ simplify to

$$c = \hbar = 1. \tag{2.0.1}$$

2.1 Construction of Positive Energy Wigner Representations

2.1.1 Implementation of symmetry transformations in quantum mechanics

Let $\underline{H}$ be the Hilbert space of a quantum mechanical system with scalar product denoted by $\langle\cdot|\cdot\rangle$ and norm $||\cdot||$ given by

$$||\psi||^2 = \langle\psi|\psi\rangle \;\; \forall \, |\psi\rangle \in \underline{H}.$$

The equivalence relation $\sim$ on $\underline{H}$, defined by

$$|\varphi\rangle \sim |\psi\rangle \Leftrightarrow \exists\, 0 \neq \lambda \in \mathbb{C} : \; |\varphi\rangle = \lambda\, |\psi\rangle \;\; \forall \, |\varphi\rangle\,, |\psi\rangle \in \underline{H}, \tag{2.1.1}$$

allows mapping any nonzero vector $0 \neq |\psi\rangle \in \underline{H}$ to the *state of the system* which is denoted by $[\psi]$ and defined as its equivalence class

$$[\psi] := \{|\varphi\rangle \in \underline{H} : \; |\varphi\rangle \sim |\psi\rangle\} \in \mathcal{P}(\underline{H}) \tag{2.1.2}$$

in the projective space $\mathcal{P}(\underline{H})$. While the *probability amplitude* $a(\varphi, \psi)$ for a transition between the states given by $|\psi\rangle$ and $|\varphi\rangle$, which is just the scalar product of the normalized vectors

$$a(\varphi, \psi) := \frac{\langle\varphi|\psi\rangle}{||\varphi||||\psi||}, \tag{2.1.3}$$

clearly depends on the phase between these vectors, the *transition probability* between the states $[\psi]$ and $[\varphi]$ themselves, which is the absolute square

$$P([\varphi], [\psi]) := |a(\varphi, \psi)|^2 \overset{(2.1.3)}{=} \frac{|\,\langle\varphi|\psi\rangle\,|^2}{||\varphi||^2||\psi||^2},$$

is independent of the representative vectors, i.e. it is well-defined with respect to the equivalence relation introduced in eq. (2.1.1).

A *symmetry transformation of the system* is given by an invertible map $T : \underline{H} \to \underline{H}$ (not necessarily linear) with the property of preserving the transition probabilities[1]

$$P([T(\varphi)], [T(\psi)]) = P([\varphi], [\psi]) \;\forall\; |\varphi\rangle\,, |\psi\rangle \in \underline{H}.$$

Wigner's Theorem [Wig31, pp. 251–254] [2] then states that the vectors $\psi \in \underline{H}$ representing $[\psi] \in \mathcal{P}(\underline{H})$ can be chosen in such a way that there is a linear operator $U : \underline{H} \to \underline{H}$ with

$$P([U\varphi], [U\psi]) = P([\varphi], [\psi]) \;\forall\; |\varphi\rangle\,, |\psi\rangle \in \underline{H}$$

and U either unitary or anti-unitary[3] for $\dim \underline{H} \geq 2$ and both alternatives possible for $\dim \mathcal{H} = 1$.

However, while Wigner's Theorem itself is already useful in the application of a single symmetry transformation T on the Hilbert space H, even more can be said about the case when T is an element of a *symmetry group* G, in particular the Poincaré group $\mathcal{P}$ of Special Relativity or one of its connected subgroups, which will be the focus of the next subsection.

2.1.2 Symmetries of special relativity

Central to *Special Relativity* [Rin06] is the concept of *spacetime*. It can be thought of as the set of *events* which take place at a certain time[4] $x^0 \in \mathbb{R}$ and position $(x^1, x^2, x^3) \in \mathbb{R}^3$ and is therefore modelled as a four-dimensional vector space $\mathbb{R}^4$, which is equipped with the *Minkowski metric* η given by

$$\mathrm{d}s^2 := \eta_{\mu\nu} x^\mu x^\nu, \text{ where } \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \tag{2.1.4}$$

a non-degenerate indefinite[5] symmetric bilinear form. The pair

$$\mathbb{M} := (\mathbb{R}^4, \eta) \tag{2.1.5}$$

is called *Minkowski space* and it is evident from eq. (2.1.4) that the choice of signature $(+, -, -, -)$ has been made here, which is generally preferred in particle physics.[6] For any two events $x, y \in \mathbb{M}$, the *Minkowski product* defined by η is denoted in the following way:

$$xy := \eta_{\mu\nu} x^\nu y^\nu \tag{2.1.6}$$

[1] As is pointed out in [Reh09], the property (2.1.1) of the map T implies that T is norm-preserving and (using the Cauchy-Schwarz inequality) maps the equivalence classes given by eq. (2.1.2) into each other. T therefore gives rise to a map $\hat{T} : \mathcal{P}(\underline{H}) \to \mathcal{P}(\underline{H})$, $[\psi] \mapsto [T\psi]$.

[2] See also [Bar64] for a more recent presentation of the theorem and [SA90] for an alternative proof.

[3] An example for the latter case, where the time reflection in Quantum Mechanics acts as an anti-unitary operator U, is given in [Wig93].

[4] The choice of units given in eq. (2.0.1) implies that space and time can be measured in the same units thanks to $c = 1$. For example, if x^0 is measured in years, then x^1, x^2, x^3 are measured in light-years.

[5] The indefiniteness of η means that the expression "metric" is not strictly justified, since a metric in the usual sense should exclusively yield positive distances.

[6] On the other hand, the convention of the opposite signature $(-, +, +, +)$ is prevalent in *General Relativity*.

The idea that the speed of light is the same for two observers who move at a relative velocity[7] is reflected by the fact that the quadratic form defined by η vanishes for the difference between any two events $x, y \in \mathbb{M}$ which are connected by a signal moving at the speed of light:

$$\begin{aligned}(x-y)^2 :=& (x-y)(x-y) \overset{(2.1.6)}{=} \eta_{\mu\nu}(x-y)^\mu(x-y)^\nu \\ \overset{(2.1.4)}{=}& (x^0-y^0)^2-(x^1-y^1)^2-(x^2-y^2)^2-(x^3-y^3)^2=0\end{aligned}$$

If this expression is negative, one says that x and y are spacelike separated. The same notion can be to subsets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{M}$, which are said to be spacelike separated if

$$(x-y)^2 < 0 \;\forall\, x \in \mathcal{O}_1, y \in \mathcal{O}_2$$

and the causal complement $\mathcal{O}'$ of a subset $\mathcal{O} \subset \mathbb{M}$ is defined by

$$\mathcal{O}' := \{y \in \mathbb{M} : \, (x-y)^2 < 0 \;\forall\, x \in \mathcal{O}\}.$$

The quadratic form is invariant under the action of the *Poincaré group*, the semidirect product $\mathcal{P} = \mathcal{L} \ltimes \mathbb{M}$, where $\mathcal{L} = \mathrm{O}(1,3)$ is the group of *Lorentz transformations* acting on events in $\mathbb{M}$ by matrix multiplication. $\mathbb{M}$ itself is regarded as an additive abelian group in this context, therefore the semidirect multiplication law reads

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$$

for $(\Lambda_1, a_1), (\Lambda_2, a_2) \in \mathcal{P}$.

2.1.3 Implementation of the Poincaré group in quantum mechanics

In the following, the strongly continuous representations of the proper orthochronous Poincaré group

$$\mathcal{P}_+^\uparrow = \mathcal{L}_+^\uparrow \ltimes \mathbb{M}, \tag{2.1.7}$$

where $\mathcal{L}_+^\uparrow$ is the connected component of the unit element in $\mathcal{L}$, and its twofold covering group

$$\mathcal{P}^c = \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{M} \tag{2.1.8}$$

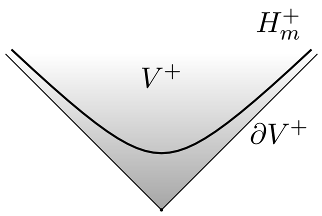
will be discussed.[8] The covering homomorphism $\Lambda : \; \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$ for the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ is given by the following assignment:[9]

$$\begin{gathered}\underset{\smile}{(\Lambda(A)x)} = A\underset{\smile}{x}A^\dagger \text{ or alternatively } (p\Lambda(A))^{\smile} = A^\dagger \overset{\smile}{p} A \;\forall A \in \mathrm{SL}(2,\mathbb{C}), x, p \in \mathbb{M}, \text{ where} \\ \underset{\smile}{x} = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} \text{ and } \overset{\smile}{p} = \begin{pmatrix} p_0 + p_3 & p_1 - \mathrm{i}p_2 \\ p_1 + \mathrm{i}p_2 & p_0 - p_3 \end{pmatrix} =: \begin{pmatrix} p_- & \overline{\mathrm{p}} \\ \mathrm{p} & p_+ \end{pmatrix} \\ \text{respectively, for all } x, p \in \mathbb{M}.\end{gathered} \tag{2.1.9}$$

[7] A more in-depth discussion of the causal structure of Minkowski space can be found in [TW97].

[8] While the present discussion is restricted to the massless infinite spin case, the construction for all positive energy representations can be found in [Fre00], for example. See [BR87] for a more general discussion of group representations.

[9] Multiplying a vector with a matrix from the right is always understood such that contractions resulting in a scalar are defined in an associative way: For $p \in \mathbb{M}$ and $A \in \mathrm{SL}(2,\mathbb{C})$, the product $p\Lambda(A)$ is consequently defined by $(p\Lambda(A))x = p(\Lambda(A)x)$ for all $x \in \mathbb{M}$.



- forward light cone V^+
- mass shell H_m^+
- boundary of the forward light cone ∂V^+

Figure 2.1.1: Possible momentum spectra for positive-energy representations of the Poincaré group

Its extension to a covering homomorphism $\Lambda : \mathcal{P}^c \to \mathcal{P}_+$ for the proper orthochronous Poincaré group given by

$$\Lambda(\,(A, a)\,) := (\Lambda(A), a) \;\forall\; (A, a) \in \mathcal{P}^c$$

is denoted by the same symbol. The group multiplication in $\mathcal{P}^c$ then has the semidirect product form

$$(A_1, a_1)(A_2, a_2) = (A_1 A_2, \Lambda(A_1)a_2 + a_1) \;\forall\, (A_1, a_1), (A_2, a_2) \in \mathcal{P}^c.$$

Following the notation introduced in eq. (2.1.9) the product of two elements of Minkowski space can be written as

$$px = \frac{1}{2}\mathrm{Tr}(\underset{\smile}{p}\overset{\smile}{x}) \;\forall\, x, p \in \mathbb{M}. \tag{2.1.10}$$

2.1.4 Infinite spin representations

The infinite spin representations arise in the following way in Wigner's particle classification setting: In an arbitrary unitary representation U_1 of $\mathcal{P}^c$ on $\mathcal{H}_1$, the representation of the translations

$$U_1(a) := U_1(\mathbf{1}, a) = \mathrm{e}^{\mathrm{i}Pa} \tag{2.1.11}$$

in terms of the generators P gives rise to the operator P^2 which commutes with the entire representation (a so-called Casimir operator) and hence has the form $P^2 = m^2\mathbf{1}$ in an irreducible representation, by Schur's Lemma. Together with the spectrum condition $P^0 > 0$, i.e. the requirement of positive energy, the spectrum has the form

$$H_m^+ :=\{p \in \mathbb{M} :\; p^2 = m^2, p_0 > 0\} \tag{2.1.12}$$

$$\text{or } \partial V^+ :=\{p \in \mathbb{M} :\; p^2 = 0, p_0 > 0\} \tag{2.1.13}$$

for $m > 0$ or $m = 0$, respectively, as is illustrated in Figure 2.1.1. Our concern will focus on the massless case, i.e. $m = 0$, which means that the spectrum of P is the boundary ∂V^+ of the forward light cone. Using the coordinates introduced in eq. (2.1.9), the boundary is described by the equations $p_- > 0$ and $p_+ p_- = |\mathrm{p}|^2$, hence most of the time only the independent variables p_-, p are being used, in terms of which the $\mathcal{L}_+^\uparrow$-invariant measure on ∂V^+ has the form

$$\widetilde{\mathrm{d}p} = \Theta(p_-)\frac{\mathrm{d}p_-}{p_-}\mathrm{d}^2\mathrm{p}. \tag{2.1.14}$$

The classification can be reduced to the representation theory of the little group

$$G_q := \operatorname{stab} q, \tag{2.1.15}$$

with the reference momentum $q \in \partial V^+$ given by $\check{q} = (\mathbf{1} + \sigma_3)/2$ (cf. (2.1.9)), whose elements can be parametrized by the matrices

$$[\varphi, a] := \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & \\ \mathrm{e}^{\mathrm{i}\varphi}\overline{a} & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix} \text{ where } \varphi \in \mathbb{R} \bmod 2\pi,\ a \in \mathbb{C}\ . \tag{2.1.16}$$

The mapping $\partial V^+ \to \mathrm{SL}(2, \mathbb{C})$, $p \mapsto B_p$ constitutes a family of boosts, defined by the property

$$q\Lambda(B_p) = p\ \forall p \in \partial V, \tag{2.1.17}$$

which are called Wigner boosts. A possible choice is

$$B_p = \frac{1}{\sqrt{p_-}} \begin{pmatrix} p_- & \overline{\mathrm{p}} \\ & 1 \end{pmatrix}. \tag{2.1.18}$$

Using these prerequisites, it is possible to restate the action of U_1 as a representation on the one-particle space

$$\mathcal{H}_1 := L^2(\partial V^+, \widetilde{\mathrm{d}p}) \otimes \mathcal{H}_q, \tag{2.1.19}$$

where the little Hilbert space $\mathcal{H}_q$ the representation space of G_q:

$$[U_1(A, a)\psi](p) = \mathrm{e}^{\mathrm{i}pa} D(R(A, p))\psi(p\Lambda(A)) \tag{2.1.20}$$

with

$$R(A, p) = B_p A B^{-1}_{p\Lambda(A)} \text{ the Wigner rotation} \tag{2.1.21}$$

and D a representation of G_q on $\mathcal{H}_q$, which is constructed as follows: Using the explicit parametrization of G_q in (2.1.16), the group multiplication

$$[\varphi_2, a_2][\varphi_1, a_1] = [\varphi_1 + \varphi_2, \mathrm{e}^{\mathrm{i}2\varphi_2} a_1 + a_2] \tag{2.1.22}$$

shows that G_q is isomorphic to the twofold covering group $\widetilde{E(2)}$ of the Euclidean group $E(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ of rigid motions in the Euclidean plane, identified with $\mathbb{C}$, with covering homomorphism $\lambda : \widetilde{E(2)} \to E(2)$ obtained from $\lambda : U(1) \to \mathrm{SO}(2)$ and group multiplication

$$[\varphi_1, a_1][\varphi_2, a_2] = [\varphi_1 + \varphi_2, \lambda(\varphi_1)a_2 + a_1]\ .$$

Again, an arbitrary unitary representation $D : \widetilde{E(2)} \to \mathrm{End}(\mathcal{H}_q)$ gives via $D(a) := D([0, a]) = \mathrm{e}^{\mathrm{i}Ka}$ rise to a Casimir operator K^2, which satisfies $K^2 = \kappa^2\mathbf{1}$ for an irreducible representation. The value of κ^2 distinguishes the infinite spin case from the helicity representations by the requirement $\kappa^2 \neq 0$. The value of κ is considered to be fixed to a positive value. Now the elements of $\mathcal{H}_q$ can be written as functions $v : S^2 \to \mathbb{C}$ in $L^2(\kappa S^1, \mathrm{d}\nu(k))$, where the measure $\mathrm{d}\nu(k)$ on the circle κS^1 is defined by restricting the Lebesgue measure on $\mathbb{R}^2$, i.e. for every integrable function $f\ :\ \kappa S^1 \to \mathbb{C}$ and $0 < \epsilon < \kappa$, we have

$$\int \mathrm{d}\nu(k)\, f(k) = \int \mathrm{d}^2 k \delta(k^2 - \kappa^2)\tilde{f}(k) = \frac{1}{2\kappa}\int_0^{2\pi} \mathrm{d}\alpha\, \tilde{f}\left(\kappa \begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix}\right), \tag{2.1.23}$$

where $\tilde{f}\ :\ \mathbb{R}^2 \to \mathbb{C}$, $k \mapsto \chi_{[\kappa-\epsilon,\kappa+\epsilon]}(|k|) f\left(\kappa\frac{k}{|k|}\right)$ equals f when restricted to κS^1.

Since the relevant measures on ∂V^+ and κS^1 have been stated in eq. (2.1.14) and eq. (2.1.23), respectively, we will abbreviate the notation of the L^2 spaces by omitting reference to them in the following.

On $L^2(\kappa S^1)$, the representation D acts in the following way:[10]

$$[D([\varphi, a])v](k) = \mathrm{e}^{-\mathrm{i}k\cdot\overline{a}} v(k\lambda(-\varphi)) \tag{2.1.24}$$

In this equation the product "$\cdot$" is understood as the Euclidean scalar product on $\mathbb{R}^2$, identified with $\mathbb{C}$:

$$\cdot\ :\ \mathbb{C}\times\mathbb{C}\to\mathbb{R},\ (\mathrm{x},\mathrm{y})\mapsto \mathrm{x}\cdot\mathrm{y} := \Re(\mathrm{x})\Re(\mathrm{y}) + \Im(\mathrm{x})\Im(\mathrm{y}) = \Re(\overline{\mathrm{x}}\mathrm{y}) \tag{2.1.25}$$

Whenever it is clear from the context that two complex numbers are to be multiplied in the sense of eq. (2.1.25), as is the case in (2.1.24), the symbol "$\cdot$" will be omitted.

Defining the maps $\varphi :\ \widetilde{E(2)} \to \mathbb{R}$ and $a :\ \widetilde{E(2)} \to \mathbb{C}$ by $\varphi([\varphi, a]) = \varphi$ and $a([\varphi, a]) = a$, respectively, the complete representation then has the form

$$[U_1(A,a)\psi](p,k) = \mathrm{e}^{\mathrm{i}pa}\mathrm{e}^{-\mathrm{i}k\overline{a(R(A,p))}}\psi(p\Lambda(A), k\lambda(-\varphi(R(A,p))))\ \forall\,(A,a)\in\mathcal{P}^c, \psi\in\mathcal{H}_1$$

with $\psi(p,k) := \psi(p)(k)$.

2.2 Modular Localization

2.2.1 The free net of observables on Fock space

Definitions and results from [BGL02] are presented in the following: Let $W_0 \subset \mathbb{M}$ denote the standard wedge

$$W_0 := \{x \in \mathbb{M} :\ |x^0| < x^3\}, \tag{2.2.1}$$

which is invariant under the one-parameter group of boosts $\Lambda(\mathrm{e}^{\mathrm{i}\sigma_3 t})$ with $t\in\mathbb{R}$. Define the corresponding group of unitary operators on $\mathcal{H}_1$ by

$$\Delta_{W_0}^{\mathrm{i}t} := U_1(\mathrm{e}^{\mathrm{i}\sigma_3 t})$$

and the representation $J_{W_0} := U_1(R_{W_0})$ of the reflection R_{W_0} across the edge of the wedge W_0, where $U_1(R_{W_0})$ is defined as complex conjugation, thereby extending the representation U_1. For an arbitrary wedge $W = \Lambda(A)W_0 + a$ with $(A,a)\in\mathcal{P}^c$ the adjoint action

[10] Analogously to the remarks regarding eq. (2.1.9), the product $k\lambda(-\varphi)$ is defined by the equation $(k\lambda(-\varphi))\cdot a = k\cdot(\lambda(-\varphi)a)$ for all $a\in\mathbb{R}^2$.

of $U(A,a)$ on these operators yields $\Delta_W^{\mathrm{i}t}$ and J_W, respectively. Then the *Tomita operator* S_W can be defined as

$$S_W := J_W \Delta_W^{\frac{1}{2}} \tag{2.2.2}$$

and one can show that S_W is a densely defined, antilinear and closed operator with $\operatorname{ran} S_W = \operatorname{dom} S_W$ and $S_W^2 \subset 1$. This definition is motivated by the Bisognano-Wichmann Theorem which shows that the geometrically defined S_W coincides with the Tomita operator for finite-component Wightman fields [BW75]. These properties can in turn be used to show that the real linear subspace

$$\mathcal{K}(W) := \{\psi \in \operatorname{dom} S_W :\ S_W\psi = \psi\}$$

is *standard*, which means that the following properties hold:

$$\begin{aligned} \mathcal{K}(W) \cap \mathrm{i}\mathcal{K}(W) =& \{0\} \\ \overline{\mathcal{K}(W) + \mathrm{i}\mathcal{K}(W)} =& \mathcal{H} \\ J_W\mathcal{K}(W) = \mathcal{K}(W') =& \mathcal{K}(W)^{\perp}, \end{aligned}$$

where the complement denoted by $\perp$ refers to the symplectic form on $\mathcal{H}_1$ which is defined as the imaginary part of the scalar product. It is also pointed out in [MSY06] that the second condition can be interpreted as a one-particle version of the Reeh-Schlieder Theorem for wedges. The subspaces for smaller regions $\mathcal{O} \subset \mathbb{M}$ are defined as the intersections

$$\mathcal{K}(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} \mathcal{K}(W). \tag{2.2.3}$$

For $1 \leq n \in \mathbb{N}$ define the bosonic n-particle Hilbert space by

$$\mathcal{H}_n := \operatorname{Sym} \mathcal{H}_1^{\otimes n}, \tag{2.2.4}$$

where Sym denotes the symmetrization operation given for all $\psi \in \mathcal{H}_1^{\otimes n}$ by[11]

$$\operatorname{Sym}\psi(p_1, k_1, \ldots, p_n, k_n) := \frac{1}{n!}\sum_{\sigma \in S_n} \psi(p_{\sigma(1)}, k_{\sigma(1)}, \ldots, p_{\sigma(n)}, k_{\sigma(n)})$$

with S_n the symmetric group of degree n, and the bosonic Fock space over $\mathcal{H}_1$ by

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \tag{2.2.5}$$

where $\mathcal{H}^0 := \mathbb{C}$ denotes the subspace for the normalized vacuum vector $\Omega \in \mathcal{H}^0$. The creation and annihilation operators for a vector $\psi \in \mathcal{H}_1$ are denoted by $a^\dagger(\psi)$ and $a(\psi)$, respectively, such that

$$\begin{aligned} [a(\varphi), a(\psi)] = [a^\dagger(\varphi), a^\dagger(\psi)] =& 0,\ [a(\varphi), a^\dagger(\psi)] = \langle \varphi, \psi\rangle\, \mathbf{1}\ \forall\, \varphi, \psi \in \mathcal{H}_1 \\ \text{and } a(\psi)\Omega =& 0\ \forall\, \psi \in \mathcal{H}_1. \end{aligned} \tag{2.2.6}$$

[11] The notation is adapted to the case of infinite spin, where the one-particle subspace $\mathcal{H}_1$ is defined by eq. (2.1.19).

With the Weyl operators defined by

$$V(\psi) := \exp(\mathrm{i}[a^\dagger(\psi) + a(\psi)]) \in \mathcal{B}(\mathcal{H})\ \forall\, \psi \in \mathcal{H}_1$$

the operator algebra for $\mathcal{O} \subset \mathbb{M}$

$$\mathcal{A}(\mathcal{O}) := \{V(\psi) :\ \psi \in \mathcal{K}(\mathcal{O})\}'', \tag{2.2.7}$$

where the prime $'$ denotes the commutant in $\mathcal{B}(\mathcal{H})$, is a von Neumann algebra by the von Neumann Double Commutant Theorem. Alternatively these algebras can be obtained as the intersection

$$\tilde{\mathcal{A}}(\mathcal{O}) := \bigcap_{W \supset \mathcal{O} \text{ wedge}} \mathcal{A}(W). \tag{2.2.8}$$

If $\mathcal{K}(\mathcal{O})$ is standard, for example in the case of a free massive scalar field, the definitions of $\mathcal{A}(\mathcal{O})$ and $\tilde{\mathcal{A}}(\mathcal{O})$ agree, which is described in [MSY06] as a functorial relation between the spaces $\mathcal{K}$ and the algebras, while in general the definition in eq. (2.2.7), where the intersections are formed first, is more restrictive than the one in eq. (2.2.8).

2.2.2 Construction of string-localized one-particle states for all positive-energy representations

This section and the following one summarize the construction of string-localized one-particle vectors and free fields from [MSY06].

Regarding the one-particle space $\mathcal{H}_1$ (cf. eq. (2.1.19)) of wave functions which depend on the momentum $p \in \partial V^+$ and the infinite spin variable $k \in \kappa S^2$, an extra dependence on the variable $e \in H$ with

$$H := \{e \in \mathbb{M} \mid e^2 = -1\}, \tag{2.2.9}$$

the hyperboloid of normalized spacelike direction vectors, can be introduced to define a stringlike intertwiner $u_1 :\ \partial V^+ \times H \to \mathcal{H}_q$, which transforms in a covariant way

$$D(R(A,p))u_1(p\Lambda(A), e) = u_1(p, \Lambda(A)e) \tag{2.2.10}$$

by choosing a 2-dimensional parametrization

$$\xi\ :\ \mathbb{C} \to \Gamma_q,\ z \mapsto \xi(z), \text{ where } \undertilde{\xi}(z) = \begin{pmatrix} |z|^2 & \overline{z} \\ z & 1 \end{pmatrix} \tag{2.2.11}$$

of the G_q-orbit $\Gamma_q := \{p \in \partial V^+ \mid pq = 1\}$ and defining

$$u_1(p,e)(k) = \int \mathrm{d}^2 z\, \mathrm{e}^{\mathrm{i}kz} v(\xi(z)\Lambda(B_p)e) \tag{2.2.12}$$

with a function v analytic in the upper half-plane, which translates to u being analytic for e in the forward tuboid[12], provided v has a distributional real boundary value. Extending the

[12]This means that the complexified version of e satisfies $e^2 = 1$ and its imaginary part is contained in the interior of the forward light cone V^+, see [MSY06].

representation D of G_q by the little group representation $D(j_0)$ of an the extra reflection j_0 at the edge of W_0 by complex conjugation one can define the conjugate intertwiner

$$u_{1c}(p,e)(k) = D(j_0)u_1(-pj_0, j_0 e)(k), \tag{2.2.13}$$

which satisfies the same intertwiner equation, but is an antilinear distribution. For smearing functions $f \in \mathcal{S}(\mathbb{M})$ and $h \in \mathcal{D}(H)$ the stringlike intertwiner (2.2.12) and its conjugate (2.2.13) lead to the one-particle states

$$\begin{aligned}\psi(f,h)(p,k) :=& \hat{f}(p) \int \mathrm{d}\sigma(e)\, h(e) u_1(p,e)(k)\\ \psi_c(f,h)(p,k) :=& \hat{f}(p) \int \mathrm{d}\sigma(e)\, h(e) u_{1c}(p,e)(k)\end{aligned}$$

with σ denoting the $\mathcal{L}_+^\uparrow$-invariant measure on H. The vector $\psi(f,h) + \psi_c(\overline{f}, \overline{h})$ is modular localized in $\operatorname{supp} f + \mathbb{R}^+ \operatorname{supp} h$, i.e.

$$\psi(f,h) + \psi_c(\overline{f}, \overline{h}) \in \mathcal{K}(\operatorname{supp} f + \mathbb{R}^+ \operatorname{supp} h) \tag{2.2.14}$$

which can be shown using the analyticity and boundedness properties of the intertwiners.

2.2.3 String-localized quantum fields

Let $\mathcal{H}$ denote again the bosonic Fock space over the one-particle space $\mathcal{H}_1$ for infinite spin, as defined in eq. (2.2.5). The creation and annihilation operators fulfilling the canonical commutation relations given in eq. (2.2.6) can be restated as the following integrals over the corresponding operators at sharp momentum p and infinite spin variable k:

$$\begin{aligned}a(\varphi) &= \int \widetilde{\mathrm{d}p} \int \mathrm{d}\nu(k)\, \varphi(p,k) a(p,k)\\ a^\dagger(\psi) &= \int \widetilde{\mathrm{d}p} \int \mathrm{d}\nu(k)\, \psi(p,k) a^\dagger(p,k)\end{aligned}$$

With the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^4)$ given by

$$\hat{f}(p) = \int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}px} f(x)\ \forall\, p \in \partial V^+, \tag{2.2.15}$$

the string localized infinite spin fields are constructed as operator valued distributions defined by

$$\Phi(x,e) = \int \widetilde{\mathrm{d}p} \int \mathrm{d}\nu(k)\, \mathrm{e}^{\mathrm{i}px} u_1(p,e)(k) a^\dagger(p,k) + \mathrm{e}^{-\mathrm{i}px} \overline{u_{1c}(p,e)(k)} a(p,k) \tag{2.2.16}$$

for $x \in \mathbb{M}$ and $e \in H$ (cf. eq. (2.2.9). This definition yields the smeared field operator

$$\Phi(f,h) = \int \widetilde{\mathrm{d}p} \int \mathrm{d}\nu(k)\, \hat{f}(p) u_1(p,e)(k) a^\dagger(p,k) + \hat{f}(-p) \overline{u_{1c}(p,e)(k)} a(p,k) \tag{2.2.17}$$

for $f \in \mathcal{S}(\mathbb{M})$ and $h \in \mathcal{D}(H)$. It is defined on the domain spanned vectors with finite particle number in $\mathcal{H}$, i.e. with only finitely many nonvanishing components in the direct sum in eq. (2.2.5).

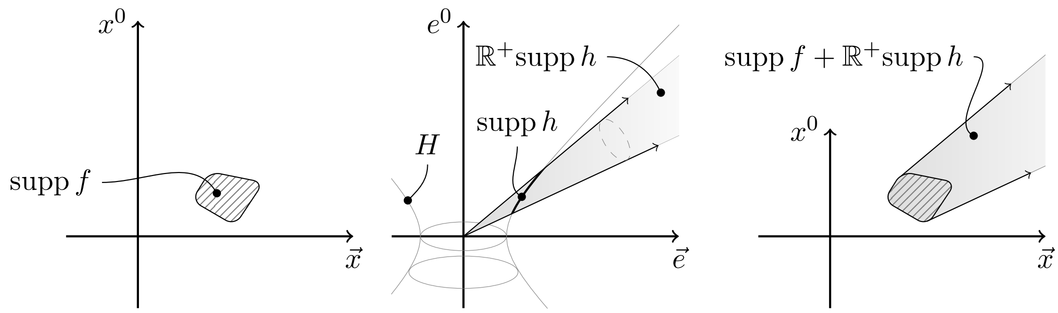


Figure 2.2.1: Localization region of a smeared string-field

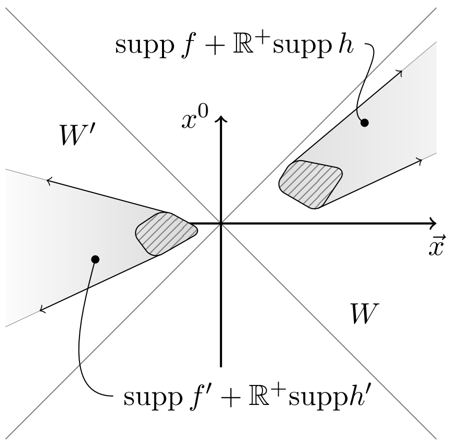


Figure 2.2.2: Spacelike separated truncated cones

The localization region of the string-field $\Phi(f,h)$ becomes a truncated cone of the form $\operatorname{supp} f + \mathbb{R}^+\operatorname{supp} h$ upon smearing in both variables, as is illustrated in fig. 2.2.1. Covariance of this field

$$U(A,a)\Phi(f,h)U^\dagger(A,a) = \Phi(f_{(\Lambda(A),a)}, h_{\Lambda(A)}), \qquad (2.2.18)$$
$$\text{where } f_{(\Lambda,a)}(x) = f(\Lambda^{-1}(x-a)) \text{ and } h_\Lambda(e) = h(\Lambda^{-1}e),$$

is a consequence of the intertwiner equation (2.2.10). The adjoint field is related to the original expression by

$$\Phi(f,h)^\dagger = \Phi_c(\overline{f}, \overline{h}) \qquad (2.2.19)$$

String-localization, i.e. local commutativity for string-fields smeared with functions which are supported in spacelike separated truncated cones (cf. Figure 2.2.2) is a consequence of eq. (2.2.14) and can be shown using the analyticity properties of the string-intertwiners directly for the commutator of these fields as well [MSY04].

Remark 1. Regarding the argument in [Sch08] that localization is lost if there are non-polynomial factors inside the momentum-space integral (which would lead to a convolution in position space with a function not concentrated at the origin), it should be noted that while $u_1(p,e)(k)$ (cf. eq. (2.2.12)) is in general a non-polynomial expression in p, the analyticity properties (depending on e) constrain the corresponding smearing in position space in such a way that locality is preserved in the sense of string-localization. This argument shows that a more in-depth analysis of the occurring non-analytic momentum dependencies is necessary if one is interested in the resulting localization properties in position space.

It is shown in [MSY06] that the single-particle vectors these fields create from the vacuum also have the Reeh-Schlieder property, i.e. the norm closure of the span of these vectors is the whole one-particle space:

Theorem 1 (Reeh-Schlieder Theorem for string-localized vectors)**.** *Let $\psi \in \mathcal{H}_1$, $\mathcal{O} \subset \mathbb{M}$ bounded and open and $\mathcal{U} \subset H$ bounded and open in H. For all $N \in \mathbb{N}$ there are M_N functions $f_N^i \in \mathcal{S}(\mathbb{M})$ and $h_N^i \in \mathcal{D}(H)$, $i = 1, \ldots, M_N$ and such that*

$$\operatorname{supp} f_N^i \subset \mathcal{O}, \operatorname{supp} h_N^i \subset \mathcal{U} \;\forall\, N \in \mathbb{N}, i = 1, \ldots, M_N \;\textit{and}\; \lim_{N\to\infty} \sum_{i=1}^{M_N} \Phi(f_N^i, h_N^i)\Omega = \psi.$$

Convergence is understood in the topology of $\mathcal{H}_1$, i.e. for all $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that

$$\left|\left|\sum_{i=1}^{M_N} \Phi(f_N^i, h_N^i)\Omega - \psi\right|\right|_{\mathcal{H}_1} < \epsilon \;\forall\, N \geq N_0.$$

Proof. The proof of the Reeh-Schlieder Theorem for string-localized vectors can be found in[MSY06, Proposition 3.2, Part 0)]. □

Remark 2. The Reeh-Schlieder Theorem for Wightman fields is discussed in [SW64]. One essential ingredient is the positive energy requirement for the representation of the translation group, also known as the spectrum condition. It is reflected in the fact that in Theorem 1 only the part of the string-field defined in eq. (2.2.17) which contains the creation operator contributes to the vectors generated from the vacuum Ω. An analogous approximation procedure for the string-fields, which acts on the forward and backward light cone simultaneously, will be discussed in Lemma 12.

2.2.4 Compactly localized two-particle states

Definition 1 (Construction of two-particle intertwiners)**.** According to [MSY06, 6.3], the two-particle intertwiner u_2 is given by the formula

$$u_2(p, \tilde{p})(k, \tilde{k}) := \int \mathrm{d}^2 z\, \mathrm{e}^{\mathrm{i}kz} \int \mathrm{d}^2 \tilde{z}\, \mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}} F(A(p, \tilde{p}, z, \tilde{z})) \;\forall\, p, \tilde{p} \in \partial V^+, k, \tilde{k} \in \kappa S^1, \quad (2.2.20)$$

$$\text{where } A(p, \tilde{p}, z, \tilde{z}) := \xi(z)\Lambda(B_p B_{\tilde{p}}^{-1})\xi(\tilde{z}) \;\forall\, p, \tilde{p} \in \partial V^+, z, \tilde{z} \in \mathbb{R}^2 \quad (2.2.21)$$

with a real-valued function $F \in \mathcal{S}(\mathbb{R})$.

The following two lemmas show that while u_2 is a covariant expression in the infinite spin representation, it diverges for certain configurations:

Lemma 1 (Two-particle intertwiner equation)**.** *This intertwiner u_2 satisfies a scalar intertwiner equation simultaneously in both variable sets:*

$$D(R(A,p)) \otimes D(R(A,\tilde{p}))u_2(p\Lambda(A),\tilde{p}\Lambda(A)) = u_2(p,\tilde{p}) \tag{2.2.22}$$

Proof. Comparing (2.2.20) with the definition (2.2.12) of $u_1(p,e)$, the Lorentz transformation $\Lambda(A)$ will occur twice and cancel in (2.2.21), instead of acting on e. □

Remark 3. The intertwiner equation shown in Lemma 1 implies that two-particle states constructed from its solutions u_2 transform as scalars which is in turn sufficient to show that these states can be modular localized in compact regions.

Lemma 2. *Let $\mathcal{O} \subset \mathbb{M}$ be a compact region in Minkowski space and $g \in \mathcal{S}(\mathbb{M}^{\times 2})$ a real-valued smearing function with* $\operatorname{supp} g \subset \mathcal{O}^{\times 2}$*. For any solution $u_2 \in L^2_{\mathrm{loc}}((\partial V^+)^{\times 2}) \otimes \mathcal{H}_q^{\otimes 2}$ of eq.* (2.2.22) *which is polynomially bounded, i.e.*

$$||u_2(p,\tilde{p})||_{\mathcal{H}_q^{\otimes 2}} \leq M(p,\tilde{p})$$

for some polynomial M defined on $(\partial V^+)^{\times 2}$, the function

$$\psi(p,k,\tilde{p},\tilde{k}) := \hat{g}(p,\tilde{p})u_2(p,\tilde{p})(k,\tilde{k}) \tag{2.2.23}$$

is modular localized in $\mathcal{O}$, which means that $\psi(g) \in \mathcal{K}_2(\mathcal{O})$, where the two-particle subspaces $\mathcal{K}_2$ are defined analogously to eq. (2.2.3) *but with the Tomita operator S_W, as defined in eq.* (2.2.2)*, replaced by its second quantization.*

Proof. Lemma 2 is a slightly reformulated version of [MSY06][Proposition 6.4]. □

Remark 4. In [MSY06] it is pointed out that the modular localization discussed in Lemma 2 implies that operators on the Fock space $\mathcal{H}$, which create vectors of the form given in eq. (2.2.23) in the two-particle space $\mathcal{H}_2$, satisfy local commutation relations in the vacuum state, which is also emphasized [Kö11]. However, for modular localized states in the n-particle space $\mathcal{H}_n$ for $n > 1$, local commutativity for a field operator creating such states from the vacuum is not guaranteed in states different from the vacuum, unlike the correspondence between one-particle states and field operators presented in section 2.2.1.

Contrary to the statement made in [MSY06][Lemma 6.3], part of the problems the intertwiner defined in eq. (2.2.20) causes in terms of the locality of two-particle observables (cf. Definition 3) can be seen by evaluating it at coinciding momenta:

Lemma 3 (Singularity of two-particle intertwiners)**.** *For $p \in \partial V^+$, the two-particle intertwiner $u_2(p,p)$ is not an element of $L^2(\kappa S^1)$.*

Proof. For $p = \tilde{p}$, the expression A, which occurs in 2.2.20, reduces to

$$\begin{aligned} A(p,p,z,\tilde{z}) = \xi(z)\xi(\tilde{z}) &\overset{(2.1.10)}{=} \frac{1}{2}\mathrm{Tr}\underset{\smile}{\xi}(z)\overset{\frown}{\xi}(\tilde{z}) \overset{(2.2.11)}{=} \frac{1}{2}\mathrm{Tr}\begin{pmatrix} 1 & -\overline{z} \\ -z & |z|^2 \end{pmatrix}\begin{pmatrix} |\tilde{z}|^2 & \overline{\tilde{z}} \\ \tilde{z} & 1 \end{pmatrix} \\ &= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} |\tilde{z}|^2 - \overline{z}\tilde{z} & \overline{\tilde{z} - z} \\ z\tilde{z}\overline{z - \tilde{z}} & |z|^2 - z\overline{\tilde{z}} \end{pmatrix} = \frac{|z-\tilde{z}|^2}{2} . \end{aligned}$$

Since this expression is invariant to simultaneous translations in $z, \tilde{z}$, the Fourier-transform in 2.2.20 becomes singular:

$$
\begin{aligned}
u_2(p,p)(k,\tilde{k}) &= \int \mathrm{d}^2 z\, \mathrm{e}^{\mathrm{i}kz} \int \mathrm{d}^2 \tilde{z}\, \mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}} F(|z-\tilde{z}|^2/2) \\
&= \int \mathrm{d}^2 z\, \mathrm{e}^{\mathrm{i}(k+\tilde{k})z} \underbrace{\int \mathrm{d}^2 \tilde{z}\, \mathrm{e}^{\mathrm{i}\tilde{k}\tilde{z}} F(|\tilde{z}|^2/2)}_{=:c} \\
&= c(2\pi)^2 \delta(k+\tilde{k}) \text{ (in the sense of distributions)}
\end{aligned}
$$

which means that the integral in u_2 diverges whenever $k = -\tilde{k}$. Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\underline{H} := L^2(\mathbb{R}^2, \mathrm{d}\nu)^{\otimes 2}$ and a real-valued function $0 \neq f \in \underline{H}$. Using polar coordinates for the variables $k, \tilde{k}$; one obtains:

$$
\begin{aligned}
\langle f, u_2(p,p) \rangle &= c(2\pi)^2 \int \mathrm{d}^2 k\, \delta(k^2 - \kappa^2) \int \mathrm{d}^2 \tilde{k}\, \delta(\tilde{k}^2 - \kappa^2) \delta(k + \tilde{k}) f(k, k') \\
&= \frac{c(2\pi)^2}{\kappa^2} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\tilde{\varphi}\, \delta(\kappa(\cos\varphi - \cos\tilde{\varphi})) \delta(\kappa(\sin\varphi - \sin\tilde{\varphi})) f(\varphi, \tilde{\varphi})
\end{aligned}
$$

If also $f(\varphi, \tilde{\varphi}) > 0$ is assumed on a neighborhood of the diagonal, this leads to a divergence of the scalar product. By the Cauchy-Schwarz inequality, $||u_2(p,p)||$ has to diverge as well. $\square$

2.3 Some Results on Lebesgue Spaces and Complex Analysis

This section is used to collect various mathematical theorems which are used throughout the discussion and which can be found in the cited literature. Where appropriate, the notation used for their statements is slightly adapted to fit the use case at hand.

In order to state the Lebesgue Differentiation Theorem in a form that is useful for the intended application, the following definition is helpful:

Definition 2 (Set of integration variables)**.** The set $Q := \mathbb{R}^2 \times \kappa S^1 \times \partial V^+ \times \kappa S^1$ captures all variables on which the two-particle states defined in eq. (2.2.23) depend, except for the p_--component (cf. (2.1.9)) of $p \in \partial V^+$. Accordingly, a point $q_0 \in Q$ is denoted in the form $q_0 := (\mathrm{p}_0, k_0, p_0, \tilde{k}_0) \in Q$. For such a point and $\epsilon > 0$ define the sets

$$
\begin{aligned}
B^1_\epsilon(\mathrm{p}_0, k_0) : &= \{\mathrm{p} \in \mathbb{R}^2 : ||\mathrm{p} - \mathrm{p}_0||_{\mathbb{R}^2} < \epsilon\} \\
&\quad \times \{k \in \kappa S^1 : d_{\kappa S^1}(k, k_0) < \epsilon\} \\
B^2_\epsilon(\tilde{p}_0, \tilde{k}_0) : &= \{\tilde{p} \in \partial V^+ : |\tilde{p}_- - \tilde{p}_{0-}| < \epsilon,\, ||\tilde{\mathrm{p}} - \tilde{\mathrm{p}}_0|| < \epsilon\} \\
&\quad \times \{\tilde{k} \in \kappa S^1 : d_{\kappa S^1}(\tilde{k} - \tilde{k}_0) < \epsilon\} \\
B_\epsilon(q_0) : &= B^1_\epsilon(\mathrm{p}_0, k_0) \times B^2_\epsilon(\tilde{p}_0, \tilde{k}_0)
\end{aligned}
\tag{2.3.1}
$$

with the Euclidean norm $||\cdot||_{\mathbb{R}^2}$ on $\mathbb{R}^2$ and the distance $d_{\kappa S^1}$ on κS^1 which is given by the angular position, i.e.

$$
d_{\kappa S^1}\left(\kappa \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \kappa \begin{pmatrix} \cos\tilde{\alpha} \\ \sin\tilde{\alpha} \end{pmatrix}\right) := |\alpha - \tilde{\alpha}|,
$$

an assignment that becomes well-defined by demanding that $\alpha, \tilde{\alpha} \in \mathbb{R}$ are chosen in such a way that $d_{\kappa S^1}$ becomes minimal for the given points on κS^1.

Theorem 2 (Lebesgue Differentiation Theorem)**.** *Let $\tilde{f}$ be a locally Lebesgue integrable function on the set Q, i.e. $\underline{\tilde{f}} \in L^1(Q)$ with the equivalence class*

$$\underline{\tilde{f}} = \{\tilde{g} : Q \to \mathbb{C} : |\{x \in Q | \tilde{f}(x) \neq \tilde{g}(x)\}| = 0\}.$$

1. *The points $q_0 \in Q$, where the following integral converges to zero, are called Lebesgue points. The set of these points is denoted by L:*

$$q_0 \in L \Leftrightarrow \lim_{\epsilon \to 0} \int \mathrm{d}q \, \frac{\chi_{B_\epsilon(q_0)}(q)}{|B_\epsilon(q_0)|} |\tilde{f}(q) - \tilde{f}(q_0)| = 0$$

Then almost all points in Q are Lebesgue Points: $\mu(Q \backslash L) = 0$.

2. *If $q_0 \in L$ is a Lebesgue point, the value $\tilde{f}(q_0)$ can be obtained from the limit of the average of $\tilde{f}$ over $B_\epsilon(q_0)$ for $\epsilon \to 0$:*

$$q_0 \in L \Rightarrow \lim_{\epsilon \to 0} \frac{1}{|B_\epsilon(q_0)|} \int \mathrm{d}q \, \chi_{B_\epsilon(q_0)}(q) \tilde{f}(q) = \tilde{f}(q_0)$$

Proof. 1. The corresponding statement with open balls instead of the more general sets $B_\epsilon(q_0)$ is known as *Lebesgue's Theorem*. By the remarks following its proof in [Tes13, Theorem 9.6], the sets $B_\epsilon(q_0)$ are admissible as well. See also [Rud87].

2. Convergence of the sequence of averages is a straightforward consequence of the previous statement:

$$\begin{aligned} \left| \left(\int \mathrm{d}q \, \frac{\chi_{B_\epsilon(q_0)}(q)}{|B_\epsilon(q_0)|} \tilde{f}(q) \right) - \tilde{f}(q_0) \right| &= \left| \int \mathrm{d}q \, \frac{\chi_{B_\epsilon(q_0)}}{|B_\epsilon(q_0)|}(q) \left(\tilde{f}(q) - \tilde{f}(q_0) \right) \right| \\ &\leq \int \mathrm{d}q \, \frac{\chi_{B_\epsilon(q_0)}(q)}{|B_\epsilon(q_0)|} \left| \tilde{f}(q) - \tilde{f}(q_0) \right| \end{aligned}$$

□

Lemma 4 (Bros-Epstein-Glaser Lemma)**.** *Let Γ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in \mathcal{S}'(\mathbb{R}^n)$ a tempered distribution with support in $\overline{\Gamma}$. Then there exists a polynomially bounded continuous function G with* $\operatorname{supp} G \subseteq \overline{\Gamma}$ *and a partial differential operator D such that $T = DG$. [RS75, Thm. IX.15]*

Only one direction of the following theorem is needed, however the full statement is found in [RS75, Thm. IX.16].

Theorem 3 (Reed-Simon Theorem on the Fourier transform for tempered distributions)**.** *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ be supported in the cone $\overline{\Gamma}_{a,\theta}$ with apex $a \in \mathbb{R}^n$ and opening angle $0 < \theta < \frac{\pi}{2}$. Then the distributional Fourier transform $\hat{T}$ is the boundary value in the sense of $\mathcal{S}'(\mathbb{R}^n)$ of a function $\hat{T}(s + \mathrm{i}t)$ in the tube $\mathbb{R}^n + \mathrm{i}\Gamma_{a,\frac{\pi}{2}-\theta}$ and satisfies the estimate*

$$|\hat{T}(s + \mathrm{i}t)| \leq |P(s + \mathrm{i}t)|(1 + (\operatorname{dist}(t, \partial\overline{\Gamma_{a,\frac{\pi}{2}-\theta}}))^{-N})$$

for a suitable polynomial P and positive integer N with dist *denoting the Euclidean distance.*

Chapter 3

A No-Go Theorem for Compact Localization in a Class of Infinite Spin Observables

3.1 Assumptions and Statement of the Theorem

3.1.1 Definition of two-particle observables

Definition 3 (Two-particle observable)**.** An operator-valued distribution B on $\mathcal{S}(\mathbb{M}^{\times 2})$ defined by

$$B(g) = \int \widetilde{\mathrm{d}p} \int \widetilde{\mathrm{d}\tilde{p}} \int \mathrm{d}\nu(k) \int \mathrm{d}\nu(\tilde{k})\, \hat{g}(p,\tilde{p}) u_2(p,\tilde{p})(k,\tilde{k}) a^\dagger(p,k) a^\dagger(\tilde{p},\tilde{k}) \\ + \hat{g}(-p,-\tilde{p}) u_{2c}(p,\tilde{p})(k,\tilde{k}) a(p,k) a(\tilde{p},\tilde{k}) \\ + \hat{g}(p,-\tilde{p}) u_0(p,\tilde{p})(k,\tilde{k}) a^\dagger(p,k) a(\tilde{p},\tilde{k}) \\ + \hat{g}(-p,\tilde{p}) u_{0c}(p,\tilde{p})(k,\tilde{k}) a^\dagger(\tilde{p},\tilde{k}) a(p,k)\ \forall\, g \in \mathcal{S}(\mathbb{M}^{\times 2}) \tag{3.1.1}$$

with fixed coefficient functions u_2, u_{2c}, u_0, u_{0c} is called a *Two-particle observable* if the following conditions are met: (cf. Axioms in [SW64, Chapter 3])

I **Domain and Continuity**
For all $g \in \mathcal{S}(\mathbb{M}^{\times 2})$, $B(g)$ is defined on the domain $\mathcal{D}$ of vectors which is generated by products of the string-fields $\Phi(f,h)$ (cf. eq. 2.2.17). The latter enjoy the Reeh-Schlieder property, which is shown in [MSY06, Thm. 3.3], hence $\mathcal{D}$ is dense in the Fock space $\mathcal{H}$.

For fixed vectors $\phi, \psi \in \mathcal{D}$, the map

$$g \in \mathcal{S}(\mathbb{M}^{\times 2}) \mapsto \langle \phi, B(g)\psi \rangle \in \mathbb{C}$$

is a tempered distribution. One says that the assignment $g \mapsto B(g)$ is an *operator-valued distribution.*

The coefficient functions, which occur in eq. (3.1.1), are locally square integrable,

$$u_2, u_{2c}, u_0, u_{0c} \in L^2_{\mathrm{loc}}((\partial V^+)^{\times 2}) \otimes \mathcal{H}_q^{\otimes 2}, \tag{3.1.2}$$

and polynomially bounded (cf. Lemma 2).

II **Transformation Law**
For $p, \tilde{p} \in \partial V^+$ and $A \in \mathrm{SL}(2,\mathbb{C})$, the two-particle intertwiner equation (cf. Lemma lemma 1) holds almost everywhere in the sense of the product measure $\widetilde{\mathrm{d}p}\widetilde{\mathrm{d}\tilde{p}}\mathrm{d}\nu(k)\mathrm{d}\nu(\tilde{k})$:

$$D(R(A,p)) \otimes D(R(A,\tilde{p}))u_2(p\Lambda(A), \tilde{p}\Lambda(A)) = u_2(p,\tilde{p}) \tag{3.1.3}$$

Note that this equation has only been assumed for u_2, which will be discussed in Remark 5.

For fixed vectors $\phi, \psi \in \mathcal{D}$, B is also assumed to satisfy

$$\langle B(g)\varphi, \psi\rangle = \langle \varphi, B(\overline{g})\psi\rangle \ \ \forall g \in \mathcal{S}(\mathbb{M}^{\times 2}). \tag{3.1.4}$$

In particular, $B(g)$ is a symmetric operator, whenever g is a real-valued test function.

III **Relative locality**
If the smearing functions $f \in \mathcal{S}(\mathbb{M})$, $g \in \mathcal{S}(\mathbb{M}^{\times 2})$ and $h \in \mathcal{D}(H)$ are chosen in such a way that the smeared string given by f, h is spacelike separated from the region given by g, i.e

$$(x + \lambda e - y_{1,2})^2 < 0 \ \forall \ x \in \operatorname{supp} f, e \in \operatorname{supp} h, \lambda \in \mathbb{R}^+, (y_1, y_2) \in \operatorname{supp} g,$$

then the associated fields commute:

$$[\Phi(f,h), B(g)] = 0 \tag{3.1.5}$$

Remark 5. $B(g)$ is in particular defined on the vacuum vector Ω, hence $B(g)\Omega$ yields a two-particle wave function given by

$$(p,k,\tilde{p},\tilde{k}) \mapsto \hat{g}(p,\tilde{p})u_2(p,\tilde{p})(k,\tilde{k}) \tag{3.1.6}$$

Part II of Definition 3 appears as the most restrictive assumption. It has been chosen to be compatible with the candidates for observables with compact localization from [MSY06] as well as [Sch08], but assuming only those properties that have been used in the proof of [MSY06][Prop. 6.4], which establishes modular localization of the vector $B(g)\Omega$ from the intertwiner property of u_2, the only coefficient function from Definition 3 appearing in eq. (3.1.6). [1]

It should be noted that, while modular localization of $B(g)\Omega$ in $\operatorname{supp} g$ is a necessary requirement for $B(g)$ to be localized in the same region as an operator, as is pointed out in the same work, it is not clear that all compactly modular localized two-particle vectors have to be of the above form.

The notion of localization for a smearing function $g \in \mathcal{S}(\mathbb{M}^{\times 2})$ implied by part III of Definition 3 can be restated in the form that $B(g)$ is assumed to be localized in a causally closed region $\mathcal{O} = \mathcal{O}'' \subset \mathbb{M}$ if $\operatorname{supp} g \subset \mathcal{O} \times \mathcal{O}$.

[1] The idea to decompose more general operators into a possibly infinite sum of products of creation and annihilation operators is also referred to as the Araki expansion, see for example [Ara63] and more recently [BC13].

An immediate consequence of the intertwiner property of u_2 is the following lemma:

Lemma 5 (Covariance of two-particle states)**.** *Let $g \in \mathcal{S}(\mathbb{M}^{\times 2})$. The two-particle state which $B(g)$ creates from the vacuum Ω transforms in the following way:*

$$U(A,a)B(g)\Omega = B(g_{(\Lambda(A),a)})\Omega \tag{3.1.7}$$

Proof. Let $\varphi_2 \in \mathcal{H}_2$. In the following calculations, the integrals over $p, \tilde{p}, k, \tilde{k}$ in the sense of eq. (3.1.1) are omitted in the notation. It follows

$$\begin{aligned}
\langle \varphi_2 | B(f)\Omega\rangle &\overset{(3.1.1)}{=} \overline{\varphi_2(p,\tilde{p})(k,\tilde{k})}\hat{f}(p,\tilde{p})u_2(p,\tilde{p})(k,\tilde{k}) \\
\Rightarrow \langle \varphi_2 | U(A,a)B(f)\Omega\rangle &\overset{(2.1.20)}{=} \overline{\varphi_2(p,\tilde{p})(k,\tilde{k})}\mathrm{e}^{\mathrm{i}(p+\tilde{p})a}\hat{f}(p\Lambda(A),\tilde{p}\Lambda(A)) \\
&\qquad [D(R(A,p)) \otimes D(R(A,\tilde{p}))u_2(p\Lambda(A),\tilde{p}\Lambda(A))](k,\tilde{k}) \\
&\overset{(2.2.15)/(3.1.3)}{=} \overline{\varphi_2(p,\tilde{p})(k,\tilde{k})}\hat{f}_{(\Lambda(A),a)}u_2(p,\tilde{p})(k,\tilde{k}) = \langle \varphi_2 | B(f_{(\Lambda(A),a)})\Omega\rangle
\end{aligned}$$

Since φ_2 was arbitrary, the claim follows. □

3.1.2 Statement of the theorem

Theorem 4 (No-Go Theorem for two-particle observables)**.** *Let B be a* Two-particle observable *in the sense of Definition 3. Then $B(g) = 0$ for all $g \in \mathcal{S}(\mathbb{M}^{\times 2})$.*

We first give a brief summary of the proof strategy:

The non-trivial part of the commutator in assumption III of Definition 3 is encoded in particular in its transition matrix element from Ω to the one-particle subspace $\mathcal{H}_1 \subset \mathcal{F}$. Applying a suitable one-parameter subgroup of the translations to $\Phi(f,h)$ turns this matrix element into a function of the translation parameter, which is polynomially bounded as a consequence of the string-fields being operator-valued tempered distributions and has half-sided support due to assumption III. Its distributional Fourier transform is then the $\mathcal{S}'$-boundary value of a function which is analytic in the upper half-plane.

On the other hand, using the Reeh-Schlieder Theorem (cf. Theorem 1) together with assumption III allows an extension of assumption II about the intertwining properties of the function u_2 to an analogous statement about u_0 and u_{0c} in Lemma 8. These properties imply that the analytic continuations of these functions in the components of $p, \tilde{p}$ have essential singularities located both in the upper and lower half-plane.

Only a vanishing of the matrix element can simultaneously fulfill the discussed analyticity- and singularity structure, which in turn implies that $B(g)$ itself has to vanish as an operator.

Remark 6. It should be noted that the singularities which are shown to imply the vanishing of the operator $B(g)$ are a consequence of the two-particle intertwiner equation eq. (3.1.3) from part II of Definition 3. Making this symmetry a part of the definition of the two-particle observables considered in the present thesis is motivated by the fact that the corresponding two-particle states (cf. Lemma 2) are at present the only known candidates for modular localization in a compact region for an infinite spin representation. The

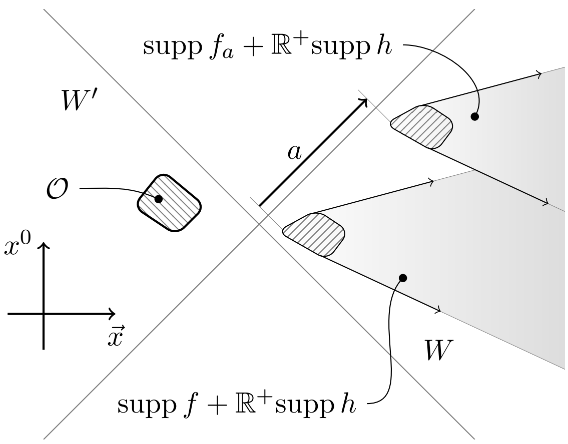


Figure 3.1.1: Relative locality between a compactly localized observable and a string-field

fact that this restriction is the only part of the argument that explicitly requires compact localization demonstrates that only the spacelike separation between the localization region defined by g and the region in which the string-field is localized is relevant for part III of Definition 3. For example, if the truncated cone defined by the smearing functions for the string-field is included in a wedge W, $B(g) = 0$ can be shown for g localized in the causal complement W' of this wedge.

3.1.3 An overview of the proof

In the proof that only $B(g) = 0$ is a solution to the simultaneous requirements of covariance (as expressed by the intertwiner equation (3.1.3)) and relative locality (cf. eq. (3.1.5)) the central object of consideration is the commutator between a string-field $\Phi(f, h)$, where f and h, the smearing functions for the apex and the spacelike directions, are chosen such that the smeared string given by these functions is localized in the standard wedge W, and $B(g)$, which is taken to be localized without loss of generality in a region $\mathcal{O}$ defined by the support of the smearing function g (cf. Definition 3) with $\mathcal{O} \subset W'$, the causal complement of W.

Since the string-field $\Phi(f, h)$ is linear in the creation and annihilation operators, while $B(g)$ is quadratic, the commutator is again linear in these operators and it is sufficient to discuss its matrix element between the vacuum vector $|\Omega\rangle$ and an arbitrary one-particle vector $|\phi\rangle \in \mathcal{H}_1$. Denoting by f_a the lightlike translated version of f, where $a > 0$ corresponds to a translation within W, consider the function γ on $\mathbb{R}$ which is given by

$$\gamma(a) = \langle\phi| \, [B(g), \Phi(f_a, h)] \, |\Omega\rangle \, . \tag{3.1.8}$$

Among the properties of this function, which are discussed in Lemma 9, the most important one is an easy consequence from the requirement of relative locality, namely that γ vanishes for $a > 0$ (cf. fig. 3.1.1), i.e. only a translation in the opposite direction will translate the apex into the causal influence region of $B(g)$. This property of half-sided support implies that the Fourier transform $\hat{\gamma}$, given by

$$\hat{\gamma}(z) = \int \mathrm{d}a\, \mathrm{e}^{-\mathrm{i}za} \gamma(a), \tag{3.1.9}$$

can be defined as an analytic function on the open upper half-plane H^+, which is discussed in more detail in Lemma 10. This kind of regularity is at conflict with the form of the two-particle intertwiner which is found in Lemma 7, because the exponential prefactors[2] encountered there reappear in the explicit form of eq. (3.1.8),

$$\begin{aligned} \gamma(a) = \int \mathrm{d}p_-\, \mathrm{e}^{\frac{\mathrm{i}}{2} a p_-} \int \mathrm{d}^2\mathrm{p} \int \mathrm{d}\nu(k)\, \hat{f}(p) \tilde{u}_1(p,h)(k) \\ \int \widetilde{\mathrm{d}\tilde{p}} \int \mathrm{d}\nu(\tilde{k})\, \underbrace{\exp\left(\mathrm{i}k \cdot \left(\overline{\mathrm{p}} - \overline{\tilde{\mathrm{p}}} \frac{p_-}{\tilde{p}_-}\right)^{-1}\right) \exp\left(-\mathrm{i}\tilde{k} \cdot \left(\overline{\tilde{\mathrm{p}}} - \overline{\mathrm{p}} \frac{\tilde{p}_-}{p_-}\right)^{-1}\right) S(p, k, \tilde{p}\tilde{k})}_{=: I(p,k,\tilde{p},\tilde{k})}, \end{aligned} \tag{3.1.10}$$

where the function I contains the dependence of the matrix element on the operator $B(g)$. Only the singular prefactors have been stated explicitly, while the remaining part has been absorbed into the function S. The function I is therefore fixed for the following discussion, but the other parts of the integrand are arbitrary up to the mentioned localization requirements for f and h. The crucial question is whether despite the singularities in the exponential factors there are nontrivial choices for the function I and consequently the observable B such that the Fourier transform $\hat{\gamma}$ actually has the analyticity properties that have previously been derived from relative locality.

In order to illustrate the basic strategy to obtain the negative answer to this question, suppose that the one-particle vector $|\phi\rangle$ and the functions f and h are chosen in a particularly singular way such that the integrals in $\mathrm{p}, k, \tilde{p}, \tilde{k}$ reduce to evaluation of the integrand at certain points and yield the following function for the matrix element of the commutator:

$$\begin{aligned} \gamma_{\mathrm{p}_0,k_0,\tilde{p}_0,\tilde{k}_0}(a) &= \langle \tilde{p}_0, \tilde{k}_0 |\, B(g)\, | f_a, \mathrm{p}_0, k_0 \rangle - \langle f_a, \mathrm{p}_0, k_0 | \otimes \langle \tilde{p}_0, \tilde{k}_0 |\, B(g)\, |\Omega\rangle \\ &= \int \mathrm{d}p_-\, \mathrm{e}^{\frac{\mathrm{i}}{2} a p_-} \hat{f}(p_-, \mathrm{p}_0) \exp\left(\mathrm{i}k_0 \cdot \left(\overline{\mathrm{p}_0} - \overline{\tilde{\mathrm{p}}_0} \frac{p_-}{\tilde{p}_{0-}}\right)^{-1}\right) \\ &\qquad \exp\left(-\mathrm{i}\tilde{k}_0 \cdot \left(\overline{\tilde{\mathrm{p}}_0} - \overline{\mathrm{p}_0} \frac{\tilde{p}_{0-}}{p_-}\right)^{-1}\right) S(p_-, \mathrm{p}_0, \tilde{p}_0, k_0\tilde{k}_0) \end{aligned} \tag{3.1.11}$$

Another effect of the intertwiner equation (3.1.3) that manifests itself in the form of γ is that the function S only depends on the product of k and $\tilde{k}$ but not on the absolute position on the circle that is determined by these variables. Therefore the Fourier transform (cf. eq. (3.1.9)) of this result has a simple relation to the one obtained with $k_0, \tilde{k}_0$ replaced by

[2] The occurrence of singularities from these prefactors in the variable p_- is evident from the fact that the explicit from of the Euclidean scalar products in the exponents are fractions with real polynomials in the denominator which are given by the Minkowski product of p and $\tilde{p}$. Since these are lightlike vectors, almost all choices of $\mathrm{p}, \tilde{p}$ will yield a strictly positive polynomial for $p_- \in \mathbb{R}$, therefore its roots are located in $\mathbb{C}\backslash\mathbb{R}$ and mapped into each other by a reflection across the real axis.

$k_1 := k_0\lambda$ and $\tilde{k}_1 := \tilde{k}_0\lambda^{-1}$ ($\lambda \in \mathrm{SO}(2)$), namely

$$\hat{\gamma}_{\mathrm{p}_1,k_1,\tilde{p}_1,\tilde{k}_1}(z) = \exp\left(\mathrm{i}(k_1 - k_0)\cdot\left(\overline{\mathrm{p}_0} - \overline{\tilde{\mathrm{p}}_0}\frac{2z}{\tilde{p}_{0-}}\right)^{-1}\right) \exp\left(-\mathrm{i}(\tilde{k}_1 - \tilde{k}_0)\cdot\left(\overline{\tilde{\mathrm{p}}_0} - \overline{\mathrm{p}_0}\frac{\tilde{p}_{0-}}{2z}\right)^{-1}\right)\hat{\gamma}_{\mathrm{p}_0,k_0,\tilde{p}_0,\tilde{k}_0}(z). \tag{3.1.12}$$

The essential singularities in z imply that the left hand side of this equation can only be analytic if $\hat{\gamma}_{\mathrm{p}_0,k_0,\tilde{p}_0,\tilde{k}_0}$ vanishes on H^+, which by eq. (3.1.11) implies that S has to vanish everywhere as well. This is equivalent to $B(g) = 0$.

This summary using singular choices for $|\phi\rangle$ and $\Phi(f, h)$ is not mathematically rigorous, but it emphasizes the main idea. In the actual proof, compactly supported functions δ_ϵ of width $\epsilon > 0$ converging weakly to δ-distributions for $\epsilon \to 0$ are chosen around the points $q := (\mathrm{p}, k, \tilde{p}, \tilde{k})$ instead. For $|\phi\rangle$, this can be done because the chosen functions are in particular square-integrable and thus the singular choice of $\tilde{p}, \tilde{k}$ can be approximated by simply choosing suitable vectors $|\phi\rangle$.

However, more work has to be done to construct for fixed $\epsilon > 0$ sequences of smearing functions $(f_{q_0,\epsilon,N}, h_{q_0,\epsilon,N})_{N\in\mathbb{N}}$ such that the terms in the commutator involving matrix elements of the string-field $\Phi(f_{q_0,\epsilon,N}, h_{q_0,\epsilon,N})$ reproduce such a choice for $N \to \infty$ for the variables p and k as well, which is the statement of Lemma 12. The resulting sequence of functions

$$\gamma_{q_0,\epsilon,N}(a) = \int \mathrm{d}p_-\, \mathrm{e}^{\frac{\mathrm{i}}{2}p_- a} K_{q_0,\epsilon,N}(p_-), \tag{3.1.13}$$

with $K_{q_0,\epsilon,N}$ defined such that eq. (3.1.13) reproduces the function γ defined in eq. (3.1.10) with f and h chosen as the sequence elements, has the support properties discussed before and simplifies in a way similar to eq. (3.1.11) because

$$K_{q_0,\epsilon,N}(p_-) \overset{N\to\infty}{\to} \int \mathrm{d}\mu(q)\, \delta_{q_0,\epsilon}(q) c_{q_0}(q) K(p_-, q) \tag{3.1.14}$$

in the sense of L^1, as is discussed in eq. (3.2.28), with c_{q_0} some continuous nonvanishing function, $\mathrm{d}\mu$ the Lebesgue measure for the variables contained in q and a function K which is directly related to the definition of $B(g)$ by (cf. eq. (3.1.10))

$$K(p_-, q) = \frac{1}{p_-}\hat{f}(p) I(p_-, q) \tag{3.1.15}$$

and it is therefore sufficient to show $K = 0$ in the sense of L^1. The function $f \in \mathcal{S}(\mathbb{M})$ is chosen such that $\operatorname{supp} f \in W_0$. Denote by $\tilde{\gamma}_{q_0,\epsilon}$ the function given by eq. (3.1.13) with N sufficiently big, which depends on $\epsilon > 0$ (cf. Definition 5). For $\epsilon \to 0$ it is shown in Lemma 14 that the sequence of holomorphic Fourier transforms (cf. eq. (3.1.9))

$$\hat{\gamma}_{q_0,\epsilon}(z) = \int \mathrm{d}a\, \mathrm{e}^{\mathrm{i}za} \tilde{\gamma}_{q_0,\epsilon}(a) \tag{3.1.16}$$

converges in a suitable sense. In particular, the limiting function is analytic on H^+ again. The discussion of singularities which appear in eq. (3.1.12) in the informal argument is then carried out in Lemma 16: The fact that the difference

$$\hat{\gamma}_{q_1,\epsilon}(z) - P_{q_0,q_1}(z)\hat{\gamma}_{q_0,\epsilon}(z),$$

where P_{q_0,q_1} is a shorthand notation for the essential singularities appearing when q_0 and q_1 are related by the rotation R discussed previously, converges to zero for $z \in \mathbb{R}$, is extended to the result that the same difference vanishes in the limit $\epsilon \to 0$ on H^+. However, a discussion of these singularities, which is hinted in the remarks to eq. (3.1.12), but carried out in greater detail in Lemma 17, shows that the limiting function for $\hat{\gamma}_{q_0\epsilon}$ has to vanish on H^+. Finally, Lemma 18 shows that its boundary value, the function $K(p_-, q_0)$ has to vanish as well, and since q_0 was arbitrary, this concludes the proof.

The actual proof consists of multiple steps: Lemma 6 and Lemma 7 provide the form of the general solutions of the one- and two-particle intertwiner equations, respectively. Lemma 8 is then used to extend the these results to the other coefficient functions which occur in Definition 3. These results are useful to compute the explicit form of the mentioned matrix element.

The geometric situation shown in Figure 3.1.1, in particular the support properties of the functions f, h as well as g imply corresponding support properties for the function γ when the one-parameter group of lightlike translations is applied to the string-field. These are worked out in Lemma 9 and the corresponding properties for of its Fourier transform $\hat{\gamma}$ are discussed in Lemma 10.

The necessary techniques to effectively restrict the integrals in eq. (3.1.10) to an arbitrary point at which the integrand can be evaluated are introduced in Lemma 11 which summarizes some applications of the Lebesgue Differentiation Theorem in a suitable form and Lemma 12 which shows that the choice of the sequence of smearing functions that is required for the definition (3.1.13) is in fact possible. Lemma 13 together with Definition 5 then shows that the limit indicated in eq. (3.1.14) can be achieved. The corresponding behaviour of the Fourier transform $\hat{\gamma}_{q_0,\epsilon}$ in the limit $\epsilon \to 0$ is investigated in Lemma 14 and it is shown that the limit function is analytic on the upper half-plane H^+.

With the help of Lemma 15, which demonstrates how a distributional limit on the real boundary can be extended to H^+ itself, the difference between $\hat{\gamma}$ for two points q_0 and q_1, which are chosen in a suitable way in order to evaluate the singularities in the function I at different positions but lead to the same value of the factor S, can be considered on H^+, which is done in Lemma 16. A crucial step in studying the behaviour of this extension in a neighbourhood of the singularities is discussed in Lemma 17 to conclude that the the functions $\hat{\gamma}_{q_0}$ have to vanish on all of H^+. Finally, Lemma 18 shows that the real boundary value $K(\cdot, q_0)$ has to vanish as well, which by eq. (3.1.15) yields the claimed result.

3.1.4 The structure of intertwiners

The following lemma establishes the general form of a string-localized one-particle intertwiner, in the sense of eq. (2.2.10). Carrying out the construction employed in the proof of [MSY06, Lemma B 3 ii)] constitutes its proof. Like Lemma 7, the purpose of this characterization is to exhibit the analytic properties of the commutator which is going to be studied in the proof of Theorem 4.

It is implicitly assumed that the solutions are continuous functions such that it makes sense to evaluate them at any given point. A slight variation of the proof strategy, taking into account the fact that intertwiners are actually equivalence classes in the sense of Lebesgue integrability, will be applied in the proof of Lemma 7.

Lemma 6 (One-particle intertwiners)**.** *Let $u_1(p,e)(k)$ be a solution of the one-particle intertwiner equation* (2.2.10)*. Then there is a function F_1, defined on the interior of the upper half-plane, such that:*

1. *The intertwiner u_1 is given by*

$$u_1(p,e)(k) = \exp\left(\mathrm{i}k \cdot \frac{\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}}{2pe}\right) F_1(pe). \tag{3.1.17}$$

2. *A choice of the function F_1 can be made in such a way that u_1 is polynomially bounded in p, analytic in e for $\Im(e) \in V^+$ and bounded by an inverse power at the boundary.*

Remark 7. The existence of one-particle intertwiners, as defined in [MSY06, Definition 3.1] and constructed in the same work, already implies that the choice in the second part of Lemma 6 is possible, since they are solutions of eq. (2.2.10) and have the required analyticity and boundedness properties. The proof of the lemma is done in such a way that these properties are captured by a corresponding characterization of the function F_1 in eq. (3.1.17). Uniqueness of the intertwiners up to a function of this form, i.e. depending on the product pe only, is also shown in [MSY06, Theorem 3.3]. The explicit formula given in Lemma 6 is therefore simply a particular way to define the one-particle intertwiner, which will be useful for the proof of Lemma 12.

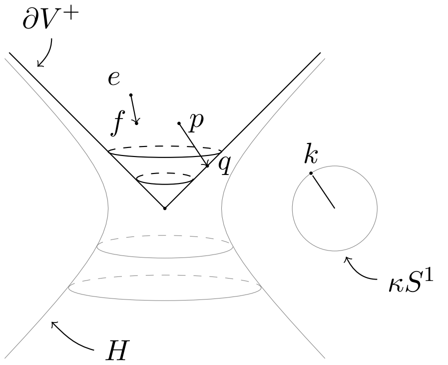
Proof. Ad 1. In the special case $A = B_p^{-1}$, the Wigner rotation $R(A,p)$ becomes trivial

$$R(B_p^{-1}, p) \stackrel{(2.1.21)}{=} B_p B_p^{-1} B^{-1}_{p\Lambda(B_p^{-1})} \stackrel{(2.1.17)}{=} B_q \stackrel{(2.1.18)}{=} \mathbf{1} \tag{3.1.18}$$

and therefore the intertwiner equation (2.2.10) reduces to

$$\begin{aligned} u_1(q, \Lambda(B_p)e) &\stackrel{(2.1.17)}{=} u_1(p\Lambda(B_p^{-1}), \Lambda(B_p)e) \\ &\stackrel{(2.2.10)}{=} D(R(B_p^{-1}, p))^{-1} u_1(p,e) \stackrel{(3.1.18)}{=} u_1(p,e), \end{aligned} \tag{3.1.19}$$

which means that the intertwiner at arbitrary momentum is given by its value at the reference momentum q. The geometric situation is illustrated in fig. 3.1.2. The special



- Lorentz transformation
  $B_p \in \mathrm{SL}(2,\mathbb{C})$ (Wigner boost)
- variable transformations
  $q = p\Lambda(B_p^{-1})$, $f = \Lambda(B_p)e$,
  k invariant
- intertwiner equation
  $u_1(q, \Lambda(B_p)e) = u_1(p, e)$

Figure 3.1.2: Wigner boost transforms the momentum to the reference momentum

case of $A \in G_q$, i.e. $\Lambda(A)$ leaving q invariant, results in

$$R(A,q) \overset{(2.1.21)}{=} B_q A B^{-1}_{q\Lambda(A)} \overset{(2.1.17)}{=} A. \tag{3.1.20}$$

In the following the abbreviation $f := \Lambda(B_p)e$ is used. For $A = [0, \bar{\mathrm{f}}/f_+] \in G_q$ this vector is transformed into the $\mathrm{f} = 0$ plane while remaining, by definition of the little group, in the plane given by $q \cdot e = \mathrm{const}$ as is indicated in fig. 3.1.3:

$$\begin{aligned}(\Lambda([0,\mathrm{f}/f_+])f)_{\backsim} &\overset{(2.1.9)}{=} [0,\mathrm{f}/f_+]\underset{\backsim}{f}[0,\mathrm{f}/f_+]^\dagger \overset{(2.1.16)}{=} \begin{pmatrix} 1 & 0 \\ \mathrm{f}/f_+ & 1 \end{pmatrix} \begin{pmatrix} f_+ & -\bar{\mathrm{f}} \\ -\mathrm{f} & f_- \end{pmatrix} \begin{pmatrix} 1 & \bar{\mathrm{f}}/f_+ \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} f_+ & 0 \\ 0 & -1/f_+ \end{pmatrix} =: \underset{\backsim}{f}^+ \ (\text{since } f_+ f_- - |\mathrm{f}|^2 = f^2 = e^2 = -1).\end{aligned} \tag{3.1.21}$$

Hence intertwiner equation (2.2.10) yields

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}k\cdot\frac{\mathrm{f}}{f_+}} u_1(q,f)(k) &\overset{(2.1.16)}{=} [D([0,\mathrm{f}/f_+])u_1(q,f)](k) \overset{(3.1.20)}{=} [D(R([0,\mathrm{f}/f_+],q))u_1(q,f)](k) \\ &\overset{(2.2.10)}{=} u_1(q,\Lambda([0,\mathrm{f}/f_+])f)(k) \overset{(3.1.21)}{=} u_1\left(q,f^+\right)(k).\end{aligned} \tag{3.1.22}$$

Finally, for $A = [\varphi, 0] \in G_q \cap \mathrm{stab}\, f^+$ for all $\varphi \in \mathbb{R} \bmod 2\pi$, eq. (2.2.10) becomes

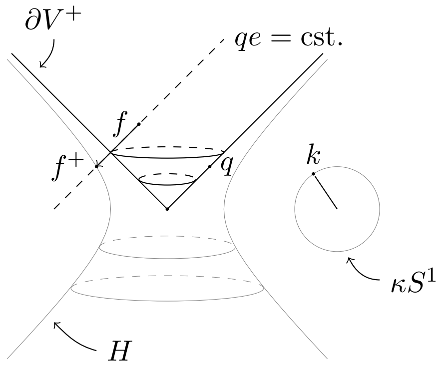
$$\begin{aligned}u_1(q,f^+)(k\lambda(-\varphi)) &\overset{(2.1.16)}{=} D([\varphi,0])u_1(q,f^+)(k) \overset{(3.1.20)}{=} D(R([\varphi,0],q))u_1(q,f^+)(k) \\ &\overset{(2.2.10)}{=} u_1(q,\Lambda([\varphi,0])f^+)(k) = u_1(q,f^+)(k),\end{aligned} \tag{3.1.23}$$

which by choosing φ such that $l := k\lambda(-\varphi)$ is an arbitrary reference point (cf fig. 3.1.4) implies that $u_1(q,f^+)(k)$ is in fact independent of the variable k and the definition

$$F_1(f_+/2) := u_1(q,f^+)(k) \tag{3.1.24}$$

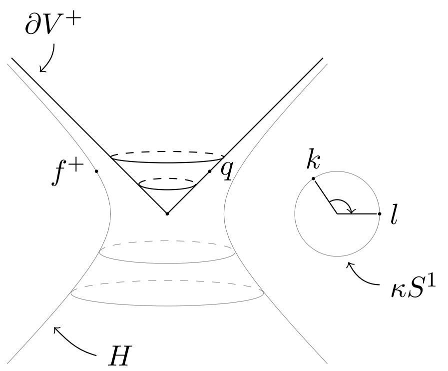
is consistent. Combining this definition with the previous equations yields

$$u_1(p,e) \overset{(3.1.19)}{=} u_1(q, \underbrace{\Lambda(B_p)e}_{=f}) \overset{(3.1.22)}{=} \mathrm{e}^{\mathrm{i}k\cdot\frac{\mathrm{f}}{f_+}} u_1(q,f^+)(k) \overset{(3.1.24)}{=} \mathrm{e}^{\mathrm{i}k\cdot\frac{\mathrm{f}}{f_+}} F_1(f_+) \tag{3.1.25}$$



- Lorentz transformation
  $[0, \bar{\mathrm{f}}/f_+] \in G_q$ (Wigner rotation)
- variable transformations
  q and k invariant
  $f^+ = \Lambda([0, \bar{\mathrm{f}}/f_+])f$
- intertwiner equation
  $\mathrm{e}^{-\mathrm{i}k \cdot \frac{\mathrm{f}}{f_+}} u_1(q, f)(k) = u_1\left(q, f^+\right)(k)$

Figure 3.1.3: Wigner rotation moves the string direction to its reference position



- Lorentz transformation
  $[\varphi, 0] \in G_q \cap \operatorname{stab} f^+$ (rotation)
- variable transformations
  q and f^+ invariant, $l = k\lambda(-\varphi)$
- intertwiner equation
  $u_1(q, f^+)(k\lambda(-\varphi)) = u_1(q, f^+)(k)$

Figure 3.1.4: Application of a rotation to the infinite spin variable

which is translated to the statement of the present lemma by evaluating the substitution before eq. (3.1.21):

$$\begin{pmatrix} f_+ & -\bar{\mathrm{f}} \\ -\mathrm{f} & f_- \end{pmatrix} \overset{(2.1.9)}{=} \underset{\backsim}{f} = (\Lambda(B_p)e) \overset{(2.1.9)}{=} B_p \underset{\backsim}{e} B_p^\dagger \overset{(2.1.18)}{=} \frac{1}{p_-} \begin{pmatrix} p_- & \bar{\mathrm{p}} \\ & 1 \end{pmatrix} \begin{pmatrix} e_+ & -\bar{\mathrm{e}} \\ -\mathrm{e} & e_- \end{pmatrix} \begin{pmatrix} p_- & \\ \mathrm{p} & 1 \end{pmatrix}$$

$$\overset{(2.1.25)}{=} \begin{pmatrix} e_+p_- + e_-p_+ - 2\mathrm{e}\cdot\mathrm{p} & -\bar{\mathrm{e}} + \frac{e_-}{p_-}\bar{\mathrm{p}} \\ -\mathrm{e} + \frac{e_-}{p_-}\mathrm{p} & \frac{e_-}{p_-} \end{pmatrix} \text{ (using } p_+p_- = |\mathrm{p}|^2) \tag{3.1.26}$$

Since $f_+ = 2pe$ by eq. (2.1.10), a combination of eq. (3.1.25) and eq. (3.1.26) yields the claim in the form of eq. (3.1.17).

Ad 2.) For $p \in \partial V^+$ and $e^2 = -1$ with $\Im(e) \in V^+$, the exponential prefactor in eq. (3.1.17) is analytic even for $e \in H^c$, except at $pe = 0$, where it contains an essential singularity. Discussing the possible form of F_1 consistent with the claim therefore amounts to estimating the growth behaviour of this prefactor close to the singularity.

A rotation A in eq. (2.2.10) mapping $\Im(e)$ to a vector whose spatial part is proportional to the 3-axis shows that $\Im(e_1) = \Im(e_2) = 0$ may be assumed without loss of generality: The representation D is unitary and the Wigner rotation appearing there does not depend on e. Combining this with $\Im(e) \in V^+$ and $e_+e_- - |\mathrm{e}|^2 = e^2 = -1$ yields

$$\Im(e_+), \Im(e_-) > 0 \text{ and } \Im(e_+) = -\frac{|\mathrm{e}|^2 - 1}{\Im(e_-)} \Rightarrow |\mathrm{e}|^2 < 1. \tag{3.1.27}$$

The equation $p \cdot e = 0$ is equivalent to $r := \frac{e_-}{p_-}|\mathrm{p}|$ coinciding with one of the roots of

$$\begin{aligned} r^2 - 2(\mathrm{e}\cdot\mathrm{p}^0)r + |\mathrm{e}|^2 - 1 &= \frac{r}{|\mathrm{p}|}\left(e_-\frac{|\mathrm{p}|^2}{p_-} + \frac{|\mathrm{e}|^2-1}{e_-}p_- - 2\mathrm{e}\cdot\mathrm{p}\right) \\ \text{(using } p_+p_- = |\mathrm{p}|^2 \text{ and } e_+e_- = |\mathrm{e}|^2 - 1) \quad &= \frac{r}{|\mathrm{p}|}\left(e_+p_- + e_-p_+ - 2\mathrm{e}\cdot\mathrm{p}\right) \\ &\overset{(2.1.10)}{=} 2pe\frac{r}{|\mathrm{p}|}, \end{aligned}$$

with $\mathrm{p}^0 := \mathrm{p}/|\mathrm{p}|$. These are located at

$$r = \mathrm{e}\cdot\mathrm{p}^0 \pm \sqrt{(\mathrm{e}\cdot\mathrm{p}^0)^2 + 1 - |\mathrm{e}|^2}$$

and these values are bounded by 2, which can be seen from eq. (3.1.27). Hence the following rational function of r, which appears in the exponential of eq. (3.1.17), becomes

$$k \cdot \frac{\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}}{2pe} = \frac{k\cdot\mathrm{e} - (k\cdot\mathrm{p}^0)r}{(r-r_+)(r-r_-)}\frac{r}{|\mathrm{p}|} \tag{3.1.28}$$

and is dominated for $r \to r_\pm$ by $C/(2p\cdot e)$, where C is the maximum of the function

$$\kappa S^1 \times D^1 \times S^1 \to \mathbb{R},\ (k, \mathrm{e}, \mathrm{p}^0) \mapsto k\cdot\mathrm{e} - (k\cdot\mathrm{p}^0)\left(\mathrm{e}\cdot\mathrm{p}^0 \pm \sqrt{(\mathrm{e}\cdot\mathrm{p}^0)^2 + 1 - |\mathrm{e}|^2}\right)$$

To find out the actual value of C, denote the angle between k and e by φ_1, the angle between e and p^0 by φ_2 and $\lambda := |\mathrm{e}|$. C/κ is now the maximum of the function

$$\begin{aligned} &[-\pi,\pi) \times [0,1) \times [-\pi,\pi) \to \mathbb{R} \\ &(\varphi_1, \lambda, \varphi_2) \mapsto \lambda\cos\varphi_1 - \cos(\varphi_2 - \varphi_1)\left(\lambda\cos\varphi_2 \pm \sqrt{1 - \lambda^2\sin^2\varphi_2}\right) \end{aligned} \tag{3.1.29}$$

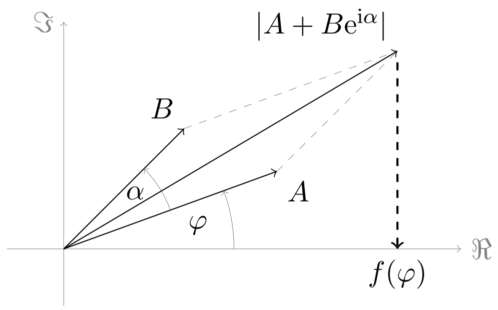


Figure 3.1.5: Geometric interpretation of (3.1.30)

For fixed φ_1, λ, the structure of this function may be conveniently discussed by considering another one-dimensional function

$$f(\varphi) = A\cos\varphi + B\cos(\varphi + \alpha) = \Re\left((A + Be^{i\alpha})e^{i\varphi}\right) \tag{3.1.30}$$

with $A, B, \alpha \in \mathbb{R}$, which has the geometrically intuitive property (cf. fig. 3.1.5)

$$\max_{\varphi} |f(\varphi)|^2 = |A + Be^{i\alpha}|^2 = A^2 + B^2 + 2AB\cos\alpha. \tag{3.1.31}$$

Applying this reasoning to (3.1.29), i.e. setting

$$A := \lambda,\ B := -\left(\lambda\cos\varphi_2 \pm \sqrt{1 - \lambda^2\sin^2\varphi_2}\right) \text{ and } \alpha := -\varphi_2$$

and determining the maximum modulus when varying over φ_1, yields

$$\begin{aligned}
\frac{C^2}{\kappa^2} \overset{(3.1.31)}{=} & \lambda^2 + \left(\lambda\cos\varphi_2 \pm \sqrt{1 - \lambda^2\sin^2\varphi_2}\right)^2 \\
& -2\lambda\left(\lambda\cos\varphi_2 \pm \sqrt{1 - \lambda^2\sin^2\varphi_2}\right)\cos\varphi_2 \\
= & \lambda^2 + \lambda^2\cos^2\varphi_2 \pm 2\lambda\cos\varphi_2\sqrt{1 - \lambda^2\sin^2\varphi_2} + 1 - \lambda^2\sin^2\varphi_2 \\
& -2\lambda^2\cos^2\varphi_2 \mp 2\lambda\cos\varphi_2\sqrt{1 - \lambda^2\sin^2\varphi_2} \\
= & 1 + \lambda^2(1 + \cos^2\varphi_2 - \sin^2\varphi_2 - 2\cos^2\varphi_2) = 1,
\end{aligned}$$

in other words, the values, that the part $k \cdot \left(\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right)$ can attain, range from $-\kappa$ to κ and it is possible to exhaust them by varying the angle between e and p^0.

The space of admissible functions F_1 is thus characterized by the requirement that the phase in eq. (3.1.28) is dominated for $p \cdot e \to 0$ and the boundedness and analyticity assumptions of u_1 in the claim: Defining the function F_{1r} by

$$F_1(p \cdot e) = \exp\left(-i\frac{\kappa}{2pe}\right)F_{1r}(pe), \tag{3.1.32}$$

the boundedness assumptions on u_1 translate to those on F_{1r} in both directions:

Suppose F_{1r} fulfills the boundedness assumptions for u_1. Approaching $pe = 0$ with $\Im(e) \in V^+$ gives $\Im(pe) > 0$ and therefore the prefactor in eq. (3.1.32) leads to a decay.

By the above estimate, this decay cannot be turned into a divergence by the prefactor in (3.1.17) and therefore the assumptions still hold for the corresponding u_1.

Conversely, if a choice u_1 compatible with these assumptions is given, any F_{1r} which is subject only to weaker bounds would give a contradiction for some suitable choice of φ_1, which is possible because $C = \kappa$ was an optimal bound. Phrased in terms of the prefactor in (3.1.32), exhausting the bound will precisely cancel, but not overcompensate its decay at $pe = 0$. $\square$

Remark 8. One example, which constitutes an admissible choice for F_{1r} in the proof of Lemma 6, would be the function given by $F_{1r}(pe) = 1$. By eq. (3.1.17) and eq. (3.1.32), the resulting one-particle intertwiner then has the form

$$u_1(p,e)(k) = \exp\left(\mathrm{i}\frac{k \cdot \left(\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right) - \kappa}{2pe}\right).$$

The way in which the general solution to the one-particle intertwiner equation eq. (2.2.10) has been derived in Lemma 6 can be understood in a geometric way by successively transforming the variables p, e and k into their reference positions. However, it has been implicitly assumed that evaluating the intertwiner u_1 at any given point yields the correct value on the entire orbit of this point with respect to the various Lorentz transformations that have been applied throughout the proof while u_1, as a locally square-integrable function is in fact only defined up to a set of measure zero. In the following lemma a similar reasoning is applied to achieve the analogous result for the solutions u_2 of the two-particle intertwiner equation eq. (3.1.3). In this case the fact that these solutions are only defined up to a null-set has been accounted for by repeatedly applying Lemma 19.

Lemma 7 (General two-particle intertwiners)**.** *Let $u_2(p,\tilde{p})(k,\tilde{k})$ be the function given in Definition 3, which satisfies eq.* (3.1.3)*. Then there is a function $F_2 : \mathbb{R}^2 \to \mathbb{C}$ such that*

$$\begin{aligned} u_2(p,\tilde{p})(k,\tilde{k}) = \exp\left(-\mathrm{i}k \cdot \left(\bar{\mathrm{p}} - \bar{\tilde{\mathrm{p}}}\frac{p_-}{\tilde{p}_-}\right)^{-1}\right) \exp\left(-\mathrm{i}\tilde{k} \cdot \left(\bar{\tilde{\mathrm{p}}} - \bar{\mathrm{p}}\frac{\tilde{p}_-}{p_-}\right)^{-1}\right) \\ F_2\left((k\tilde{k})^{-1}\left(\mathrm{p} - \tilde{\mathrm{p}}\frac{p_-}{\tilde{p}_-}\right)\left(\tilde{\mathrm{p}} - \mathrm{p}\frac{\tilde{p}_-}{p_-}\right)\right) \end{aligned} \tag{3.1.33}$$

in the sense of L^2_{loc}.

Remark 9. As an intermediate consistency check, it may be helpful to verify the number of independent variables that u_2 depends on, according to eq. (3.1.33): As a function of two momenta $p, \tilde{p} \in \partial V^+$ and two continuous spin-variables $k, \tilde{k} \in \kappa S^1$, u_2 depends a priori on 8 real parameters. The intertwiner equation (3.1.3) imposes 6 conditions on u_2, corresponding to the number of generators of $\mathrm{SL}(2,\mathbb{C})$. The remaining freedom in the definition of u_2 under these conditions is just the dependence of F_2 on 2 real coordinates.

Proof of Lemma 7. Step 1: Choice of suitable coordinates
Starting from the coordinates $p = (p_-, \mathrm{p})$ defined in eq. (2.1.9) with the measure defined

in eq. (2.1.14), while k is integrated using the measure $\mathrm{d}\nu(k)$, likewise for the variables $\tilde{p}, \tilde{k}$, one can perform a transformation to more suitable coordinates, given by

$$\begin{aligned} t &:= \ln p_-,\ \alpha := \arg\, k,\ \mathrm{s} := \mathrm{e}^{-\mathrm{i}\alpha}\left(\overline{\tilde{\mathrm{p}}}\frac{p_-}{\tilde{p}_-} - \overline{\mathrm{p}}\right)^{-1}, \\ s_+ &:= 2p\tilde{p} \text{ and } \psi := 2\arg \mathrm{s} + \alpha - \arg \tilde{k} \pm \pi. \end{aligned} \tag{3.1.34}$$

It is our aim now to apply the intertwiner equation in these coordinates and show that if the set of variables $v := (t, \mathrm{p}, \mathrm{s}, \alpha)$ is translated to a reference position in this manner, the phase factors in eq. (3.1.33) arise and the only remaining dependency is on the pair of variables $c := (s_+, \psi)$, captured by the function F_2. It should be noted that both values are encoded in its argument w (cf. eq. (3.1.33)):

$$\kappa^2|w| = \frac{p_-}{\tilde{p}_-}\left|\mathrm{p}\frac{\tilde{p}_-}{p_-} - \tilde{\mathrm{p}}\right|^2 = \mathrm{Tr}\begin{pmatrix} |\mathrm{p}|^2/p_- & -\overline{\mathrm{p}} \\ -\mathrm{p} & p_- \end{pmatrix}\begin{pmatrix} \tilde{p}_- & \overline{\tilde{\mathrm{p}}} \\ \tilde{\mathrm{p}} & |\tilde{\mathrm{p}}|^2/\tilde{p}_- \end{pmatrix} \overset{(2.1.10)}{=} 2p\tilde{p} \overset{(3.1.34)}{=} s_+ \tag{3.1.35}$$

$$\begin{aligned} \arg w &= \arg\left(\tilde{\mathrm{p}} - \mathrm{p}\frac{\tilde{p}_-}{p_-}\right) + \arg\left(\mathrm{p} - \tilde{\mathrm{p}}\frac{p_-}{\tilde{p}_-}\right) - \arg k - \arg \tilde{k} \\ &= (\arg s + \alpha) + (\arg(-s) + \alpha) - \alpha - \arg \tilde{k} = 2\arg s \pm \pi + \alpha - \arg \tilde{k} \overset{(3.1.34)}{=} \psi \end{aligned}$$

The inverse transformation to eq. (3.1.34) is given by

$$\begin{aligned} p_- &= \mathrm{e}^t,\ \tilde{p}_- = |\mathrm{s}|^2 s_+ p_-,\ \tilde{\mathrm{p}} = (\bar{\mathrm{s}}\mathrm{p} + \mathrm{e}^{\mathrm{i}\alpha})\mathrm{s}s_+, \\ k &= \kappa\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix} \text{ and } \tilde{k} = \kappa\begin{pmatrix}\cos\beta & \sin\beta\end{pmatrix}, \text{ where } \beta = 2\arg s - \alpha - \psi \pm \pi. \end{aligned} \tag{3.1.36}$$

Step 2: The measure for the two-particle space in new coordinates

Finally, we determine the measure for integration in the new coordinates, denoted by $\mathrm{d}\rho(v, c)$, using the transformation formula and the fact that eq. (3.1.36) automatically ensures $p_-, \tilde{p_-} \geq 0$ iff $s_+ \geq 0$:

$$\begin{aligned} \mathrm{d}\rho(v, c) &= \widetilde{\mathrm{d}p}\widetilde{\mathrm{d}\tilde{p}}\mathrm{d}\nu(k)\mathrm{d}\nu(\tilde{k}) \overset{(2.1.14)}{=} \Theta(p_-)\frac{\mathrm{d}p_-}{p_-}\mathrm{d}^2\mathrm{p}\Theta(\tilde{p}_-)\frac{\mathrm{d}\tilde{p}_-}{\tilde{p}_-}\mathrm{d}^2\tilde{\mathrm{p}}\mathrm{d}\nu(k)\mathrm{d}\nu(\tilde{k}) \\ &= \frac{1}{p_-\tilde{p}_-}\det\left(\frac{\partial(p_-, \mathrm{p}, \tilde{p}_-, \tilde{\mathrm{p}}, k, \tilde{k})}{\partial(t, \mathrm{p}, s_+, \mathrm{s}, \alpha, \psi)}\right)\Theta(s_+)\mathrm{d}t\mathrm{d}^2\mathrm{p}\mathrm{d}s_+\mathrm{d}^2\mathrm{s}\mathrm{d}\alpha\mathrm{d}\psi \end{aligned} \tag{3.1.37}$$

For the purpose of an easier calculation the variable r is introduced by the following substitution:

$$\mathrm{r} = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{s} \Rightarrow \mathrm{s} = \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{r} \tag{3.1.38}$$

$$\Rightarrow \det\frac{\partial(\mathrm{r}, \alpha)}{\partial(\mathrm{s}, \alpha)} = \underbrace{\det\frac{\partial(\mathrm{r}', \mathrm{r}'')}{\partial(|\mathrm{r}|, \arg \mathrm{r})}}_{=|\mathrm{r}|}\underbrace{\det\frac{\partial(\arg \mathrm{r}, \alpha)}{\partial(\arg \mathrm{s}, \alpha)}}_{=1}\underbrace{\det\frac{\partial(|\mathrm{s}|, \arg \mathrm{s})}{\partial(\mathrm{s}', \mathrm{s}'')}}_{=|\mathrm{s}|^{-1}} = 1 \tag{3.1.39}$$

All dependencies on s in the Jacobian which occurs in eq. (3.1.37) may be expressed in terms of r: Plugging eq. (3.1.38) into eq. (3.1.36) yields

$$\tilde{p}_- = |\mathrm{r}|^2 s_+ p_-,\ \tilde{\mathrm{p}} = (\bar{\mathrm{r}}\mathrm{p} + 1)\mathrm{r}s_+ \text{ and } \beta = 2\arg \mathrm{r} + \alpha - \psi \pm \pi \tag{3.1.40}$$

The Jacobian eq. (3.1.37) can be further simplified, because the form of eq. (3.1.36) with the alterations made in eq. (3.1.40), where p_- depends only on t, p is unchanged, k depends only on α and no variable except $\tilde{k}$ depends on ψ, essentially leads to a block structure. Some nonvanishing elements outside of the block structure are indicated by square brackets, but these do not influence the value of the determinant by Leibniz' formula. Also, the notation $\tilde{\mathrm{p}} = \tilde{\mathrm{p}}' + \mathrm{i}\tilde{\mathrm{p}}''$ and $\mathrm{r} = \mathrm{r}' + \mathrm{i}\mathrm{r}''$ is used:

$$
\begin{aligned}
&\det \frac{\partial(p_-, \mathrm{p}, \tilde{p}_-, \tilde{\mathrm{p}}, k, \tilde{k})}{\partial(t, \mathrm{p}, s_+, \mathrm{s}, \alpha, \psi)} = \det \frac{\partial(p_-, \mathrm{p}, \tilde{p}_-, \tilde{\mathrm{p}}, k, \tilde{k})}{\partial(t, \mathrm{p}, s_+, \mathrm{r}, \alpha, \psi)} \underbrace{\det \frac{\partial(\mathrm{r}, \alpha)}{\partial(\mathrm{s}, \alpha)}}_{\overset{(3.1.39)}{=} 1} \\
&\overset{(3.1.36)/(3.1.40)}{=} \begin{vmatrix}
\frac{\partial p_-}{\partial t} & & & & & & \\
& \mathbf{1}_{\mathbb{R}^2} & & & & & \\
& & \frac{\partial \tilde{p}_-}{\partial s_+} & \frac{\partial \tilde{p}_-}{\partial \mathrm{r}'} & \frac{\partial \tilde{p}_-}{\partial \mathrm{r}''} & & \\
& & \frac{\partial \tilde{\mathrm{p}}'}{\partial s_+} & \frac{\partial \tilde{\mathrm{p}}'}{\partial \mathrm{r}'} & \frac{\partial \tilde{\mathrm{p}}'}{\partial \mathrm{r}''} & & \\
& & \frac{\partial \tilde{\mathrm{p}}''}{\partial s_+} & \frac{\partial \tilde{\mathrm{p}}''}{\partial \mathrm{r}'} & \frac{\partial \tilde{\mathrm{p}}''}{\partial \mathrm{r}''} & & \\
& & & & & \frac{\partial k}{\partial \alpha} & \\
& & & \left[\frac{\partial \tilde{k}}{\partial \mathrm{r}'}\right] & \left[\frac{\partial \tilde{k}}{\partial \mathrm{r}''}\right] & \left[\frac{\partial \tilde{k}}{\partial \alpha}\right] & \frac{\partial \tilde{k}}{\partial \psi}
\end{vmatrix}
= \frac{\partial p_-}{\partial t} \frac{\partial k}{\partial \alpha} \frac{\partial \tilde{k}}{\partial \psi} \underbrace{\begin{vmatrix}
\frac{\partial \tilde{p}_-}{\partial s_+} & \frac{\partial \tilde{p}_-}{\partial \mathrm{r}'} & \frac{\partial \tilde{p}_-}{\partial \mathrm{r}''} \\
\frac{\partial \tilde{\mathrm{p}}'}{\partial s_+} & \frac{\partial \tilde{\mathrm{p}}'}{\partial \mathrm{r}'} & \frac{\partial \tilde{\mathrm{p}}'}{\partial \mathrm{r}''} \\
\frac{\partial \tilde{\mathrm{p}}''}{\partial s_+} & \frac{\partial \tilde{\mathrm{p}}''}{\partial \mathrm{r}'} & \frac{\partial \tilde{\mathrm{p}}''}{\partial \mathrm{r}''}
\end{vmatrix}}_{=:J} \\
&\overset{(3.1.36)/(2.1.23)}{=} p_- \frac{1}{2\kappa} \left(-\frac{1}{2\kappa}\right) J
\end{aligned}
\tag{3.1.41}
$$

We now calculate the remaining Jacobian J.

$$
\begin{aligned}
\frac{1}{s_+^2 p_-} J &\overset{(3.1.40)}{=} \frac{1}{s_+^2 p_-} \begin{vmatrix}
|\mathrm{r}|^2 & 2\mathrm{r}' s_+ & 2\mathrm{r}'' s_+ \\
|\mathrm{r}|^2 \mathrm{p}' + \mathrm{r}' & (2\mathrm{r}'\mathrm{p}' + 1) s_+ & 2\mathrm{r}''\mathrm{p}' s_+ \\
|\mathrm{r}|^2 \mathrm{p}'' + \mathrm{r}'' & 2\mathrm{r}'\mathrm{p}'' s_+ & (2\mathrm{r}''\mathrm{p}'' + 1) s_+
\end{vmatrix} \\
&= \begin{vmatrix}
|\mathrm{r}|^2 & 2\mathrm{r}' & 2\mathrm{r}'' \\
|\mathrm{r}|^2 \mathrm{p}' + \mathrm{r}' & 2\mathrm{r}'\mathrm{p}' + 1 & 2\mathrm{r}''\mathrm{p}' \\
|\mathrm{r}|^2 \mathrm{p}'' + \mathrm{r}'' & 2\mathrm{r}'\mathrm{p}'' & 2\mathrm{r}''\mathrm{p}'' + 1
\end{vmatrix} = \left| \begin{pmatrix} 1 \\ \mathrm{p}' \\ \mathrm{p}'' \end{pmatrix} \begin{pmatrix} |\mathrm{r}|^2 & 2\mathrm{r}' & 2\mathrm{r}'' \end{pmatrix} + \begin{pmatrix} 0 & & \\ \mathrm{r}' & 1 & \\ \mathrm{r}'' & & 1 \end{pmatrix} \right| \\
&= \begin{vmatrix}
|\mathrm{r}|^2 & 2\mathrm{r}' & 2\mathrm{r}'' \\
\mathrm{r}' & 1 & \\
\mathrm{r}'' & & 1
\end{vmatrix} = |\mathrm{r}|^2 - 2\mathrm{r}'^2 - 2\mathrm{r}''^2 = -|\mathrm{r}|^2 \Rightarrow J = -|\mathrm{r}|^2 s_+^2 p_- \overset{(3.1.36)}{=} -s_+ \tilde{p}_-
\end{aligned}
\tag{3.1.42}
$$

In the step to the last line, the only nonvanishing contributions to the determinant come from those terms where the number of contributing rows from each matrix does not exceed its rank. Thus there can be at most one row from the first matrix and two from the second one. Since the first row in the second matrix vanishes, only one combination remains.

Combining equations (3.1.37), (3.1.41) and (3.1.42) yields the final form of the measure for the new coordinates:

$$
\begin{aligned}
\mathrm{d}\rho(v, c) &= \frac{1}{p_- \tilde{p}_-} \left(-\frac{p_-}{4\kappa^2}\right) (-s_+ \tilde{p}_-) \Theta(s_+) \mathrm{d}t \mathrm{d}^2\mathrm{p} \mathrm{d}s_+ \mathrm{d}^2\mathrm{s} \mathrm{d}\alpha \mathrm{d}\psi \\
&= \frac{1}{4\kappa^2} \mathrm{d}t \mathrm{d}^2\mathrm{p} \mathrm{d}^2\mathrm{s} \mathrm{d}\alpha \; \Theta(s_+) s_+ \mathrm{d}s_+ \mathrm{d}\psi
\end{aligned}
\tag{3.1.43}
$$

We will conveniently use the same symbol for the function u_2 in the new coordinates,

$$
u_2(t, \mathrm{p}, \mathrm{s}, \alpha, s_+, \psi) = u_2(p, \tilde{p})(k, \tilde{k}),
\tag{3.1.44}
$$

where we have implicitly chosen an element of the equivalence class in L^2_{loc} given by u_2.
Step 3: Lorentz transformation to the reference point
The strategy is now as follows: For each component of $v = (t, \mathrm{p}, \mathrm{s}, \alpha)$, starting from t and proceeding in the stated order, the following steps are performed:

- Construct a Lorentz transformation which acts as a translation on that component, by virtue of the intertwiner equation (3.1.3). For the variable s, this equation also introduces phase factors.

- Using this result, the value of the function u_2 for an arbitrary value of the component is related to its value at some reference value of the component, for almost every value of the remaining components.

- Apply Lemma 19 to deduce that there is an L^2_{loc}-equivalent element, for which the previous statement is true everywhere.

Once this is done, the function u_2 will be determined by its value at $v = 0$ and the resulting form turns out to be precisely the statement of the present lemma.

Step 3.1: Lorentz boost for the variable t

We begin with a 1-parameter group of Lorentz boosts given by $A_1(\Delta_1) := \mathrm{e}^{\sigma_3\Delta_1/2}$ and evaluate its action on the original variables. Only those variables, whose value changes in the process, are mentioned explicitly.

$$\begin{aligned}(p\Lambda(A_1(\Delta_1)))^{\backsim} &\overset{(2.1.9)}{=} \begin{pmatrix} \mathrm{e}^{\Delta_1/2} & \\ & \mathrm{e}^{-\Delta_1/2}\end{pmatrix} \begin{pmatrix} p_- & \overline{\mathrm{p}} \\ \mathrm{p} & |\mathrm{p}|^2/p_-\end{pmatrix} \begin{pmatrix} \mathrm{e}^{\Delta_1/2} & \\ & \mathrm{e}^{-\Delta_1/2}\end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{\Delta_1}p_- & \overline{\mathrm{p}} \\ \mathrm{p} & |\mathrm{p}|^2/\mathrm{e}^{\Delta_1}p_-\end{pmatrix} \Rightarrow p_- \mapsto \mathrm{e}^{\Delta_1}p_- \end{aligned} \tag{3.1.45}$$

If follows analogously that $\tilde{p}_- \mapsto \mathrm{e}^{\Delta_1}\tilde{p}_-$. In order to determine the little group action on $k, \tilde{k}$ as well as possible phase factors, the resulting Wigner rotations, as introduced in eq. (2.1.20), have to be calculated:

$$\begin{aligned} R(A_1(\Delta_1), p) &= B_p A_1(\Delta_1) B^{-1}_{p\Lambda(A_1(\Delta_1))} \\ &\overset{(2.1.18)}{=} \frac{1}{\mathrm{e}^{\Delta_1/2}p_-}\begin{pmatrix} p_- & \overline{\mathrm{p}} \\ & 1\end{pmatrix}\begin{pmatrix} \mathrm{e}^{\Delta_1/2} & \\ & \mathrm{e}^{-\Delta_1/2}\end{pmatrix}\begin{pmatrix} 1 & -\overline{\mathrm{p}} \\ & \mathrm{e}^{\Delta_1}p_-\end{pmatrix} = \mathbf{1}\end{aligned}$$

Again, an analogous calculation yields $R(A_1(\Delta_1), \tilde{p}) = \mathbf{1}$. Using the intertwiner equation (3.1.3) and the identification (3.1.44) these results imply

$$\begin{aligned} u_2(t, \mathrm{p}, \mathrm{s}, \alpha, s_+, \psi) &\overset{(3.1.44)}{=} u_2(p, \tilde{p})(k, \tilde{k}) \\ &\overset{(3.1.3)}{=} [D(R(A_1(\Delta_1), p)) \otimes D(R(A_1(\Delta_1), \tilde{p}))u_2(p\Lambda(A_1(\Delta_1)), \tilde{p}\Lambda(A_1(\Delta_1))](k, \tilde{k}) \\ &\overset{(3.1.44)}{=} u_2(t + \Delta_1, \mathrm{p}, \mathrm{s}, \alpha, s_+, \psi)\ \rho\text{-almost everywhere.}\end{aligned}$$

In the last step it has been used that our results for the transformation of $p_-, \tilde{p}_-$ translate via eq. (3.1.34) to $t \mapsto t + \Delta_1$ while s is invariant, since it does not depend on $p_-, \tilde{p}_-$ individually, but only on their ratio,

$$\mathrm{s} \overset{(3.1.34)}{=} \mathrm{e}^{-\mathrm{i}\alpha} \left(\overline{\tilde{\mathrm{p}}} \frac{p_-}{\tilde{p}_-} - \overline{\mathrm{p}} \right)^{-1} \overset{(3.1.45)}{\mapsto} \mathrm{e}^{-\mathrm{i}\alpha} \left(\overline{\tilde{\mathrm{p}}} \frac{\mathrm{e}^{\Delta_1} p_-}{\mathrm{e}^{\Delta_1} \tilde{p}_-} - \overline{\mathrm{p}} \right)^{-1} = \mathrm{s}.$$

s_+ is Lorenz-invariant by construction, a fact that is used also in the following steps. ψ is invariant, because it depends only on s, which has just been shown to be invariant and $k, \tilde{k}$ which are invariant because the Wigner rotations are trivial here.

Lemma 19 now implies that for almost all values of the remaining variables $\mathrm{p}, \mathrm{s}, \alpha, s_+, \psi$,

$$u_2(t, \mathrm{p}, \mathrm{s}, \alpha, s_+, \psi) = u_2(0, \mathrm{p}, \mathrm{s}, \alpha, s_+, \psi), \tag{3.1.46}$$

hence u_2 is even equivalent to a function for which this equation holds everywhere.

Step 3.2: Wigner boost for the variable p

A 2-parameter group of Lorentz transformations is given by

$$\mathbb{C} \ni \Delta_2 \mapsto A_2(\Delta_2) := \begin{pmatrix} 1 & \overline{\Delta_2} \\ & 1 \end{pmatrix},$$

but its action on the variables has to be evaluated only at $t = 0$:

$$\begin{aligned}
(p\Lambda(A_2(\Delta_2)))^{\backsim} &\overset{(2.1.9)}{=} \begin{pmatrix} 1 & \\ \Delta_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \overline{\mathrm{p}} \\ \mathrm{p} & |\mathrm{p}|^2 \end{pmatrix} \begin{pmatrix} 1 & \overline{\Delta_2} \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \overline{\mathrm{p} + \Delta_2} \\ \mathrm{p} + \Delta_2 & |\mathrm{p} + \Delta_2|^2 \end{pmatrix} \\
&\Rightarrow \mathrm{p} \mapsto \mathrm{p} + \Delta_2 \\
(\tilde{p}\Lambda(A_2(\Delta_2)))^{\backsim} &\overset{(2.1.9)}{=} \begin{pmatrix} 1 & \\ \Delta_2 & 1 \end{pmatrix} \begin{pmatrix} \tilde{p}_- & \overline{\tilde{\mathrm{p}}} \\ \tilde{\mathrm{p}} & |\tilde{\mathrm{p}}|^2/\tilde{p}_- \end{pmatrix} \begin{pmatrix} 1 & \overline{\Delta_2} \\ & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \\ \Delta_2 & 1 \end{pmatrix} \begin{pmatrix} \tilde{p}_- & \tilde{p}_- \overline{\Delta_2} + \overline{\tilde{\mathrm{p}}} \\ \tilde{\mathrm{p}} & \tilde{\mathrm{p}} \overline{\Delta_2} + |\tilde{\mathrm{p}}|^2/\tilde{p}_- \end{pmatrix} \\
&= \begin{pmatrix} \tilde{p}_- & \overline{\tilde{\mathrm{p}} + \tilde{p}_- \Delta_2} \\ \tilde{\mathrm{p}} + \tilde{p}_- \Delta_2 & |\tilde{\mathrm{p}} + \tilde{p}_- \Delta_2|^2/\tilde{p}_- \end{pmatrix} \\
&\Rightarrow \tilde{\mathrm{p}} \mapsto \tilde{\mathrm{p}} + \tilde{p}_- \Delta_2
\end{aligned} \tag{3.1.47}$$

Just like in the previous step, the Wigner rotations are trivial:

$$\begin{aligned}
R(A_2(\Delta_2), p) &\overset{(2.1.18)}{=} \begin{pmatrix} 1 & \overline{\mathrm{p}} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \overline{\Delta_2} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -\overline{(\mathrm{p} + \Delta_2)} \\ & 1 \end{pmatrix} = \mathbf{1} \\
R(A_2(\Delta_2), \tilde{p}) &\overset{(2.1.18)}{=} \frac{1}{\tilde{p}_-} \begin{pmatrix} p_- & \overline{\tilde{\mathrm{p}}} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \overline{\Delta_2} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -\overline{(\tilde{\mathrm{p}} + \tilde{p}_- \Delta_2)} \\ & p_- \end{pmatrix} = \mathbf{1}
\end{aligned}$$

This time, the intertwiner equation (3.1.3) gives

$$u_2(0, \mathrm{p}, \mathrm{s}, \alpha, s_+, \psi) = u_2(0, \mathrm{p} + \Delta_2, \mathrm{s}, \alpha, s_+, \psi),$$

where evaluation at $t=0$ is justified by the equivalence of u_2 to a function which is t-invariant, which has been the conclusion of the previous step. We convince ourselves that s has been left unchanged in this step as well:

$$\mathrm{s} \overset{(3.1.34)}{=} \mathrm{e}^{-\mathrm{i}\alpha}\left(\overline{\tilde{\mathrm{p}}}\frac{1}{\tilde{p}_-}-\overline{\mathrm{p}}\right)^{-1} \overset{(3.1.47)}{\mapsto} \mathrm{e}^{-\mathrm{i}\alpha}\left(\overline{(\tilde{\mathrm{p}}+\tilde{p}_-\Delta_2)}\frac{1}{\tilde{p}_-}-\overline{(\mathrm{p}+\Delta_2)}\right)^{-1} = \mathrm{s}$$

In order to see that s_+, ψ are invariant, the same reasoning as in the previous step can be applied.

Applying Lemma 19 again, it follows for almost all values of $\mathrm{s}, \alpha, s_+, \psi$,

$$u_2(0,\mathrm{p},\mathrm{s},\alpha,s_+,\psi) = u_2(0,0,\mathrm{s},\alpha,s_+,\psi), \tag{3.1.48}$$

hence u_2 can further be assumed to satisfy this equation everywhere.

Step 3.3: Spatial rotation for the variable α

In this step a 1-parameter group of rotations

$$\mathbb{R} \ni \Delta_3 \mapsto A_3(\Delta_3) := \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Delta_3/2} & \\ & \mathrm{e}^{-\mathrm{i}\Delta_3/2} \end{pmatrix}$$

is used to resolve the dependence on α. Its action may now be evaluated at $t=0$, $\mathrm{p}=0$.

$$\begin{aligned}
(p\Lambda(A_3(\Delta_3)))^{\backsim} &\overset{(2.1.9)}{=} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\Delta_3/2} & \\ & \mathrm{e}^{\mathrm{i}\Delta_3/2} \end{pmatrix}\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Delta_3/2} & \\ & \mathrm{e}^{-\mathrm{i}\Delta_3/2} \end{pmatrix} = \overset{\backsim}{p} \\
(\tilde{p}\Lambda(A_3(\Delta_3)))^{\backsim} &\overset{(2.1.9)}{=} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\Delta_3/2} & \\ & \mathrm{e}^{\mathrm{i}\Delta_3/2} \end{pmatrix}\begin{pmatrix} \tilde{p}_- & \overline{\tilde{\mathrm{p}}} \\ \tilde{\mathrm{p}} & \tilde{p}_+ \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Delta_3/2} & \\ & \mathrm{e}^{-\mathrm{i}\Delta_3/2} \end{pmatrix} \\
&= \begin{pmatrix} \tilde{p}_- & \mathrm{e}^{-\mathrm{i}\Delta_3}\overline{\tilde{\mathrm{p}}} \\ \mathrm{e}^{\mathrm{i}\Delta_3}\tilde{\mathrm{p}} & \tilde{p}_+ \end{pmatrix} \Rightarrow \tilde{\mathrm{p}} \mapsto \mathrm{e}^{\mathrm{i}\Delta_3}\tilde{\mathrm{p}}
\end{aligned} \tag{3.1.49}$$

While the momenta $p, \tilde{p}$ and therefore by eq. (3.1.34) also s are invariant, the Wigner rotations act as pure rotations on $k, \tilde{k}$:

$$\begin{aligned}
R(A_3(\Delta_3), p) &\overset{(2.1.18)}{=} A_3(\Delta_3) = [\Delta_3/2, 0] \\
R(A_3(\Delta_3), \tilde{p}) &\overset{(2.1.18)}{=} \frac{1}{\tilde{p}_-}\begin{pmatrix} \tilde{p}_- & \overline{\tilde{\mathrm{p}}} \\ & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Delta_3/2} & \\ & \mathrm{e}^{-\mathrm{i}\Delta_3/2} \end{pmatrix}\begin{pmatrix} 1 & -\mathrm{e}^{-\mathrm{i}\Delta_3}\overline{\tilde{\mathrm{p}}} \\ & \tilde{p}_- \end{pmatrix} \\
&= \frac{1}{\tilde{p}_-}\begin{pmatrix} \tilde{p}_- & \overline{\tilde{\mathrm{p}}} \\ & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\Delta_3/2} & -\mathrm{e}^{-\mathrm{i}\Delta_3/2}\overline{\tilde{\mathrm{p}}} \\ & \mathrm{e}^{-\mathrm{i}\Delta_3/2}\tilde{p}_- \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Delta_3/2} & \\ & \mathrm{e}^{-\mathrm{i}\Delta_3/2} \end{pmatrix} \\
&\overset{(2.1.16)}{=} [\Delta_3/2, 0]
\end{aligned}$$

By eq. (2.1.24) these Wigner rotations act on the arguments of u_2 as a rotation in

$k, \tilde{k}$, which acts as a translation in α, while leaving s, s_+, ψ invariant:

$$
\begin{aligned}
k \overset{(2.1.24)}{\mapsto} k\lambda(-\Delta_3/2) \Rightarrow \alpha &\overset{(3.1.34)}{=} \arg k \mapsto \alpha + \Delta_3 \\
\Rightarrow \ \mathrm{s} &\overset{(3.1.34)}{=} \mathrm{e}^{-\mathrm{i}\alpha} \frac{\tilde{p}_-}{\overline{\tilde{\mathrm{p}}}} \overset{(3.1.49)}{\mapsto} \mathrm{e}^{-\mathrm{i}(\alpha+\Delta_3)} \frac{\tilde{p}_-}{\overline{\mathrm{e}^{\mathrm{i}\Delta_3}\tilde{\mathrm{p}}}} = \mathrm{s} \\
\tilde{k} \overset{(2.1.24)}{\mapsto} \tilde{k}\lambda(-\Delta_3/2) \Rightarrow \beta &\overset{(3.1.36)}{=} \arg \tilde{k} \mapsto \beta + \Delta_3 \\
\Rightarrow \psi &\overset{(3.1.34)}{=} 2\arg \mathrm{s} + \alpha - \arg \tilde{k} \pm \pi \\
&\mapsto 2\arg \mathrm{s} + (\alpha + \Delta_3) - (\arg \tilde{k} + \Delta_3) \pm \pi = \psi
\end{aligned}
$$

Since λ is a double covering, a factor of 2 occurs in the first line. The rotation angle picks up an extra sign, because in eq. (2.1.24) λ multiplies a row vector from the right.

Now the intertwiner equation eq. (3.1.3) leads to

$$u_2(0, 0, \mathrm{s}, \alpha, s_+, \psi) = u_2(0, 0, \mathrm{s}, \alpha + \Delta_3, s_+, \psi),$$

which by Lemma 19 implies for almost all values of s, s_+, ψ,

$$u_2(0, 0, \mathrm{s}, \alpha, s_+, \psi) = u_2(0, 0, \mathrm{s}, 0, s_+, \psi), \tag{3.1.50}$$

hence it can be assumed that this equation for u_2 holds for all values.

Step 3.4: Wigner rotation for the variable s

In this step the phase factors, which appear in the statement of the present lemma, are deduced from the intertwiner equation. While A_1 and A_2 have the form of Wigner boosts, the groups to reach the remaining variables' reference positions belong to the Little Group.

The relevant 2-parameter group of Lorenz transformations is now given by

$$\mathbb{C} \ni \Delta_3 \mapsto A_3(\Delta_3) := \begin{pmatrix} 1 & \\ \Delta_3 & 1 \end{pmatrix}.$$

The action of this group may now be restricted to $t = 0$, $\mathrm{p} = 0$, as well as $\alpha = 0$,

$$
\begin{aligned}
(p\Lambda(A_3(\Delta_3)))^{\backsim} &\overset{(2.1.9)}{=} \begin{pmatrix} 1 & \overline{\Delta}_3 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ \Delta_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} = \overset{\backsim}{p} \\
(\tilde{p}\Lambda(A_3(\Delta_3)))^{\backsim} &\overset{(2.1.9)}{=} \begin{pmatrix} 1 & \overline{\Delta}_3 \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{p}_- & \overline{\tilde{\mathrm{p}}} \\ \tilde{\mathrm{p}} & |\tilde{\mathrm{p}}|^2/\tilde{p}_- \end{pmatrix} \begin{pmatrix} 1 & \\ \Delta_3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \overline{\Delta}_3 \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{p}_- + \overline{\tilde{\mathrm{p}}}\Delta_3 & \overline{\tilde{\mathrm{p}}} \\ \tilde{\mathrm{p}} + |\tilde{\mathrm{p}}|^2\Delta_3/\tilde{p}_- & |\tilde{\mathrm{p}}|^2/\tilde{p}_- \end{pmatrix} \\
&= \begin{pmatrix} \tilde{p}_-|1 + \overline{\tilde{\mathrm{p}}}\Delta_3/\tilde{p}_-|^2 & \overline{\tilde{\mathrm{p}}}(1 + \tilde{\mathrm{p}}\overline{\Delta}_3/\tilde{p}_-) \\ \tilde{\mathrm{p}}(1 + \overline{\tilde{\mathrm{p}}}\Delta_3/\tilde{p}_-) & |\tilde{\mathrm{p}}|^2/\tilde{p}_- \end{pmatrix} \\
&\Rightarrow \tilde{p}_- \mapsto \tilde{p}_-|1 + \overline{\tilde{\mathrm{p}}}\Delta_3/\tilde{p}_-|^2,\ \tilde{\mathrm{p}} \mapsto \tilde{\mathrm{p}}(1 + \overline{\tilde{\mathrm{p}}}\Delta_3/\tilde{p}_-),
\end{aligned} \tag{3.1.51}
$$

which results in the following action on s:

$$\mathrm{s} \overset{(3.1.34)}{=} \frac{\tilde{p}_-}{\bar{\tilde{\mathrm{p}}}} \overset{(3.1.51)}{\mapsto} \frac{\tilde{p}_-}{\bar{\tilde{\mathrm{p}}}}(1+\bar{\tilde{\mathrm{p}}}\Delta_3/\tilde{p}_-) = \mathrm{s}+\Delta_3$$

As the following calculation shows, this is the first step where non-trivial Wigner rotations occur:

$$R(A_3(\Delta_3), p) = \begin{pmatrix} 1 & \\ \Delta_3 & 1 \end{pmatrix} \overset{(2.1.16)}{=} [0, \overline{\Delta_3}], \text{ since } t = \mathrm{p} = 0 \Rightarrow B_p = \mathbf{1}$$

$$R(A_3(\Delta_3), \tilde{p}) = \frac{1}{\tilde{p}_-|1+\bar{\tilde{\mathrm{p}}}\Delta_3/\tilde{p}_-|}\begin{pmatrix} \tilde{p}_- & \bar{\tilde{\mathrm{p}}} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ \Delta_3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\bar{\tilde{\mathrm{p}}}(1+\tilde{\mathrm{p}}\overline{\Delta}_3/\tilde{p}_-) \\ & \tilde{p}_-|1+\bar{\tilde{\mathrm{p}}}\Delta_3/\tilde{p}_-|^2 \end{pmatrix}$$

$$= \frac{1}{\tilde{p}_-|1+\bar{\tilde{\mathrm{p}}}\Delta_3/\tilde{p}_-|}\begin{pmatrix} \tilde{p}_- & \bar{\tilde{\mathrm{p}}} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & -\bar{\tilde{\mathrm{p}}}(1+\tilde{\mathrm{p}}\overline{\Delta}_3/\tilde{p}_-) \\ \Delta_3 & \tilde{p}_-(1+\tilde{\mathrm{p}}\overline{\Delta}_3/\tilde{p}_-) \end{pmatrix}$$

$$= \frac{1}{|\tilde{p}_-+\bar{\tilde{\mathrm{p}}}\Delta_3|}\begin{pmatrix} \tilde{p}_-+\bar{\tilde{\mathrm{p}}}\Delta_3 & \\ \Delta_3 & \tilde{p}_-+\tilde{\mathrm{p}}\overline{\Delta}_3 \end{pmatrix} \overset{(3.1.36)}{=} \frac{1}{|\mathrm{s}||\mathrm{s}+\Delta_3|}\begin{pmatrix} \bar{\mathrm{s}}(\mathrm{s}+\Delta_3) & \\ \Delta_3/s_+ & \overline{\mathrm{s}(\mathrm{s}+\Delta_3)} \end{pmatrix}$$

$$\overset{(2.1.16)}{=} \left[\arg(\bar{\mathrm{s}}(\mathrm{s}+\Delta_3)), \frac{\bar{\mathrm{s}}(\mathrm{s}+\Delta_3)}{|\mathrm{s}|^2|\mathrm{s}+\Delta_3|^2}\frac{\overline{\Delta}_3}{s_+}\right] = [\arg(\bar{\mathrm{s}}(\mathrm{s}+\Delta_3)), \overline{\Delta}_3/(s_+\overline{\mathrm{s}(\mathrm{s}+\Delta_3)})]$$

By eq. (2.1.24), it is clear that the action of $R(A(\Delta_3), p)$ leaves k and therefore by eq. (3.1.36) equivalently α invariant, while

$$\begin{aligned} \tilde{k} &\overset{(2.1.24)}{\mapsto} \tilde{k}\lambda(-\arg(\bar{\mathrm{s}}(s+\Delta_3))) \Leftrightarrow \arg\tilde{k} \mapsto \arg\tilde{k} + 2\arg(\bar{\mathrm{s}}(\mathrm{s}+\Delta_3)) \\ \Rightarrow \psi &\overset{(3.1.34)}{=} 2\arg\mathrm{s} - \arg\tilde{k} \pm \pi \\ &\mapsto 2\arg(\mathrm{s}+\Delta_3) - (\arg\tilde{k} + 2\arg(\bar{\mathrm{s}}(\mathrm{s}+\Delta_3))) \pm \pi = \psi, \end{aligned}$$

hence ψ is invariant as well.

These results lead to the following form of the intertwiner equation, where both $k, \tilde{k}$ and the new variables are both used to keep the notation simple:

$$\begin{aligned} u_2(0,0,\mathrm{s},0,s_+,\psi) \overset{(2.1.24)}{=} & \exp(-\mathrm{i}k\Delta_3)\exp\left(-\mathrm{i}\tilde{k}\frac{\Delta_3}{s_+\bar{\mathrm{s}}(\mathrm{s}+\Delta_3)}\right) \\ & u_2(0,0,\mathrm{s}+\Delta_3,0,s_+,\psi) \end{aligned}$$

This equation becomes singular whenever s or $\mathrm{s}+\Delta_3$ vanish. Therefore, in contrast to the previous steps, where t, p and α have been translated to 0, Lemma 19 is applied this time to first determine u_2 from its value at $\mathrm{s} = \mathrm{s}_0 \neq 0$, which is still chosen arbitrarily, i.e. the following equation holds for almost all α, s_+, ψ:

$$\begin{aligned} u_2(0,0,\mathrm{s},0,s_+,\psi) = & \exp(-\mathrm{i}k(\mathrm{s}_0-\mathrm{s}))\exp\left(-\mathrm{i}\tilde{k}\frac{\mathrm{s}_0-\mathrm{s}}{s_+\bar{\mathrm{s}}\mathrm{s}_0}\right) \\ & u_2(0,0,\mathrm{s}_0,0,s_+,\psi) \end{aligned} \tag{3.1.52}$$

The second phase factor is continuous only at $\mathrm{s} \neq 0$, but the limit $\mathrm{s} \to 0$ can be performed, provided that the origin is approached from the correct direction.

Because s is now changed on the lhs. of eq. (3.1.52), the dependency of $\tilde{k}$ on the new variables has to be resolved at this point:

$$\begin{aligned}
&\arg\tilde{k} \overset{(3.1.36)}{=} 2\arg\mathrm{s} - \psi \pm \pi \\
\Rightarrow\ &\tilde{k}\cdot\frac{\mathrm{s}_0-\mathrm{s}}{s_+\bar{\mathrm{s}}\mathrm{s}_0} = -\kappa\mathrm{e}^{-\mathrm{i}\psi}\frac{\mathrm{s}}{\bar{\mathrm{s}}}\cdot\frac{\mathrm{s}_0-\mathrm{s}}{s_+\bar{\mathrm{s}}\mathrm{s}_0} \overset{(2.1.25)}{=} -\kappa\mathrm{e}^{-\mathrm{i}\psi}\cdot\frac{\mathrm{s}_0-\mathrm{s}}{s_+\mathrm{s}\mathrm{s}_0}
\end{aligned} \tag{3.1.53}$$

It is now possible to state the correct direction in terms of $\arg\mathrm{s}$ such that the first term in eq. (3.1.53) vanishes, and perform the limit $|\mathrm{s}|\to 0$:

$$\begin{aligned}
\arg\mathrm{s} \overset{!}{=} \psi \pm \frac{\pi}{2} \overset{(3.1.53)}{\Rightarrow} \lim_{|\mathrm{s}|\to 0}\tilde{k}\cdot\frac{\mathrm{s}_0-\mathrm{s}}{s_+\bar{\mathrm{s}}\mathrm{s}_0} = \kappa\mathrm{e}^{-\mathrm{i}\psi}\cdot\frac{1}{s_+\mathrm{s}_0} \\
\Rightarrow u_2(0,0,0,0,s_+,\psi) = \exp\left(-\mathrm{i}k\mathrm{s}_0\right)\exp\left(-\mathrm{i}\kappa\mathrm{e}^{-\mathrm{i}\psi}\cdot\frac{1}{s_+\mathrm{s}_0}\right) \\
u_2(0,0,\mathrm{s}_0,0,s_+,\psi)
\end{aligned} \tag{3.1.54}$$

Combining eq. (3.1.52) and eq. (3.1.54), which hold for almost all α, s_+, ψ, relates the value of u_2 at any s to the one at $\mathrm{s}=0$:

$$\begin{aligned}
u_2(0,0,\mathrm{s},0,s_+,\psi) &= \exp\left(\mathrm{i}k\mathrm{s}\right)\exp\left(-\mathrm{i}\left(\tilde{k}\cdot\frac{\mathrm{s}_0-\mathrm{s}}{s_+\bar{\mathrm{s}}\mathrm{s}_0} + \kappa\mathrm{e}^{-\mathrm{i}\psi}\cdot\frac{1}{s_+\mathrm{s}_0}\right)\right) \\
&\quad u_2(0,0,0,0,s_+,\psi) \\
&\overset{(3.1.53)}{=} \exp\left(\mathrm{i}\kappa\Re(\mathrm{s})\right)\exp\left(-\mathrm{i}\kappa\mathrm{e}^{-\mathrm{i}\psi}\cdot\frac{1}{s_+\mathrm{s}}\right)u_2(0,0,0,0,s_+,\psi)
\end{aligned} \tag{3.1.55}$$

The fact that $\alpha=0$ implies $k\cdot\mathrm{s} = \kappa\Re(\mathrm{s})$ has also been used in the last step.

Equations (3.1.46), (3.1.48), (3.1.50) and (3.1.55), may be combined to show that u_2 is equivalent to a function which satisfies

$$\begin{aligned}
u_2(p,\tilde{p})(k,\tilde{k}) &\overset{(3.1.44)}{=} u_2(t,\mathrm{p},\mathrm{s},\alpha,s_+,\psi) \\
&= \exp\left(\mathrm{i}\kappa\Re(\mathrm{s})\right)\exp\left(-\mathrm{i}\kappa\mathrm{e}^{-\mathrm{i}\psi}\cdot\frac{1}{s_+\mathrm{s}}\right)u_2(0,0,0,0,s_+,\psi).
\end{aligned}$$

By eq. (3.1.34) and the remarks following it, the definition

$$F_2(w) := u_2(0,0,0,0,\kappa^2|w|,\arg w)$$

yields eq. (3.1.33), which was our aim to show. □

A similar result to Lemma 7, which showed the general solution of eq. (3.1.3) to be of the form eq. (3.1.33), can be obtained for the other functions, which occur in Definition 3.

Lemma 8 (Conjugate two-particle intertwiners)**.** *Let u_0, u_{0c} and u_{2c} be the functions given in Definition 3. Then there are functions F_0 and F_{0c} such that in addition to eq. (3.1.33) the following equations hold in the sense of L^2_{loc}:*

$$u_{2c}(p,\tilde p)(k,\tilde k) = \mathrm{e}^{+\mathrm{i}k\cdot\frac{1}{\overline{\mathrm{p}}-\overline{\tilde{\mathrm{p}}}\frac{p_-}{\tilde p_-}}}\,\mathrm{e}^{+\mathrm{i}\tilde k\cdot\frac{1}{\overline{\tilde{\mathrm{p}}}-\overline{\mathrm{p}}\frac{\tilde p_-}{p_-}}}\,\overline{F_2\left((k\tilde k)^{-1}\left(\mathrm{p}-\tilde{\mathrm{p}}\frac{p_-}{\tilde p_-}\right)\left(\tilde{\mathrm{p}}-\mathrm{p}\frac{\tilde p_-}{p_-}\right)\right)} \tag{3.1.56}$$

$$u_{0}(p,\tilde p)(k,\tilde k) = \mathrm{e}^{-\mathrm{i}k\cdot\frac{1}{\overline{\mathrm{p}}-\overline{\tilde{\mathrm{p}}}\frac{p_-}{\tilde p_-}}}\,\mathrm{e}^{+\mathrm{i}\tilde k\cdot\frac{1}{\overline{\tilde{\mathrm{p}}}-\overline{\mathrm{p}}\frac{\tilde p_-}{p_-}}}\,F_0\left((k\tilde k)^{-1}\left(\mathrm{p}-\tilde{\mathrm{p}}\frac{p_-}{\tilde p_-}\right)\left(\tilde{\mathrm{p}}-\mathrm{p}\frac{\tilde p_-}{p_-}\right)\right) \tag{3.1.57}$$

$$u_{0c}(p,\tilde p)(k,\tilde k) = \mathrm{e}^{+\mathrm{i}k\cdot\frac{1}{\overline{\mathrm{p}}-\overline{\tilde{\mathrm{p}}}\frac{p_-}{\tilde p_-}}}\,\mathrm{e}^{-\mathrm{i}\tilde k\cdot\frac{1}{\overline{\tilde{\mathrm{p}}}-\overline{\mathrm{p}}\frac{\tilde p_-}{p_-}}}\,F_{0c}\left((k\tilde k)^{-1}\left(\mathrm{p}-\tilde{\mathrm{p}}\frac{p_-}{\tilde p_-}\right)\left(\tilde{\mathrm{p}}-\mathrm{p}\frac{\tilde p_-}{p_-}\right)\right) \tag{3.1.58}$$

Proof. The form of u_{2c}, claimed in eq. (3.1.56), is proven first, because it is the least complicated case. Let $f \in \mathcal{S}(\mathbb{M}^{\times 2})$ and $g_2 \in \mathcal{H}_2$. Using the reality condition assumed in Definition 3 and suppressing the integrations in the notation, the equation

$$\begin{aligned}
&\overline{g_2(p,\tilde p)(k,\tilde k)}\hat f(p,\tilde p)u_2(p,\tilde p)(k,\tilde k) \overset{(3.1.1)}{=} \langle g_2|B(f)\Omega\rangle \overset{(3.1.4)}{=} \langle g_2|B(\overline{f})^\dagger\Omega\rangle \\
\overset{(3.1.1)}{=}\,&\overline{g_2(p,\tilde p)(k,\tilde k)\hat{\overline{f}}(-p,-\tilde p)u_{2c}(p,\tilde p)(k,\tilde k)} = \overline{g_2(p,\tilde p)(k,\tilde k)}\hat f(p,\tilde p)\overline{u_{2c}(p,\tilde p)(k,\tilde k)}
\end{aligned}$$

shows that $u_2 = \overline{u_{2c}}$, because f and g_2 can be arbitrarily chosen. Lemma 7 then implies eq. (3.1.56).

Equations (3.1.57) and (3.1.58) are shown by relating the covariance properties of the terms in (3.1.1), which involve u_0 and u_{0c}, to the results of Lemma 5. It is our first aim to show intertwiner equations for u_0, u_{0c}, which are analogous to (3.1.3).

To this end, an arbitrary open region $\mathcal{O}$ with nonempty causal complement $\mathcal{O}'$ and an arbitrary covering element $A \in \mathrm{SL}(2,\mathbb{C})$ of a Lorentz transformation are chosen. Its representation on the Fock space $\mathcal{H}$ is abbreviated by $U(A) := U(A,0)$. The quantity of interest is then the matrix element between one-particle states of $U(A)B(g)U(A)^\dagger$, where g is a smearing function with $\operatorname{supp} g \subseteq \mathcal{O}$. One of the states is an arbitrary vector $g_1 \in \mathcal{H}_1$, while the other one is generated from Ω by a string-field $\Phi(f,h)$, where the smearing functions are such that $\operatorname{supp} f + \mathbb{R}^+ \operatorname{supp} h \subseteq \Lambda(A)\mathcal{O}^\perp$. With $f_\Lambda := f_{(\Lambda,0)}$, it follows:

$$\begin{aligned}
&\int \widetilde{\mathrm{d}p} \int \widetilde{\mathrm{d}\tilde p} \int \mathrm{d}\nu(k) \int \mathrm{d}\nu(\tilde k)\, \overline{g_1(\tilde p,\tilde k)}\hat f(p)u_1(p,h)(k) \\
&\Big(\hat g(\tilde p\Lambda(A), -p\Lambda(A))[D(R(A,\tilde p)) \otimes \overline{D}(R(A,p))u_0(\tilde p\Lambda(A), p\Lambda(A))](\tilde k, k) \\
&+\hat g(-p\Lambda(A), \tilde p\Lambda(A))[\overline{D}(R(A,p)) \otimes D(R(A,\tilde p))u_{0c}(p\Lambda(A), \tilde p\Lambda(A))](k,\tilde k)\Big) \\
&= \langle g_1|[U(A)B(g)U(A)^\dagger]\Phi(f,h)\Omega\rangle = \langle g_1|U(A)B(g)[U(A)^\dagger\Phi(f,h)U(A)]U(A)^\dagger\Omega\rangle \\
&\overset{(2.2.18)}{=} \langle g_1|U(A)B(g)\Phi(f_{\Lambda^{-1}}, h_{\Lambda^{-1}})U(A)^\dagger\Omega\rangle = \langle g_1|U(A)\Phi(f_{\Lambda^{-1}}, h_{\Lambda^{-1}})B(g)U(A)^\dagger\Omega\rangle \\
&= \langle g_1|[U(A)\Phi(f_{\Lambda^{-1}}, h_{\Lambda^{-1}})U(A)^\dagger]U(A)B(g)\Omega\rangle \overset{(2.2.18)/(3.1.7)}{=} \langle g_1|\Phi(f,h)B(g_{\Lambda(A)})\Omega\rangle \\
&= \langle g_1|B(g_{\Lambda(A)})\Phi(f,h)\Omega\rangle = \int \widetilde{\mathrm{d}p} \int \widetilde{\mathrm{d}\tilde p} \int \mathrm{d}\nu(k) \int \mathrm{d}\nu(\tilde k)\, \overline{g_1(\tilde p,\tilde k)}\hat f(p)u_1(p,h)(k) \\
&\Big(\hat g(\tilde p\Lambda(A), -p\Lambda(A))u_0(\tilde p, p)(\tilde k, k) + \hat g(-p\Lambda(A), \tilde p\Lambda(A))u_{0c}(p,\tilde p)(k,\tilde k)\Big)
\end{aligned}$$

The support properties of the involved smearing functions and assumption III (relative locality) have been used, whenever B and Φ have been exchanged in the calculation. It is evident from the resulting equation, that the desired intertwiner equation is contained in the integrand. However, the choice of $g_1 \in \mathcal{H}_1$ was arbitrary and Theorem 1 guarantees that vectors of the form $\Phi(f,h)\Omega$, even with the mentioned support requirements with regard to f, h are dense in $\mathcal{H}_1$. This implies the equation

$$\begin{aligned}
&\Big(\hat{g}(\tilde{p}\Lambda(A), -p\Lambda(A))[D(R(A,\tilde{p})) \otimes \overline{D}(R(A,p))u_0(\tilde{p}\Lambda(A), p\Lambda(A))](\tilde{k}, k) \\
&+\hat{g}(-p\Lambda(A), \tilde{p}\Lambda(A))[\overline{D}(R(A,p)) \otimes D(R(A,\tilde{p}))u_{0c}(p\Lambda(A), \tilde{p}\Lambda(A))](k, \tilde{k})\Big) \\
&=\hat{g}(\tilde{p}\Lambda(A), -p\Lambda(A))u_0(\tilde{p}, p)(\tilde{k}, k) + \hat{g}(-p\Lambda(A), \tilde{p}\Lambda(A))u_{0c}(p, \tilde{p})(k, \tilde{k})
\end{aligned}$$

in the sense of L^2_{loc}, which equivalently reads

$$\begin{aligned}
0 = \hat{g}(\tilde{p}\Lambda(A), -p\Lambda(A))\Big(&[D(R(A,\tilde{p})) \otimes \overline{D}(R(A,p))u_0(\tilde{p}\Lambda(A), p\Lambda(A))](\tilde{k}, k) \\
&-u_0(\tilde{p}, p)(\tilde{k}, k)\Big) \\
+\hat{g}(-p\Lambda(A), \tilde{p}\Lambda(A))\Big(&[\overline{D}(R(A,p)) \otimes D(R(A,\tilde{p}))u_{0c}(p\Lambda(A), \tilde{p}\Lambda(A))](k, \tilde{k}) \\
&-u_{0c}(p, \tilde{p})(k, \tilde{k})\Big).
\end{aligned}$$

Since $g \in C_0^\infty(\mathcal{O})$ was arbitrary, it now suffices to show that this equation can only hold for all possible choices of g, if the brackets vanish almost everywhere.

A sufficient set of choices for g can be constructed in the following way: Pick $g_a, g_b \in C_0^\infty(\mathcal{O})$ of product form with $g_a \neq g_b$ and define $g_\pm(x, \tilde{x}) := g_a(x)g_b(\tilde{x}) \pm g_a(\tilde{x})g_b(x)$, both of which are nonvanishing. Their Fourier transforms $\hat{g}_\pm$ in the sense of (2.2.15) are therefore products of nonzero entire analytic functions of $p, \tilde{p}$, hence their sets of roots are unions of isolated planes, which are orthogonal to the coordinate axes, hence the Lorentz-invariant measure of the restriction of these sets to ∂V^+ is zero. Substitution of $g_\pm$ into the previous equation consequently yields two equations in which the dependency on $\hat{g}_\pm$ reduces to a prefactor due to $\hat{g}_\pm(\tilde{p}, p) = \pm\hat{g}_\pm(p, \tilde{p})$. Dividing by it is possible almost everywhere by the previous argument, hence a non-degenerate system of equations is obtained, which has the following unique solution:

$$\begin{aligned}
D(R(A,\tilde{p})) \otimes \overline{D}(R(A,p))u_0(\tilde{p}\Lambda(A), p\Lambda(A)) &= u_0(\tilde{p}, p)(\tilde{k}, k) \\
\overline{D}(R(A,p)) \otimes D(R(A,\tilde{p}))u_{0c}(p\Lambda(A), \tilde{p}\Lambda(A)) &= u_{0c}(p, \tilde{p})(k, \tilde{k})
\end{aligned}$$

Once these equations are established, one may proceed analogously to the proof of Lemma 7. The only difference is that the occurrence of $\overline{D}$ instead of D leads to a different sign in one of the exponents of eq. (3.1.52), corresponding to the sign pattern in equations (3.1.57) and (3.1.58), while the definition at the end of the proof concerns F_0 and F_{0c}, respectively. □

Remark 10. Lemma 6 and Lemma 7 describe the general solutions of the stringlike intertwiner equation and the two-particle intertwiner equation, respectively. The intertwiners discussed in [MSY06], although constructed in a different manner, are therefore special cases of these sets of solutions.

3.2 Proof of the Theorem

3.2.1 Preliminaries

As a first step, the geometric situation illustrated in Figure 3.1.1 is fixed. Define the lightlike vector $n \in \mathbb{M}$ by

$$n_{\pm} = \pm 1, \mathrm{n} = 0. \tag{3.2.1}$$

Let $f \in \mathcal{S}(\mathbb{M})$ be a smearing function with $\operatorname{supp} f \subset W_0 + bn$ (cf. (2.2.1)) for some constant $b > 0$ and $h \in \mathcal{D}(H)$ with $\operatorname{supp} h \subset W_0$ as well. The object of interest is the commutator of the string-field $\Phi(f, h)$, localized inside the truncated cone $\operatorname{supp} f + \mathbb{R}^+ \operatorname{supp} h \subset W_0 + bn$, and the two-particle observable $B(g)$, where $g \in \mathcal{S}(\mathbb{M}^{\times 2})$ is a smearing function with $\operatorname{supp} g \subset W_0'^{\times 2}$.

By eq. (3.1.1), $B(g)$ is quadratic in the creation and annihilation operators, while $\Phi(f, h)$, as defined in eq. (2.2.17), is linear in this sense. Hence the Leibniz formula implies that the commutator consists of two terms, creating and annihilating one particle, respectively. Considering the adjoint of the commutator interchanges these terms but eq. (3.1.7) and eq. (3.1.4) show that the result is the same up to a sign if all smearing functions are replaced by their complex conjugates, which preserves their support properties. Without loss of generality, it is therefore sufficient to study the matrix element of the creation operator.

In addition, let $\phi \in \mathcal{H}_1$ an arbitrary one-particle. The function

$$\gamma \;:\; \mathbb{R} \to \mathbb{C},\; a \mapsto \langle \phi, [B(g), U(an)\Phi(\tilde{f}, h)U^\dagger(an)]\Omega \rangle \,,$$

whose definition can be restated using eq. (2.2.18) restricted to the translation subgroup of $\mathcal{P}^c$

$$\gamma(a) = \langle \phi, [B(g), \Phi(\tilde{f}_a, h)]\Omega \rangle \,, \text{ where } f_s := f_{(\mathbf{1}, sn)}, \tag{3.2.2}$$

then inherits some support, boundedness and continuity properties from Definition 3, which are the subject of the next lemma.

3.2.2 Properties of the matrix element

Lemma 9 (Position space properties)**.** *The function γ, as defined in eq.* (3.2.2)*, has the following properties:*

1. ***Support:*** $\operatorname{supp} \gamma \subseteq (-\infty, -b]$

2. ***Boundedness:*** *There are constants $C, L > 0$ and $N \in \mathbb{N}$ such that*

$$|\gamma(a)| \leq C \left(\frac{1}{L} \chi_{[-L,0]-b}(a) + \frac{1}{(N-1)!} |a + b|^{N-1} \right) \; \forall a < -b.$$

3. ***Continuity:*** *γ is a continuous function.*

Proof. The proof uses the various assumptions made in Definition 3.

1. Let $a > -b$ and $x \in \operatorname{supp} f_a, e \in \operatorname{supp} h, (y_1, y_2) \in \operatorname{supp} g$. Applying the support properties of the functions f, h and g shows that the vector

$$x + \lambda e - y_{1,2} = \underbrace{x - (a+b)n}_{\in (\operatorname{supp} f) - bn \subseteq W_0} + \underbrace{(a+b)}_{>0} n + \lambda e - y_{1,2}$$

are spacelike for $\lambda > 0$. By the locality assumption III, $\gamma(a) = 0$ in this case.

2. The tempered distribution condition in assumption I of Definition 3 yields the following bound, where $C > 0$ and $M_\alpha, M_\beta \in \mathbb{N}$:

$$|\gamma(a)| \leq C \sum_{\substack{|\alpha| \leq M_\alpha \\ |\beta| \leq M_\beta}} ||\tilde{f}_a||_{\alpha,\beta}, \text{ with the seminorms}$$

$$||f||_{\alpha,\beta} = \sup_{x \in \mathbb{R}^4} |x^\alpha \partial^\beta f(x)| \; \forall \, f \in \mathcal{S}(\mathbb{R}^4), \; \alpha, \beta \text{ multi-indices.} \tag{3.2.3}$$

In order to split this estimate into a polynomial bound on a and absorb everything else into a global constant, the argument of $\tilde{f}_a$ as well as the multi-index α are split uniquely into $x = x_\| n + x_\perp$ with $x_\| \in \mathbb{R}$ and $\alpha = \alpha_\| n + \alpha_\perp$, respectively. The same symbol n is used for the multi-index component corresponding to the n-direction, i.e.

$$(x_\| n + x_\perp)^{\alpha_\| n + \alpha_\perp} = x_\|^{\alpha_\|} x_\perp^{\alpha_\perp}.$$

It follows

$$\begin{aligned}
||\tilde{f}_a||_{\alpha,\beta} &\overset{(3.2.3)}{=} \sup_{x \in \mathbb{R}^4} \left| x^\alpha \partial^\beta \tilde{f}_a(x) \right| = \sup_{x \in \mathbb{R}^4} \left| x^\alpha \partial^\beta \tilde{f}_{\mu^*+b}(x - na) \right| \\
&= \sup_{x \in \mathbb{R}^4} \left| (x + na)^\alpha \partial^\beta \tilde{f}_{\mu^*+b}(x) \right| = \sup_{x \in \mathbb{R}^4} \left| (x_\| + a)^{\alpha_\|} x_\perp^{\alpha_\perp} \partial^\beta \tilde{f}_{\mu^*+b}(x) \right| \\
&= \sup_{x \in \mathbb{R}^4} \left| \sum_{i=0}^{\alpha_\|} \binom{a_\|}{i} a^i x_\|^{\alpha_\| - i} x_\perp^{\alpha_\perp} \partial^\beta \tilde{f}_{\mu^*+b}(x) \right| \\
&\leq \sum_{i=0}^{\alpha_\|} |a|^i \underbrace{\sup_{x \in \mathbb{R}^4} \left| \binom{a_\|}{i} x_\|^{\alpha_\| - i} x_\perp^{\alpha_\perp} \partial^\beta \tilde{f}_{\mu^*+b}(x) \right|}_{=: C_{\alpha,\beta,i}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow |\gamma(a)| \leq & C \sum_{\substack{|\alpha| \leq M_\alpha \\ |\beta| \leq M_\beta}} \sum_{i=0}^{\alpha_\|} C_{\alpha,\beta,i} |a|^i = C \sum_{0 \leq i \leq M_\alpha} |a|^i \underbrace{\sum_{\substack{|\alpha| \leq M_\alpha \\ \alpha_\| \geq i \\ |\beta| \leq M_\beta}} C_{\alpha,\beta,i}}_{=: C_i} \\
= & C \sum_{0 \leq i \leq M_\alpha} C_i |a|^i
\end{aligned}$$

$N := M_\alpha + 1$, let $L > 0$ and $a \leq -b$. Therefore the previous estimate yields

$$\left| \frac{\gamma(a)}{\frac{1}{L} \chi_{[-L,0]}(a+b) + |a+b|^{N-1}} \right| \leq \tilde{C}$$

for some constant $\tilde{C} > 0$, because the denominator is bounded from below by some strictly positive constant and the above estimate shows that the quotient converges to CC_{M_α} for $a \to -\infty$. Let then $\epsilon > 0$ and choose a point $\tilde{L} > 0$ such that the distance of the lhs. to this limit is bounded by ϵ for all $a \in (-\infty, -b - L - \tilde{L})$. On the other hand, for the cases $a \in [-\tilde{L}, 0) - b - L$ and $a \in [-L, 0] - b$, the above estimate shows that the lhs. is bounded by a continuous function, which in the first case can be continued to a continuous function on a closed interval. In each case, the result is bounded in modulus by some constant. $\tilde{C}$ may then be defined as the maximum of the three constants obtained in this way.

3. Equation (3.2.2) shows that γ is the composition $\gamma = s \circ t$ of the maps

$$\begin{aligned} t &: \mathbb{R} \to \mathcal{S}(\mathbb{M}),\ a \mapsto \tilde{f}_a \text{ and} \\ s &: \mathcal{S}(\mathbb{M}) \to \mathbb{C},\ f \mapsto \langle \phi, [B(g), \Phi(f,h)]\Omega \rangle, \end{aligned}$$

where the translation t is continuous at $a = 0$, because for $\delta \in \mathbb{R}$ every seminorm $||\cdot||_{\alpha\beta}$ (cf. eq. (3.2.3)) of $t(a+\delta) - t(a) \in \mathcal{S}(\mathbb{R}^4)$ can be estimated against a polynomial in δ, which has a zero at $\delta = 0$. The same notation as in the previous step is being used also for the multi-index β, for example $\beta + n$ denotes one additional partial derivative in the n-direction:

$$\begin{aligned} &||t(\delta) - t(0)||_{\alpha,\beta} = \sup_{x \in \mathbb{R}^4} \left| x^\alpha \partial^\beta \left(\tilde{f}_\delta - \tilde{f} \right) \right| \\ =& \sup_{x \in \mathbb{R}^4} \left| x^\alpha \partial^\beta \left(\tilde{f}(x - \delta) - \tilde{f}(x) \right) \right| \\ =& \sup_{x \in \mathbb{R}^4} \left| x^\alpha \int_0^\delta \mathrm{d}y \, \partial^{\beta+n} \tilde{f}(x - yn) \right| \leq \sup_{x \in \mathbb{R}^4} |x^\alpha| \int_0^\delta \mathrm{d}y \left| \partial^{\beta+n} \tilde{f}(x - yn) \right| \\ \leq& \sup_{x \in \mathbb{R}^4} |x^\alpha| \int_0^\delta \mathrm{d}y \, \frac{||\tilde{f}||_{0,\beta+n} + ||\tilde{f}||_{\alpha,\beta+n}}{1 + |(x - yn)^\alpha|} \\ \leq& \underbrace{\left(||\tilde{f}||_{0,\beta+n} + ||\tilde{f}||_{\alpha,\beta+n} \right)}_{=:C} \delta \sup_{x \in \mathbb{R}^4} \max_{y \in [0,\delta]} \frac{|x^\alpha|}{1 + |(x - yn)^\alpha|} \\ =& C\delta \sup_{x \in \mathbb{R}^4} \max_{y \in [0,\delta]} \frac{|(x + yn)^\alpha|}{1 + |x^\alpha|} = C\delta \sup_{x \in \mathbb{R}^4} \frac{\left|x_\perp^{\alpha_\perp}\right|}{1 + |x^\alpha|} \max_{y \in [0,\delta]} \left| (x_\parallel + y)^{\alpha_\parallel} \right| \\ \leq& C\delta \sum_{i=0}^{\alpha_\parallel} \binom{\alpha_\parallel}{i} \sup_{x \in \mathbb{R}^4} \frac{\left|x_\perp^{\alpha_\perp}\right|}{1 + |x^\alpha|} \underbrace{\max_{y \in [0,\delta]} \left| x_\parallel^{\alpha_\parallel - i} y^i \right|}_{= \left| x_\parallel^{\alpha_\parallel - i} \right| \delta^i} = C\delta \sum_{i=0}^{\alpha_\parallel} \underbrace{\binom{\alpha_\parallel}{i} \sup_{x \in \mathbb{R}^4} \frac{\left| x^{\alpha - in} \right|}{1 + |x^\alpha|}}_{=:C_{\alpha,i} < \infty} \delta^i \\ =& C\delta \sum_{i=0}^{\alpha_\parallel} C_{\alpha,i} \delta^i, \text{ therefore } \lim_{\delta \to 0} ||t(\delta) - t(0)||_{\alpha,\beta} = 0 \ \forall \, \alpha, \beta \text{ multi-indices.} \end{aligned}$$

Finiteness of the constants $C_{\alpha,i}$, as defined in the last line, is due to the fact that the multi-index in the denominator dominates the one in the numerator. t is even continuous at every $a \in \mathbb{R}$, which can be shown analogously, replacing $\tilde{f}$ by $\tilde{f}_a$.

On the other hand, the smearing operation s of the commutator's matrix element is continuous in the smearing function f as well, because the string-fields $\Phi(\cdot, h)$ are operator-valued distributions in the endpoint of the string, which means that s is a tempered distribution.

Because γ is a composition of continuous functions in the respective sense, it has to be continuous, too.

$\square$

Analogously to the remarks before [RS75, Thm. IX.15], the holomorphic Fourier transform of γ yields its distributional Fourier transformation as a weak boundary value. As a special case contained in [RS75, Thm. IX.16], it also fulfills certain estimates (cf. Theorem 3). The various properties of the holomorphic Fourier transform of γ that are going to be needed in the present context are summarized as the subject of the following lemma:

Lemma 10 (Holomorphic Fourier transform)**.** *The holomorphic Fourier transform of a continuous polynomially bounded function* $\gamma : \mathbb{R} \to \mathbb{C}$ *with* $\operatorname{supp} \gamma \subseteq (-\infty, -b]$ *for some* $b > 0$*, which is defined by*

$$\hat{\gamma}(z) = \int \mathrm{d}a\, \mathrm{e}^{-\mathrm{i}za} \gamma(a) \;\forall\, z \in H^+, \tag{3.2.4}$$

where $H^+ := \{z \in \mathbb{C} : \Im(z) > 0\}$ *is the upper half-plane, has the following properties:*

1. ***Analyticity:*** $\hat{\gamma}$ *is an analytic function on* H^+*.*
2. ***Boundedness:*** *There are constants* $C > 0, N \in \mathbb{N}$ *such that*
$$|\hat{\gamma}(z)| \leq C\mathrm{e}^{-b\Im(z)}(1 + \Im(z)^{-N}) \;\forall\, z \in H^+$$
3. ***Distributional boundary value:*** *The sequence of distributions* $\hat{\gamma}_t \in \mathcal{S}'(\mathbb{R})$*, given by the restrictions of* $\hat{\gamma}$ *to horizontal lines of constant imaginary part* $t > 0$*,*
$$\hat{\gamma}_t : \mathcal{S}(\mathbb{R}) \mapsto \mathbb{C},\; f \mapsto \int \mathrm{d}s\, \gamma(s + \mathrm{i}t) f(s), \tag{3.2.5}$$
converges for $t \to 0$ *to the distributional Fourier transform of* γ*, denoted by the same symbol,*
$$\hat{\gamma} : \mathcal{S}(\mathbb{R}) \to \mathbb{C},\; f \mapsto \int \mathrm{d}a\, \gamma(a) \hat{f}(a) \tag{3.2.6}$$
$$\text{with } \hat{f}(a) := \int \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}sa} f(s) \text{ the Fourier transform on } \mathcal{S}(\mathbb{R}), \tag{3.2.7}$$
in the sense of $\mathcal{S}'(\mathbb{R})$*:*
$$\lim_{t \to 0} \hat{\gamma}_t(f) = \hat{\gamma}(f) \;\forall\, f \in \mathcal{S}(\mathbb{R})$$

Proof. The proof of Lemma 10 is found in chapter A in the appendix. $\square$

As a matter of convenience, all integrals over momenta and infinite spin variables are suppressed in the notation in the following.

3.2.3 Explicit form of the matrix element

An explicit form of γ is given by the following expression. In the calculation, only those terms from Definition 3 and eq. (2.2.17) which give a contribution to the matrix element have been stated explicitly. For instance, a term with only creation operators from B and Φ would map Ω to a three-particle state, which is orthogonal to $\phi \in \mathcal{H}_1$, and is therefore omitted in the notation.

$$
\begin{aligned}
\gamma(a) \overset{(3.2.2)}{=} & \langle \phi, [B(g), \Phi(f_a, h)]\Omega \rangle \\
=& \langle \Omega, \overline{\phi(q', l')} a(q', l') \big(\; [\hat{g}(p, -\tilde{p}) u_0(p, \tilde{p})(k, \tilde{k}) a^\dagger(p, k) a(\tilde{p}, \tilde{k}) \\
& + \hat{g}(-p, \tilde{p}) u_{0c}(p, \tilde{p})(k, \tilde{k}) a^\dagger(\tilde{p}, \tilde{k}) a(p, k) \,] \; \hat{f}_a(q) u_1(q, h)(l) a^\dagger(q, l) \\
& - \hat{f}_a(-q) \overline{u_{1c}(q, h)(l)} a(q, l) \; \hat{g}(p, \tilde{p}) u_2(p, \tilde{p})(k, \tilde{k}) a^\dagger(p, k) a^\dagger(\tilde{p}, \tilde{k}) \; \big) \Omega \rangle \\
\overset{(2.2.6)}{=} & \overline{\phi(\tilde{p}, \tilde{k})} \big(\; [\hat{g}(\tilde{p}, -p) u_0(\tilde{p}, p)(\tilde{k}, k) + \hat{g}(-p, \tilde{p}) u_{0c}(p, \tilde{p})(k, \tilde{k})] \hat{f}_a(p) u_1(p, h)(k) \\
& - [\hat{g}(\tilde{p}, p) u_2(\tilde{p}, p)(\tilde{k}, k) + \hat{g}(p, \tilde{p}) u_2(p, \tilde{p})(k, \tilde{k})] \hat{f}_a(-p) \overline{u_{1c}(p, h)(k)} \; \big)
\end{aligned}
$$

In accordance with the notational simplification introduced above, there are implicit integrations of $p, \tilde{p} \in \partial V^+$ with measures $\widetilde{\mathrm{d}p}$ and $\widetilde{\mathrm{d}\tilde{p}}$ and $k, \tilde{k} \in$ with measure $\mathrm{d}\nu$. An integral substitution $p \mapsto -p$ in the terms containing u_2 leads to the following expression for $\gamma(a)$:

$$
\begin{aligned}
\gamma(a) =& \overline{\phi(\tilde{p}, \tilde{k})} \hat{f}_a(p) \\
& \big(\; \Theta(p_-) [\hat{g}(\tilde{p}, -p) u_0(\tilde{p}, p)(\tilde{k}, k) + \hat{g}(-p, \tilde{p}) u_{0c}(p, \tilde{p})(k, \tilde{k})] u_1(p, h)(k) \\
& + \Theta(-p_-) [\hat{g}(\tilde{p}, -p) u_2(\tilde{p}, -p)(\tilde{k}, k) + \hat{g}(-p, \tilde{p}) u_2(-p, \tilde{p})(k, \tilde{k})] \overline{u_{1c}(-p, h)(k)} \; \big),
\end{aligned}
$$

where the p_- variable is integrated over $\mathbb{R}$, which may be simplified by also using the backward light cone

$$
\partial V^- := \{ p \in \mathbb{M} : \, p^2 = 0, p_0 < 0 \} \tag{3.2.8}
$$

and considering the pairs of intertwiners as restrictions of the same function to $\partial V^\pm$,

$$
\tilde{u}_1(p, h)(k) := \begin{cases} u_1(p, h)(k) & \text{for } p \in \partial V^+ \\ \overline{u_{1c}(-p, h)(k)} & \text{for } p \in \partial V^- \end{cases} \tag{3.2.9}
$$

since within the present choice of coordinates $\pm p_- > 0$ is equivalent to $p \in \partial V^\pm$, which in turn is equivalent to $p\tilde{p} > 0$, because $\tilde{p} \in \partial V^+$. Now Lemma 7 and Lemma 8 can be applied to the expression for $\gamma(a)$. As it turns out, the phase factors appearing in the claims of these lemmas become equal across all four terms. Abbreviating the remainder by

$$
\begin{aligned}
S(p, \tilde{p}, \psi) := & \Theta(p\tilde{p}) [\hat{g}(\tilde{p}, -p) F_0(2p\tilde{p} e^{i\psi} / \kappa^2) + \hat{g}(-p, \tilde{p}) F_{0c}(2p\tilde{p} e^{i\psi} / \kappa^2)] \\
& + \Theta(-p\tilde{p}) [\hat{g}(\tilde{p}, -p) F_2(2p\tilde{p} e^{i\psi} / \kappa^2) + \hat{g}(-p, \tilde{p}) F_2(2p\tilde{p} e^{i\psi} / \kappa^2)],
\end{aligned}
$$

where the argument of F_2 has been rewritten using (3.1.35), likewise for the arguments of F_0, F_{0c}, yields the following form:

$$\begin{aligned}\gamma(a) =& \overline{\phi(\tilde{p},\tilde{k})} \mathrm{e}^{\frac{\mathrm{i}}{2}ap_-} \hat{f}(p)\tilde{u}_1(p,h)(k) \\ & \exp\left(+\mathrm{i}k\cdot\left(\bar{\mathrm{p}}-\bar{\tilde{\mathrm{p}}}\frac{p_-}{\tilde{p}_-}\right)^{-1}\right)\exp\left(-\mathrm{i}\tilde{k}\cdot\left(\bar{\tilde{\mathrm{p}}}-\bar{\mathrm{p}}\frac{\tilde{p}_-}{p_-}\right)^{-1}\right)S(p,\tilde{p},\psi)\end{aligned}$$

The translation of f by a in the n-direction has been restated with another phase factor using

$$\hat{f}_a(p) \overset{(3.2.2)}{=} \mathrm{e}^{\mathrm{i}pan}\hat{f}(p) \overset{(3.2.1)}{=} \mathrm{e}^{\frac{\mathrm{i}}{2}ap_-}\hat{f}(p).$$

For the following steps it is convenient to abbreviate the resulting form of γ, using the following definitions (cf. (3.1.33) for the second one)

$$\Psi(p,k) := \hat{f}(p)\tilde{u}_1(p,h)(k) \tag{3.2.10}$$

$$I(p,\tilde{p},k,\tilde{k}) := \exp\left(+\mathrm{i}k\cdot\left(\bar{\mathrm{p}}-\bar{\tilde{\mathrm{p}}}\frac{p_-}{\tilde{p}_-}\right)^{-1}\right)\exp\left(-\mathrm{i}\tilde{k}\cdot\left(\bar{\tilde{\mathrm{p}}}-\bar{\mathrm{p}}\frac{\tilde{p}_-}{p_-}\right)^{-1}\right)S(p,\tilde{p},\psi)$$

$$K(p,q) := \frac{1}{p_-}\hat{f}(p)I(p,k,\tilde{p},\tilde{k}), \tag{3.2.11}$$

and to capture all variables except for p_-, k into a tuple (cf. Definition 2):

$$q := (\mathrm{p},k,\tilde{p},\tilde{k}) \in Q := \mathbb{R}^2\times\kappa S^2\times\partial V^+\times\kappa S^2 \tag{3.2.12}$$

3.2.4 Restriction of the integrations

In order to facilitate restricting the implicit integrals defining the function γ to any given point $q\in Q$, the following lemma summarizes various applications of the Lebesgue Differentiation Theorem (cf. Theorem 2).

Lemma 11 (Applications of the Lebesgue Differentiation Theorem)**.** *Let $\tilde{f}$ be a locally Lebesgue integrable function on Q, i.e. $\underline{\tilde{f}}\in L^1(Q)$*

1. *Let $b : \mathbb{R}^2\to\mathbb{R}^+$ be a C^∞-function which is non-negative ($b(\mathrm{p})\geq 0\ \forall\,\mathrm{p}\in\mathbb{R}^2$), compactly supported ($\operatorname{supp} b\subseteq\overline{B_1(0)}$) and normalized ($\int \mathrm{d}^2\mathrm{p}\, b(\mathrm{p}) = 1$). The rescaled function is defined by $b_{\mathrm{p}_0,\epsilon}(\mathrm{p}) = \epsilon^{-2}b((\mathrm{p}-\mathrm{p}_0)/\epsilon)$. Similarly, let $\delta_{k_0,\epsilon}$ be a positive, normalized smooth function on κS^1 which weakly approximates a δ-Distribution at $k_0\in\kappa S^1$ for $\epsilon\to$ With*

$$\delta_{q_0,\epsilon}(q) := b_{\mathrm{p}_0,\epsilon}(p)\delta_{k_0,\epsilon}(k)\frac{\chi_{B^2_\epsilon(\tilde{p}_0,\tilde{k}_0)}(\tilde{p},\tilde{k})}{|B^2_\epsilon(\tilde{p}_0,\tilde{k}_0)|}$$

For any Lebesgue point $q_0\in L(\tilde{f})$ the same limit as in Theorem 2 can be obtained by integrating against $\delta_{q_0,\epsilon}$ instead of averaging over $B_\epsilon(q_0)$:

$$\lim_{\epsilon\to 0}\int \mathrm{d}q\,\delta_{q_0,\epsilon}(q)\tilde{f}(q) = \tilde{f}(q_0) \tag{3.2.13}$$

2. *Let $\tilde{f} \in L^1(\mathbb{R})$ and consider the family of box kernels*

$$b^{(m,n)} := b_{\frac{m}{2^n},\frac{1}{2^{n+1}}} = \frac{1}{2^n}\chi_{\left[\frac{2m-1}{2^{n+1}},\frac{2m+1}{2^{n+1}}\right]}$$

with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. For all $s_0 \in L(\tilde{f})$, there is a sequence $(m_{s_0,n})_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\int \mathrm{d}s\, b^{(m_{s_0,n},n)}(s)|\tilde{f}(s) - \tilde{f}(s_0)| = 0,$$

i.e. the function $\tilde{f}$ can be recovered on $\mathbb{R}$, up to a set of measure zero, from a countable number of averages.

3. *For $\tilde{f} \in L^1([-\pi,\pi])$ and the Fejér kernel*

$$\mathcal{F}_M(x) = \frac{1}{M}\sum_{N=0}^{M-1}\sum_{n=-N}^{N}\mathrm{e}^{\mathrm{i}nx} \tag{3.2.14}$$

the following limit vanishes for $x_0 \in L(f)$:[3]

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\mathrm{d}x\,\mathcal{F}_M(x-x_0)|\tilde{f}(x) - \tilde{f}(x_0)| = 0$$

Proof. 1. The normalization and positivity properties of b translate to the analogous statements for $\delta_{q_0,\epsilon}$ via (3.2.13), while the support property becomes $\operatorname{supp}\delta_{q_0,\epsilon} \subseteq \overline{B_\epsilon(q_0)}$ and the function is bounded by

$$\begin{aligned}||\delta_{q_0,\epsilon}||_\infty &= ||b_{\mathrm{p}_0,\epsilon}||_\infty||\delta_{k_0,\epsilon}||_\infty|B^2_\epsilon(\tilde{p}_0,\tilde{k}_0)|^{-1}\\ &=||b_{\mathrm{p}_0,1}||_\infty||\delta_{k_0,1}||_\infty\frac{|B^1_1(\mathrm{p}_0,k_0)|}{|B^1_\epsilon(\mathrm{p}_0,k_0)|}|B^2_\epsilon(\tilde{p}_0,\tilde{k}_0)|^{-1}\\ &=||b||_\infty||\delta_{k_0,1}||_\infty|B^1_1(\mathrm{p}_0)||B_\epsilon(q_0)|^{-1}.\end{aligned} \tag{3.2.15}$$

$||b||_\infty$ itself is finite because $b \in C^\infty(B_1(\mathrm{p}_0))$, analogously $||\delta_{k_0,1}||_\infty < \infty$ holds. Therefore a similar estimate (cf. [Tes13, Problem 8.14]) of the form

$$\begin{aligned}&\Rightarrow\left|\left(\int \mathrm{d}q\,\delta_{q_0,\epsilon}(q)\tilde{f}(q)\right) - \tilde{f}(q_0)\right| = \left|\int \mathrm{d}q\,\delta_{q_0,\epsilon}\left(\tilde{f}(q) - \tilde{f}(q_0)\right)\right|\\ &\leq \int \mathrm{d}q\,\delta_{q_0,\epsilon}(q)\left|\tilde{f}(q) - \tilde{f}(q_0)\right| \leq ||\delta_{q_0,\epsilon}||_\infty\int \mathrm{d}q\,\chi_{B_\epsilon(q_0)}(q)\left|\tilde{f}(q) - \tilde{f}(q_0)\right|\\ &\overset{(3.2.15)}{=}||b||_\infty||\delta_{k_0,1}||_\infty|B^1_1(\mathrm{p}_0,k_0)|\int \mathrm{d}q\,\frac{\chi_{B_\epsilon(q_0)}(q)}{|B_\epsilon(q_0)|}\left|\tilde{f}(q) - \tilde{f}(q_0)\right|\end{aligned}$$

shows the analogous result for integration against $\delta_{q_0,\epsilon}$.

2. For $s_0 \in \mathbb{R}$ define the sequence elements by

$$\begin{aligned}m_{s_0,n} := \min\left\{m\in\mathbb{Z}\,\middle|\, m + \frac{1}{2} \geq 2^n s_0\right\} &\Rightarrow 2^n s_0 - \frac{1}{2} \leq m_{s_0,n} < 2^n s_0 + \frac{1}{2}\\ \Rightarrow |m_{s_0,n} - 2^n s_0| \leq \frac{1}{2} &\Rightarrow \left|\frac{m_{s_0,n}}{2^n} - s_0\right| \leq \frac{1}{2^{n+1}}\\ &\Rightarrow \left|\frac{2m\pm1}{2^{n+1}} - s_0\right| \leq \frac{1}{2^n}\end{aligned} \tag{3.2.16}$$

[3]Addition on $[-\pi,\pi]$ is defined in a periodic way, i.e. by regarding the interval as a circle and identifying the endpoints.

The integral with the corresponding kernels can be estimated against the integral from part 1, up to a prefactor:

$$\int \mathrm{d}s\, b^{(m_{s_0,n},n)}(s)|\tilde{f}(s)-\tilde{f}(s_0)| = \frac{1}{2^n}\int \mathrm{d}s\, \chi_{\left[\frac{2m-1}{2^{n+1}},\frac{2m+1}{2^{n+1}}\right]}(s)|\tilde{f}(s)-\tilde{f}(s_0)|$$

$$\overset{(3.2.16)}{\leq} \frac{1}{2^n}\int \mathrm{d}s\, \chi_{\left[s_0-\frac{1}{2^n},s_0+\frac{1}{2^n}\right]}(s)|\tilde{f}(s)-\tilde{f}(s_0)| = 2\int \mathrm{d}s\, b_{s_0,\frac{1}{2^n}}(s)|\tilde{f}(s)-\tilde{f}(s_0)|$$

3. A simple calculation shows that the summation in the definition (3.2.14) of the Fejér kernel can be written as a single term:

$$\begin{aligned}\mathcal{F}_M(x) =& \frac{1}{M}\sum_{N=0}^{M-1}\sum_{n=-N}^{N}\mathrm{e}^{\mathrm{i}nx} = \frac{1}{M}\sum_{N=0}^{M-1}\mathrm{e}^{-\mathrm{i}Nx}\sum_{n=0}^{2N}\mathrm{e}^{\mathrm{i}nx}\\ =& \frac{1}{M}\sum_{N=0}^{M-1}\mathrm{e}^{-\mathrm{i}Nx}\frac{1-\mathrm{e}^{\mathrm{i}(2N+1)x}}{1-\mathrm{e}^{\mathrm{i}x}} = \frac{1}{M}\frac{1}{1-\mathrm{e}^{\mathrm{i}x}}\left(\sum_{N=0}^{M-1}\mathrm{e}^{-\mathrm{i}Nx}-\mathrm{e}^{\mathrm{i}x}\sum_{N=0}^{M-1}\mathrm{e}^{\mathrm{i}Nx}\right)\\ =& \frac{1}{M}\frac{1}{1-\mathrm{e}^{\mathrm{i}x}}\left(\frac{1-\mathrm{e}^{-\mathrm{i}Mx}}{1-\mathrm{e}^{-\mathrm{i}x}}-\mathrm{e}^{\mathrm{i}x}\frac{1-\mathrm{e}^{\mathrm{i}Mx}}{1-\mathrm{e}^{\mathrm{i}x}}\right) = \frac{1}{M}\frac{2-\mathrm{e}^{-\mathrm{i}Mx}-\mathrm{e}^{\mathrm{i}Mx}}{(1-\mathrm{e}^{-\mathrm{i}x})(1-\mathrm{e}^{\mathrm{i}x})}\\ =& \frac{1}{M}\frac{1-\cos(Mx)}{1-\cos x} = \frac{1}{M}\frac{\sin^2\frac{Mx}{2}}{\sin^2\frac{x}{2}}\end{aligned} \tag{3.2.17}$$

The shifted kernel, defined as

$$\mathcal{F}_{x_0,M}(x) := \mathcal{F}_M(x-x_0) = \sum_{n=-M+1}^{M-1}\mathcal{F}_M^{(n)}\mathrm{e}^{\mathrm{i}n(x-x_0)} \text{ with } \mathcal{F}_M^{(n)} = 1-\frac{|n|}{M},$$

is normalized in the sense of

$$\int_{-\pi}^{\pi}\mathrm{d}x\,\mathcal{F}_{x_0,M}(x) = \sum_{n=-M+1}^{M-1}\mathcal{F}_M^{(n)}\int_{-\pi}^{\pi}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}n(x-x_0)} = F_M^{(0)} = 1$$

and fulfills the radial bound

$$\frac{2}{\pi}|x| \leq \sin|x| \leq |x|\ \forall\, x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \overset{(3.2.17)}{\Rightarrow} \mathcal{F}_M(x) \leq \min\left\{\frac{\pi^2}{4}M,\frac{\pi^2}{Mx^2}\right\}.$$

Combining the estimates, which agree at $x = 2/M$, $\mathcal{F}_M$ is dominated by the function $\mathcal{F}_M^{\uparrow}$, given by

$$\mathcal{F}_M^{\uparrow}(x) := \frac{\pi^2}{M\left(\max\left\{\frac{2}{M},|x|\right\}\right)^2} = \begin{cases}\frac{\pi^2}{4}M & \text{for } |x| \leq \frac{2}{M}\\ \frac{\pi^2}{Mx^2} & \text{otherwise}\end{cases} \tag{3.2.18}$$

which is even and monotonously decreasing and can therefore, following the proof of [Sch04][Lemma 7], be linearly decomposed into box kernels

$$b_{x'} := \frac{1}{2x'}\chi_{[-x',x']}$$

for $0 < x' < \pi$, as the following calculation demonstrates:

$$\begin{aligned}\frac{4\pi^2}{M}\int_{\frac{2}{M}}^{\pi}\mathrm{d}x'\,\frac{1}{x'^2}b_{x'}(x) =& \frac{2\pi^2}{M}\int_{\max\left\{\frac{2}{M},|x|\right\}}^{\pi}\mathrm{d}x'\,\frac{1}{x'^3} = \frac{\pi^2}{M\left(\max\left\{\frac{2}{M},|x|\right\}\right)^2}-\frac{1}{M}\\ &\overset{(3.2.18)}{=} \mathcal{F}_M^{\uparrow}(x)-\frac{2\pi}{M}b_\pi(x)\ \forall\, x\in[-\pi,\pi]\end{aligned} \tag{3.2.19}$$

With the Hardy-Littlewood Maximal Function defined for all $x_0 \in [-\pi, \pi]$ by

$$[Mf](x_0) := \sup_{0<x'<\pi} \int_{-\pi}^{\pi} \mathrm{d}x \, b_{x'}(x - x_0) f(x)$$

The convolution of any $\tilde{f} \in L^1([-\pi, \pi])$ with $\mathcal{F}_M$ can therefore be estimated in the following way:

$$\begin{aligned}
&\int_{-\pi}^{\pi} \mathrm{d}x \, \mathcal{F}_M^{\uparrow}(x - x_0) |\tilde{f}(x)| \\
&\overset{(3.2.19)}{=} \frac{2\pi}{M} \lim_{\epsilon \searrow 0} \int_{\frac{2}{M}}^{\pi+\epsilon} \mathrm{d}x' \left(\frac{2\pi}{x'^2} + \delta(x' - \pi) \right) \int_{-\pi}^{\pi} \mathrm{d}x \, b_{x'}(x - x_0) |\tilde{f}(x)| \leq C[Mf](x_0) \\
&\Rightarrow \left| \int_{-\pi}^{\pi} \mathrm{d}x \, \mathcal{F}_M(x - x_0) \tilde{f}(x) \right| \leq \int_{-\pi}^{\pi} \mathrm{d}x \, \mathcal{F}_M^{\uparrow}(x - x_0) |\tilde{f}(x)| \leq C[M\tilde{f}](x_0) \quad (3.2.20)
\end{aligned}$$

In view of the standard proof of the Lebesgue Differentiation Theorem [Rud87] which depends essentially on an estimate against the Maximal Function like (3.2.20), the latter is sufficient to establish the claim.

□

Remark 11. It is briefly presented here how one can easily introduce the functions presented in Lemma 11, part 1., which are necessary for the restriction of the integral defining γ, at least for the transversal components $\mathrm{p} \in \mathbb{R}^2$ of the momentum p. However, introducing the restriction for the variable k as well will be more involved and is the topic of Lemma 12.

Let $q_0 \in Q$ be a Lebesgue point of K and define for $\epsilon > 0$ and a function b, as in Lemma 11, part 3.:

$$\begin{aligned}
\Psi_{\mathrm{p}_0,\epsilon}(p, k) &= b_{\mathrm{p}_0,\epsilon}(\mathrm{p}) \hat{f}(p) \tilde{u}_1(p, h)(k) \qquad (3.2.21) \\
\delta_{\tilde{p}_0,\tilde{k}_0,\epsilon}(\tilde{p}, \tilde{k}) &= \frac{\chi_{B^2_\epsilon(\tilde{p}_0,\tilde{k}_0)}(\tilde{p}, \tilde{k})}{\mu_2(B^2_\epsilon(\tilde{p}_0, \tilde{k}_0))}
\end{aligned}$$

While $\delta_{\tilde{p}_0,\tilde{k}_0,\epsilon} \in \mathcal{H}_1$ is as good a choice as ϕ for the definition of γ in eq. (3.2.2), the support assumptions on f, h have been necessary for the claim of Lemma 9, which on the other hand justified the assumptions of Lemma 10. However, the following calculation, using $\mathrm{d}^4x = \mathrm{d}x_+ \mathrm{d}x_- \mathrm{d}^2\mathrm{x}/2$ and $\mathrm{d}x_\pm := \mathrm{d}x_+ \mathrm{d}x_-/2$, as well as the convolution theorem for the

Fourier transform,

$$
\begin{aligned}
&\check{b}_{\mathrm{p}_0,\epsilon}(\mathrm{y}) := \frac{1}{(2\pi)^2}\int \mathrm{d}^2\mathrm{y}\, \mathrm{e}^{\mathrm{i}\mathrm{p}\mathrm{y}} b_{\mathrm{p}_0,\epsilon}(\mathrm{p}) \\
\Rightarrow &\Psi_{\mathrm{p}_0,\epsilon}(p,k) \overset{(3.2.21)}{=} b_{\mathrm{p}_0,\epsilon}(\mathrm{p})\Psi(p,k) \\
\overset{(3.2.10)}{=} &b_{\mathrm{p}_0,\epsilon}(\mathrm{p})\hat{f}(p)\tilde{u}_1(p,h)(k) \overset{(2.2.15)}{=} b_{\mathrm{p}_0,\epsilon}(\mathrm{p})\left(\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} f(x)\right)\tilde{u}_1(p,h)(k) \\
= &\left(\int \mathrm{d}^2\mathrm{y}\, \mathrm{e}^{-\mathrm{i}\mathrm{p}\mathrm{y}} \check{b}_{\mathrm{p}_0,\epsilon}(\mathrm{y})\right)\left(\int \mathrm{d}x_+\mathrm{d}x_-\, \mathrm{e}^{\frac{\mathrm{i}}{2}\left(p_-x_+ + \frac{|\mathrm{p}|^2}{p_-}x_-\right)}\int \mathrm{d}^2\mathrm{x}\, \mathrm{e}^{-\mathrm{i}\mathrm{p}\mathrm{x}} f(x_\pm,\mathrm{x})\right)\tilde{u}_1(p,h)(k) \\
= &\int \mathrm{d}x_+\mathrm{d}x_-\, \mathrm{e}^{\frac{\mathrm{i}}{2}\left(p_-x_+ + \frac{|\mathrm{p}|^2}{p_-}x_-\right)}\int \mathrm{d}^2\mathrm{x}\, \mathrm{e}^{-\mathrm{i}\mathrm{p}\mathrm{x}} \underbrace{\left(\int \mathrm{d}^2\mathrm{y}\, f(x_\pm,\mathrm{x}-\mathrm{y})\check{b}_{\mathrm{p}_0,\epsilon}(\mathrm{y})\right)}_{=:f_{\mathrm{p}_0,\epsilon}(x)}\tilde{u}_1(p,h)(k) \\
= &\left(\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px} f_{\mathrm{p}_0,\epsilon}(x)\right)\tilde{u}_1(p,h)(k) \overset{(2.2.15)}{=} \hat{f}_{\mathrm{p}_0,\epsilon}(x)\tilde{u}_1(p,h)(k)
\end{aligned}
$$

where $\check{b}_{\mathrm{p}_0,\epsilon} \in \mathcal{S}(\mathbb{R}^2)$, since it is the Fourier transform of a compactly supported C^∞-function, shows that replacing f in eq. (3.2.10) by the convolution $f_{\mathrm{p}_0,\epsilon}$ introduces the function $b_{\mathrm{p}_0,\epsilon}$ occurring in eq. (3.2.21). The crucial property of $f_{\mathrm{p}_0,\epsilon}$ is that the projection of its support to the x^0-x^3-plane is the same as that of f itself. By eq. (3.2.1), the results of Lemma 9 are therefore preserved under this replacement.

The following lemma shows that the desired form eq. (3.2.21) of the approximation to a δ-distribution obtained from the string-field eq. (2.2.17) can be achieved for the k-dependent factor as well. This result is not as easy to prove as the construction of the p-dependent factor by the previous remarks, therefore the proof is found in chapter A in the appendix.

Lemma 12 (Approximation method for string-fields)**.** *Let $\mathrm{p}_0 \in \mathbb{R}^2$, $k_0 \in \kappa S^1$, such that $\mathrm{p}_0 \nparallel k_0$ are not parallel.*[4] *For $\epsilon > 0$, consider the function*

$$\Psi_{\mathrm{p}_0,k_0,\epsilon} : (\partial V^+ \cup \partial V^-) \times \kappa S^1 \to \mathbb{C}\,,\ (p,k) \mapsto \hat{f}\left(p_-, \frac{|\mathrm{p}|^2}{p_-}\right) b_{\mathrm{p}_0,\epsilon}(\mathrm{p})\delta_{k_0,\epsilon}(k), \qquad (3.2.22)$$

where $f \in \mathcal{S}(\mathbb{M})$ is a fixed function with $\operatorname{supp} f \subset W$*. There is a sequence of sets of finitely many functions*

$$\left((f^i_{\epsilon,N}, h^i_{\epsilon,N}) \in \mathcal{S}(\mathbb{R}^4) \times \mathcal{D}(H),\ i = 1,...,M_{\epsilon,N}\right)_{N\in\mathbb{N}} \qquad (3.2.23)$$

which can be interpreted as the smearing functions of strings contained in the standard wedge W, i.e.

$$\operatorname{supp} f^i_{\epsilon,N} \subset W,\ \operatorname{supp} h^i_{\epsilon,N} \subset W \cap H\ \forall\, i = 1,...,M_{\epsilon,N}, N \in \mathbb{N},$$

which can be used to approximate $\Psi_{\mathrm{p}_0,k_0,\epsilon}$ in the sense of L^2 up to a continuous function $c_{k_0} : \mathbb{R}^2 \times \kappa S^1 \to \mathbb{C}$, for ϵ sufficiently small:

$$\lim_{N\to 0}\int \frac{\mathrm{d}p_-}{|p_-|}\mathrm{d}^2\mathrm{p}\int \mathrm{d}\nu(k)\left|\sum_{i=1}^{M_{\epsilon,N}} \hat{f}^i_{\epsilon,N}(p)\tilde{u}_1(p,h^i_{\epsilon,N})(k) - c_{k_0}(\mathrm{p},k)\Psi_{\mathrm{p}_0,k_0,\epsilon}(p,k)\right|^2 = 0 \qquad (3.2.24)$$

[4]This restriction is admissible because it excludes only a subset of measure zero in $\mathbb{R}^2 \times \kappa S^1$.

The function c_{k_0} is has the property

$$c_{k_0}(\mathrm{p}, k_0) = 1 \;\forall\, k_0 \in S^1, \mathrm{p} \in \mathbb{R}^2. \tag{3.2.25}$$

Proof. The proof of Lemma 12 is found in chapter A in the appendix. □

Definition 4. It is convenient to define the following functions, which depend on the sequence constructed in Lemma 12, in order to perform the $\epsilon \to 0$-approximation:

$$K_{q_0,\epsilon}(p_-) = \int \mathrm{d}\mu(q)\, \delta_{q_0,\epsilon}(q) c_{k_0}(\mathrm{p}, k) K(p_-, q) \tag{3.2.26}$$

$$\gamma_{q_0,\epsilon}(a) = \int \mathrm{d}p_-\, \mathrm{e}^{\frac{\mathrm{i}}{2}ap_-} K_{q_0,\epsilon}(p_-) \tag{3.2.27}$$

$$K_{q_0,\epsilon,N}(p_-) = \frac{1}{p_-} \int \mathrm{d}\mu(q) \sum_{i=1}^{M_{\epsilon,N}} \hat{f}^i_{\epsilon,N}(p) \tilde{u}_1(p, h^i_{\epsilon,N})(k) \delta_{\tilde{p}_0,\tilde{k}_0,\epsilon}(\tilde{p}, \tilde{k}) I(p, k, \tilde{p}, \tilde{k}) \tag{3.2.28}$$

$$\gamma_{q_0,\epsilon,N}(a) = \int \mathrm{d}p_-\, \mathrm{e}^{\mathrm{i}ap_-} K_{q_0,\epsilon,N}(p_-) \tag{3.2.29}$$

Remark 12. Definition 4 illustrates the approximation necessary for the restriction process: While eq. (3.2.26) contains the function $\delta_{q_0,\epsilon}$ directly, the corresponding factors in eq. (3.2.28) are, except for the part $\delta_{\tilde{p}_0,\tilde{k}_0,\epsilon}$, the sequence elements constructed in lemma 12.

It is now relatively straightforward to show that the approximation technique presented in Lemma 12 is sufficient to obtain eq. (3.2.26), the approximate integral kernel for $B(g)$ with the functions (3.2.28) which can be reached from the corresponding matrix-element of the relative commutator.

Lemma 13 (Approximation of restrictions in the relative commutator)**.**

$$\lim_{N\to\infty} ||K_{q_0,\epsilon,N} - K_{q_0,\epsilon}||_1 = 0$$

Proof. Using eq. (3.2.26)-(3.2.29) yields

$$\begin{aligned}
||K_{q_0,\epsilon,N} - K_{q_0,\epsilon}||_1^2 &= \left(\int \mathrm{d}p_-\, |K_{q_0,\epsilon,N}(p_-) - K_{q_0,\epsilon}(p_-)| \right)^2 \\
\leq \Bigg(\int \frac{\mathrm{d}p_-}{|p_-|} \int \mathrm{d}^2\mathrm{p} \int \mathrm{d}\nu(k) &\left| \sum_{i=1}^{M_{\epsilon,N}} \hat{f}^i_{\epsilon,N}(p) \tilde{u}_1(p, h^i_{\epsilon,N})(k) - c_{k_0}(\mathrm{p}, k) \Psi_{\mathrm{p}_0,k_0,\epsilon}(p, k) \right| \\
&\left| \int \widetilde{\mathrm{d}\tilde{p}} \int \mathrm{d}\nu(\tilde{k}) \delta_{\tilde{p}_0,\tilde{k}_0,\epsilon}(\tilde{p}, \tilde{k}) I(p, k, \tilde{p}, \tilde{k}) \right| \Bigg)^2 \\
\leq \int \frac{\mathrm{d}p_-}{|p_-|} \int \mathrm{d}^2\mathrm{p} \int \mathrm{d}\nu(k) &\left| \sum_{i=1}^{M_{\epsilon,N}} \hat{f}^i_{\epsilon,N}(p) \tilde{u}_1(p, h^i_{\epsilon,N})(k) - c_{k_0}(\mathrm{p}, k) \Psi_{\mathrm{p}_0,k_0,\epsilon}(p, k) \right|^2 \\
\int \frac{\mathrm{d}p_-}{|p_-|} \int \mathrm{d}^2\mathrm{p} \int \mathrm{d}\nu(k) &\left| \int \widetilde{\mathrm{d}\tilde{p}} \int \mathrm{d}\nu(\tilde{k}) \delta_{\tilde{p}_0,\tilde{k}_0,\epsilon}(\tilde{p}, \tilde{k}) I(p, k, \tilde{p}, \tilde{k}) \right|^2,
\end{aligned}$$

where the Cauchy-Schwarz inequality has been used in the last step. The right hand side of this estimate matches the sequence in eq. (3.2.24) up to a factor given by the last line, therefore the claim can be proven by applying Lemma 12. □

Once it has been established that the functions $K_{q_0,\epsilon,N}$, which originate from actual relative commutators can approximate the functions $K_{q_0,\epsilon}$, which contain the factor $\delta_{q_0,\epsilon}(q)$ explicitly in the integrand, it is convenient to pick a suitable element from the sequence:

Definition 5 (Choice of the approximation sequence)**.** For $q_0 \in Q, \epsilon > 0$, let $M \in \mathbb{N}$, such that

$$||K_{q_0,\epsilon,N} - K_{q_0,\epsilon}||_1 < \epsilon \;\forall\, M \leq N \in \mathbb{N}, \tag{3.2.30}$$

which is possible by Lemma 12, and define the functions

$$\tilde{K}_{q_0,\epsilon} := K_{q_0,\epsilon,M} \text{ and} \tag{3.2.31}$$

$$\tilde{\gamma}_{q_0,\epsilon} := \gamma_{q_0,\epsilon,M} \overset{(3.2.29)}{=} \int \mathrm{d}p_-\, \mathrm{e}^{\mathrm{i}p_- a} \tilde{K}_{q_0,\epsilon}(p_-) \tag{3.2.32}$$

with M depending on q_0 and ϵ.[5]

The holomorphic Fourier transforms of the commutator matrix elements which have been restricted in this way behave in the following way for the limit $\epsilon \to 0$:

3.2.5 Analytic continuation of the restricted matrix element

Lemma 14 (Approximation of the holomorphic Fourier transform)**.** *The family of functions parametrized by $q_0 \in Q$ and $\epsilon > 0$ and given by (cf. Definition 5)*

$$\hat{\gamma}_{q_0,\epsilon} \;:\; H^+ \to \mathbb{C},\; z \mapsto \int \mathrm{d}a\, \mathrm{e}^{-\mathrm{i}za} \tilde{\gamma}_{q_0,\epsilon}(a) \tag{3.2.33}$$

has the following property: For almost all q_0 there is an analytic function $\hat{\gamma}_{q_0}$ on H^+ such that

$$\lim_{\epsilon \to 0} \hat{\gamma}_{q_0,\epsilon}(z) = \hat{\gamma}_{q_0}(z) \;\forall\, z \in H^+. \tag{3.2.34}$$

For any compact subset $K \subset H^+$, the limit is uniform in $z \in K$, which is known as compact convergence.

Remark 13. It may be helpful when considering the proof of Lemma 14 to strengthen the assumption of local L^2-integrability eq. (3.1.2) by further assuming the coefficient functions in eq. (3.1.1) to be smooth. The function K which occurs in eq. (3.2.40) and the following equations then becomes a smooth function as well, rendering all choices of Lebesgue points unnecessary, since all points in the domain of a continuous function are Lebesgue points. Consequently, the statement eq. (3.2.34) becomes valid for all q_0 without the need to exclude a null set.

[5] Regarding the way that in eq. (3.2.32) the error ϵ coming from the approximation of the function $\gamma_{q_0,\epsilon}$ via application of Lemma 12 is linked in an arbitrary way to the width ϵ of the δ-function introduced in its definition in (3.2.27), the choice of this particular term on the right hand side of the inequality (3.2.30) has been made with simplicity in mind, but any function of $\epsilon > 0$ which vanishes as $\epsilon \to 0$ would be equally appropriate.

Proof of Lemma 14. By Lemma 9, $\operatorname{supp} \gamma_{q_0,\epsilon,N} \subseteq (-\infty, -b]$, therefore the integration range in eq. (3.2.33) can be restricted to this half-axis:

$$\hat{\gamma}_{q_0,\epsilon}(z) = \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{-\mathrm{i}za} \tilde{\gamma}_{q_0,\epsilon}(a) \tag{3.2.35}$$

Compact convergence of this sequence can be established using [Jä03, 8.3, Hilfssatz], by checking the following properties:

- Local boundedness: Using equations (3.2.27), (3.2.32) and applying the Cauchy-Schwarz inequality yields the estimate

$$\begin{aligned} |\tilde{\gamma}_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon}(a)| =& \left| \int \mathrm{d}p_-\, \mathrm{e}^{\mathrm{i}p_- a} \left(\tilde{K}_{q_0,\epsilon}(p_-) - K_{q_0,\epsilon}(p_-) \right) \right| \\ \leq& ||\tilde{K}_{q_0,\epsilon} - K_{q_0,\epsilon}||_1 \overset{(3.2.30)}{<} \epsilon \end{aligned} \tag{3.2.36}$$

which holds for each $a \in \mathbb{R}$, $q_0 \in Q$ and $\epsilon > 0$. Let $z \in H^+$, i.e. $\Im(z) > 0$. It follows

$$\begin{aligned} |\hat{\gamma}_{q_0,\epsilon}(z)| \overset{(3.2.35)}{\leq}& \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{\Im(z)a} |\tilde{\gamma}_{q_0,\epsilon}(a)| \\ \overset{(3.2.36)}{<}& \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{\Im(z)a} (|\gamma_{q_0,\epsilon}(a)| + \epsilon) = \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{\Im(z)a} |\gamma_{q_0,\epsilon}(a)| + \frac{\mathrm{e}^{-\Im(z)b}}{\Im(z)} \epsilon \\ \overset{(3.2.27)}{=}& \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{\Im(z)a} \int \mathrm{d}q\, \delta_{q_0,\epsilon}(q) \left| \int \frac{\mathrm{d}p_-}{p_-} K(p_-, q) \right| + \frac{\mathrm{e}^{-\Im(z)b}}{\Im(z)} \epsilon \\ \leq& (C_{q_0} + \epsilon) \frac{\mathrm{e}^{-\Im(z)b}}{\Im(z)} \quad \text{for some } C_{q_0} > 0,\ \mu\text{-almost all } q_0. \end{aligned} \tag{3.2.37}$$

  Finiteness of the constant C_{q_0} for any particular ϵ is a consequence of the L^2-bounds eq. (3.1.2), which by Lemma 11 implies the convergence in $\epsilon \to 0$ for μ-almost all values of q_0. Uniformity of C_{q_0} w.r.t. ϵ is then a consequence of the fact that a convergent sequence is always bounded. This establishes a pointwise bound in z. But the set

$$D_z := \left\{ z' \in H^+ \,\middle|\, |z' - z| < \frac{\Im(z)}{2} \right\}$$

  is an open neighborhood of z in H^+, i.e. $z \in D \subset H^+$ and eq. (3.2.37) translates to the following estimate of local boundedness of the sequence:

$$|\hat{\gamma}_{q_0,\epsilon}(z')| < 2(C_{q_0} + \epsilon) \frac{\mathrm{e}^{-\frac{\Im(z)}{2}b}}{\Im(z)}\ \forall\, z' \in D_z$$

- Pointwise Convergence: It is sufficient to establish that for μ-almost all q_0, the sequence $\hat{\gamma}_{q_0,\epsilon}(z)$ is Cauchy w.r.t. ϵ for all $z \in D$, where $D \subseteq H^+$ is dense. The estimate can be established in a way similar to eq. (3.2.37), including the remarks concerning the properties of the constant C_{q_0}. In fact, choose any $z \in H^+$ and $\tilde{\epsilon} > 0$. It follows

$$\begin{aligned} |\hat{\gamma}_{q_0,\epsilon}(z) - \hat{\gamma}_{q_0,\epsilon'}(z)| \overset{(3.2.35)}{\leq}& \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{\Im(z)a} |\tilde{\gamma}_{q_0,\epsilon}(a) - \tilde{\gamma}_{q_0,\epsilon'}(a)| \\ \overset{(3.2.36)}{<}& \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{\Im(z)a} (|\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon'}(a)| + \epsilon + \epsilon'). \end{aligned} \tag{3.2.38}$$

The error terms coming from eq. (3.2.36) can be estimated in a straightforward way, hence for ϵ, ϵ' sufficiently small

$$\int_{-\infty}^{-b} \mathrm{d}a \, \mathrm{e}^{\Im(z)a}(\epsilon + \epsilon') < \frac{\tilde{\epsilon}}{5}. \tag{3.2.39}$$

For any $c \in (-\infty, -b]$, the remaining integration in eq. (3.2.38) may be split into the parts $(-\infty, c]$ and $(c, -b]$. The first part can be estimated for $\epsilon, \epsilon' > 0$ sufficiently small as

$$\begin{aligned} &\int_{-\infty}^{c} \mathrm{d}a \, \mathrm{e}^{\Im(z)a} |\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon'}(a)| \\ \overset{(3.2.27)}{\leq} &\int_{-\infty}^{c} \mathrm{d}a \, \mathrm{e}^{\Im(z)a} \int \mathrm{d}q \, \left(\delta_{q_0,\epsilon}(q) + \delta_{q_0,\epsilon'}(q)\right) \int \mathrm{d}p_- \, |K(p_-, q)| \\ \leq & C_{q_0} \frac{\mathrm{e}^{\Im(z)c}}{\Im(z)}, \end{aligned} \tag{3.2.40}$$

with a suitable constant $C_{q_0} > 0$, because the result of the p_--integration is an integrable function in q, thus the result of the subsequent q-integration is a convergent sequence for μ-almost all $q_0 \in Q$ when $\epsilon, \epsilon' \to 0$. It is therefore sufficient to choose C_{q_0} as an upper bound on its limit to establish eq. (3.2.40). From the exponent if follows that if $-c > 0$ is chosen sufficiently big, then

$$\int_{-\infty}^{c} \mathrm{d}a \, \mathrm{e}^{\Im(z)a} |\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon'}(a)| < \frac{\tilde{\epsilon}}{5}. \tag{3.2.41}$$

The second part is estimated by first choosing a number $N \in \mathbb{N}$ and finitely many points

$$a_i := \frac{i}{N} c - \left(1 - \frac{i}{N}\right) b \in (c, -b], \; i = 1, \dots, N \tag{3.2.42}$$

and evaluating the function $\gamma_{q_0,\epsilon}$ at these points:

$$\begin{aligned} \gamma_{q_0,\epsilon}(a_i) &\overset{(3.2.27)}{=} \int \mathrm{d}p_- \, \mathrm{e}^{\frac{\mathrm{i}}{2} a_i p_-} \int \mathrm{d}q \, \delta_{q_0,\epsilon}(q) c_{k_0}(\mathrm{p}, k) K(p_-, q) \\ &= \int \mathrm{d}q \, \delta_{q_0,\epsilon}(q) \underbrace{\int \mathrm{d}p_- \, \mathrm{e}^{\frac{\mathrm{i}}{2} a_i p_-} c_{k_0}(\mathrm{p}, k) K(p_-, q)}_{=: K_i(q)} \end{aligned} \tag{3.2.43}$$

The order of integrations has been changed using Fubini's Theorem and $K \in L^1(\mathbb{R} \times Q)$. For each i the resulting expression is a convergent sequence when $\epsilon \to 0$ for μ-almost all $q \in Q$, since Lemma 11 can be applied to the result of the p_--integration. However, the union of countably and in this case finitely many null sets still has measure zero, i.e.

$$|Q \backslash L(K_i)| = 0 \; \forall \, i = 1, \dots, N \Rightarrow \left| Q \backslash \bigcap_{i=1}^{N} L(K_i) \right| = 0,$$

therefore the N sequences given by eq. (3.2.43) converge for almost all $q \in Q$, chosen independently from i. For these q, they are therefore Cauchy sequences, i.e. for ϵ, ϵ' sufficiently small

$$|\gamma_{q_0,\epsilon}(a_i) - \gamma_{q_0,\epsilon'}(a_i)| < -\frac{\tilde{\epsilon}}{5(b + c)} \; \forall \, i = 1, \dots, N. \tag{3.2.44}$$

The difference between a phase factor of the form encountered in eq. (3.2.43), where it has been evaluated at a specific point given by eq. (3.2.42), and one evaluated at some arbitrary point $a \in (c, -b]$ can be estimated using

$$1 - \cos 2x = 2\sin^2 x \text{ and } \sin^2 x \leq x^2 \; \forall x \in \mathbb{R}$$

$$\begin{aligned}\Rightarrow \left| \mathrm{e}^{\frac{\mathrm{i}}{2}p_- a} - \mathrm{e}^{\frac{\mathrm{i}}{2}p_- a_i} \right|^2 =& 2\left(1 - \Re\left(\mathrm{e}^{\frac{\mathrm{i}}{2}p_-(a-a_i)}\right)\right) = 2\left(1 - \cos\left(\frac{p_-}{2}(a-a_i)\right)\right) \\ =& 4\sin^2\left(\frac{p_-}{4}(a-a_i)\right) \leq \left|\frac{p_-}{2}(a-a_i)\right|^2. \end{aligned} \tag{3.2.45}$$

Let $\delta > 0$. By choosing N sufficiently big it is possible to pick $i = 1, \dots, N$ for all $a \in (c, -b]$ such that $|a - a_i| < \delta$. More precisely, in view of eq. (3.2.42), a choice with $c + b < 2\delta N$ ensures a spacing of the points a_i that is fine enough. According to this choice, define $\tilde{a} := a_i$.

$$\begin{aligned}|\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon}(\tilde{a})| \leq& \int \mathrm{d}p_- \underbrace{\left|\mathrm{e}^{\frac{\mathrm{i}}{2}ap_-} - \mathrm{e}^{\frac{\mathrm{i}}{2}a_i p_-}\right|}_{\overset{(3.2.45)}{\leq} \frac{|p_-|}{2}|a-a_i|} \int \mathrm{d}q\, \delta_{q_0,\epsilon}(q)|K(p_-,q)| \\ =& \frac{1}{2}\underbrace{|a-a_i|}_{<\delta} \int \mathrm{d}q\delta_{q_0,\epsilon}(q) \underbrace{\int \mathrm{d}p_- |p_- K(p_-,q)|}_{=:\tilde{K}(q)} \end{aligned} \tag{3.2.46}$$

Since the function $(p_-, q) \mapsto p_- K(p_-, q)$ is integrable, $\tilde{K} \in L^1(Q)$ and, again by Lemma 11, the set $L(\tilde{K})$ contains almost all $q \in Q$. For these q and the right hand side of eq. (3.2.46) converges when $\epsilon \to$ and consequently, for $\delta, \epsilon > 0$ sufficiently small,

$$|\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon}(\tilde{a})| < -\frac{\tilde{\epsilon}}{5(b+c)}. \tag{3.2.47}$$

The second part of the estimate is now established by combining these preparatory steps, using eq. (3.2.47) for ϵ replaced by ϵ' sufficiently small as well, to obtain

$$\begin{aligned}&\int_c^{-b} \mathrm{d}a\, |\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon'}(a)| \\ \leq& \int_c^{-b} \mathrm{d}a\, (\underbrace{|\gamma_{q_0,\epsilon}(\tilde{a}) - \gamma_{q_0,\epsilon'}(\tilde{a})|}_{\overset{(3.2.44)}{<} -\frac{\tilde{\epsilon}}{5(b+c)}} + \underbrace{|\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon}(\tilde{a})| + |\gamma_{q_0,\epsilon'}(a) - \gamma_{q_0,\epsilon'}(\tilde{a})|}_{\overset{(3.2.47)}{<} -\frac{2\tilde{\epsilon}}{5(b+c)}}) \\ <& -\frac{3\tilde{\epsilon}}{5(b+c)} \int_c^{-b} \mathrm{d}a = \frac{3}{5}\tilde{\epsilon}. \end{aligned} \tag{3.2.48}$$

Finally, the two parts of the integration in eq. (3.2.38), together with the error estimate eq. (3.2.39), yield

$$\begin{aligned}|\hat{\gamma}_{q_0,\epsilon}(z) - \hat{\gamma}_{q_0,\epsilon'}(z)| &\overset{(3.2.38)}{<} \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{\Im(z)a}(|\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon'}(a)| + \epsilon + \epsilon') \\ \overset{(3.2.39)}{<} & \underbrace{\int_{-\infty}^{c} \mathrm{d}a\, \mathrm{e}^{\Im(z)a}|\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon'}(a)|}_{\overset{(3.2.41)}{<} \frac{\tilde{\epsilon}}{5}} + \underbrace{\int_c^{-b} \mathrm{d}a\, \mathrm{e}^{\Im(z)a}|\gamma_{q_0,\epsilon}(a) - \gamma_{q_0,\epsilon'}(a)|}_{\overset{(3.2.48)}{<} \frac{3}{5}\tilde{\epsilon}} + \frac{\tilde{\epsilon}}{5} < \tilde{\epsilon}\end{aligned}$$

which shows that $\hat{\gamma}_{q_0,\epsilon}$ is a Cauchy sequence for $\epsilon \to 0$, because the choice of $\tilde{\epsilon} > 0$ was arbitrary.

- Analyticity: The fact that all functions $\gamma_{q_0,\epsilon,N}$ are analytic in H^+ is shown in Lemma 10.

An implication of the compact convergence, by the Weierstrass Convergence Theorem [Jä03, 8.1, Weierstraßscher Konvergenzsatz], is the analyticity part of the claim. ☐

Lemma 15 (Extension of weak-* limits on the real axis to the upper half-plane)**.** *Let $(\gamma_\epsilon)_{\epsilon>0}$ be a sequence of analytic functions on H^+ with the following properties:*

1. *The limit*
$$\lim_{\epsilon\to 0} \gamma_\epsilon = \gamma \tag{3.2.49}$$
exists in the sense of compact convergence, with γ an analytic function on H^+. (cf. Lemma 14) The sequence fulfills the uniform bound
$$|\gamma_\epsilon(z)| < C\Im(z)^{-1}\,\forall z \in H^+, \epsilon > 0 \tag{3.2.50}$$
with a suitable constant $C > 0$.

2. *For $\epsilon > 0$, the boundary limit*
$$\lim_{t\searrow 0} \gamma_\epsilon(\cdot + \mathrm{i}t) = g_\epsilon \tag{3.2.51}$$
exists and is given by a function $g_\epsilon \in L^1(\mathbb{R})$, where convergence is understood in the weak- topology.*

3. *The corresponding sequence of boundary functions $(g_\epsilon)_{\epsilon>0}$ fulfills*
$$\lim_{\epsilon\to 0} g_\epsilon = 0 \text{ in } L^1(\mathbb{R}). \tag{3.2.52}$$

Then $\gamma = 0$ on all of H^+.

Proof. By [SW64, Thm. 2-17] it is sufficient to show that $\lim_{t\searrow 0}\gamma(\cdot + \mathrm{i}t) = 0$ in the weak-* topology, i.e.
$$\lim_{t\searrow 0}\int \mathrm{d}s\,\gamma(s+\mathrm{i}t)\varphi(s) = 0\;\forall\,\varphi \in C_0^\infty(\mathbb{R}). \tag{3.2.53}$$
A summary of the relations between the various assumptions and the claim, the latter being indicated by a dashed arrow, is given by following diagram:

$$\begin{tikzcd}
\gamma_\epsilon(\cdot + \mathrm{i}t) \arrow[r, "t \searrow 0,\text{ weak-*}", "(3.2.51)"'] \arrow[d, "\epsilon \to 0"', "(3.2.49)"] & g_\epsilon \arrow[d, "\epsilon \to 0, L^1"', "(3.2.52)"] \\
\gamma(\cdot + \mathrm{i}t) \arrow[r, dashed, "t \searrow 0,\text{ weak-*}", "(3.2.53)"'] & 0
\end{tikzcd}$$

The following family of seminorms is used on the space $C_0^\infty(\mathbb{R})$ of test functions:

$$||\varphi||_{K,n} := \sup_{x\in K} |\varphi^{(n)}(x)| \;\forall\, \varphi \in C_0^\infty(\mathbb{R}), K \subset \mathbb{R} \text{ compact}, n \in \mathbb{N} \tag{3.2.54}$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ and $\tilde{\epsilon} > 0$. A bound similar to the one for γ_ϵ holds for the limiting function γ as well: Suppose $\delta := |\gamma(z)| - C\Im(z)^{-1} > 0$ for some $z \in H^+$. Then

$$|\gamma_\epsilon(z) - \gamma(z)| \geq |\gamma(z)| - |\gamma_\epsilon(z)| > \delta + C\Im(z)^{-1} - |\gamma_\epsilon(z)| \overset{(3.2.50)}{>} \delta \,\forall \epsilon > 0,$$

which directly contradicts eq. (3.2.49). Hence

$$|\gamma(z)| < C\Im(z)^{-1} \,\forall z \in H^+. \tag{3.2.55}$$

For $\epsilon > 0$ the difference of γ_ϵ and the limit γ is again an analytic function on H^+

$$\tilde{\gamma}_\epsilon := \gamma_\epsilon - \gamma \tag{3.2.56}$$

and fulfills a similar bound, due to eq. (3.2.50) and eq. (3.2.55):

$$|\tilde{\gamma}_\epsilon(z)| < 2C\Im(z)^{-1} \,\forall z \in H^+, \epsilon > 0 \tag{3.2.57}$$

Its first and second antiderivative of $\tilde{\gamma}_\epsilon$ at some $z \in H^+$ can be computed as

$$\tilde{\gamma}_{\epsilon,z^*}^{(-1)}(z) = \int_{z^*}^{z} \mathrm{d}z' \tilde{\gamma}_\epsilon(z') \text{ and} \tag{3.2.58}$$

$$\tilde{\gamma}_{\epsilon,z^*}^{(-2)}(z) = \int_{z^*}^{z} \mathrm{d}z' \tilde{\gamma}_{\epsilon,z^*}^{(-1)}(z') \tag{3.2.59}$$

respectively, starting from an arbitrary point $z^* \in H^+$ with $\Re(z^*) \in \operatorname{supp}\varphi$. The set $I := \operatorname{supp}\varphi + \mathrm{i}\Im(z^*)$ is a compact subset of H^+ and by assumption 1., the convergence (3.2.49) is uniform on I. In view of eq. (3.2.56), this may be rephrased in the following way: Let $\tilde{\epsilon}' > 0$. Then there is $\epsilon_0 > 0$ such that $|\tilde{\gamma}_\epsilon(z)| < \tilde{\epsilon}'$ for all $0 < \epsilon < \epsilon_0$ and $z \in I$. The focus is on ϵ_0 not depending on z.

The functions defined in eq. (3.2.58) and (3.2.59) fulfill the following bounds, where the integration contour can been assumed to be composed of two pieces, running parallel to the $\Re$- and $\Im$-directions respectively without changing the result due to $\tilde{\gamma}_\epsilon$ and consequently $\tilde{\gamma}_\epsilon^{(-1)}$ being analytic on H^+:

$$\begin{aligned} |\tilde{\gamma}_{\epsilon,z^*}^{(-1)}(z)| &\overset{(3.2.58)}{\leq} \int_{z^*}^{\Re(z)+\mathrm{i}\Im(z^*)} |\mathrm{d}z'| \underbrace{|\tilde{\gamma}_\epsilon(z')|}_{<\tilde{\epsilon}'} + \int_{\Re(z)+\mathrm{i}\Im(z^*)}^{z} |\mathrm{d}z'| \underbrace{|\tilde{\gamma}_\epsilon(z')|}_{\overset{(3.2.57)}{<} 2C|\Im(z')|^{-1}} \\ &< \tilde{\epsilon}'|\Re(z) - \Re(z^*)| + 2C\left|\ln\frac{\Im(z)}{\Im(z^*)}\right| \;\forall z \in I, 0 < \epsilon < \epsilon_0 \end{aligned} \tag{3.2.60}$$

Note that $z' \in I$ for the first integral, thus the preceding discussion justifies the bound $|\tilde{\gamma}_\epsilon(z)| < \tilde{\epsilon}'$. Defining $I' := \operatorname{supp}\varphi + \mathrm{i}]0, \Im(z^*)]$, the bound (3.2.60) can be used to infer the following one:

$$
\begin{aligned}
\Rightarrow |\tilde{\gamma}^{(-2)}_{\epsilon,z^*}(z)| \overset{(3.2.59)}{<} & \int_{z^*}^{z} |\mathrm{d}z'| \left(\tilde{\epsilon}'|\Re(z')-\Re(z^*)| + 2C\left|\ln\frac{\Im(z')}{\Im(z^*)}\right|\right) \\
=& \int_{z^*}^{\Re(z)+\mathrm{i}\Im(z^*)} |\mathrm{d}z'|\,\tilde{\epsilon}'|\Re(z')-\Re(z^*)| \\
&+\int_{\Re(z)+\mathrm{i}\Im(z^*)}^{z} |\mathrm{d}z'| \left(\tilde{\epsilon}'|\Re(z)-\Re(z^*)| + 2C\left|\ln\frac{\Im(z')}{\Im(z^*)}\right|\right) \\
=&\frac{\tilde{\epsilon}'}{2}|\Re(z)-\Re(z^*)|^2 + \tilde{\epsilon}'|\Re(z)-\Re(z^*)||\Im(z)-\Im(z^*)| \\
&+2C|\Im(z)\ln\Im(z) - \Im(z^*)\ln\Im(z^*) \\
&\qquad -\Im(z)+\Im(z^*) - (\Im(z)-\Im(z^*))\ln\Im(z^*)| \\
=&\tilde{\epsilon}'|\Re(z)-\Re(z^*)|\left(\frac{|\Re(z)-\Re(z^*)|}{2} + |\Im(z)-\Im(z^*)|\right) \\
&+2C\left(\Im(z)\left|\ln\frac{\Im(z)}{\Im(z^*)}\right| + |\Im(z)-\Im(z^*)|\right)\ \forall\, z\in I', 0<\epsilon<\epsilon_0
\end{aligned}
\tag{3.2.61}
$$

A choice of sufficiently small $\Im(z^*)>0$ and $\tilde{\epsilon}'>0$ therefore results in the right hand side of (3.2.61) being bounded by $\tilde{\epsilon}/(3||\varphi||_{I,2}|I|)$, consequently

$$
\begin{aligned}
\left|\int \mathrm{d}s\, \tilde{\gamma}_\epsilon(s+\mathrm{i}t)\varphi(s)\right| = & \left|\int \mathrm{d}s\, \tilde{\gamma}^{(-2)}_{\epsilon,z^*}(s+\mathrm{i}t)\varphi^{(2)}(s)\right| \leq \int \mathrm{d}s \left|\tilde{\gamma}^{(-2)}_{\epsilon,z^*}(s+\mathrm{i}t)\varphi^{(2)}(s)\right| \\
\leq & ||\varphi||_{I,2}\int \mathrm{d}s \left|\tilde{\gamma}^{(-2)}_{\epsilon,z^*}(s+\mathrm{i}t)\right| \leq ||\varphi||_{I,2}|I|\sup_{z\in I'}|\tilde{\gamma}^{-2}_{\epsilon,z^*}(z)| \\
< & \frac{\tilde{\epsilon}}{3}\ \forall\, 0<t\leq\Im(z^*), 0<\epsilon<\epsilon_0
\end{aligned}
\tag{3.2.62}
$$

by partial integration, taking into account that $\operatorname{supp}\varphi$ is compact for the vanishing of the boundary terms. Applying the definition of the weak-* topology, assumption 2. implies that if $\Im(z^*)>0$ is chosen sufficiently small, dependent on φ, then

$$
\left|\int \mathrm{d}s\,(\gamma_\epsilon(s+\mathrm{i}t)-g_\epsilon(s))\varphi(s)\right| < \frac{\tilde{\epsilon}}{3}\ \forall\, 0<t\leq\Im(z^*), 0<\epsilon<\epsilon_0. \tag{3.2.63}
$$

Using Assumption 3. it is possible to chose ϵ_0 also sufficiently small such that $||g_\epsilon||_1<\tilde{\epsilon}/3$ for $0<\epsilon<\epsilon_0$, hence

$$
\begin{aligned}
\left|\int \mathrm{d}s\, g_\epsilon(s)\varphi(s)\right| \leq & \int \mathrm{d}s\,|g_\epsilon(s)\varphi(s)| \leq ||\varphi||_{I,0}\int \mathrm{d}s\,|g_\epsilon(s)| \\
= & ||\varphi||_{I,0}||g_\epsilon||_1 < \frac{\tilde{\epsilon}}{3}\ \forall\, 0<\epsilon<\epsilon_0.
\end{aligned}
\tag{3.2.64}
$$

A combination of the estimates (3.2.62), (3.2.63) and (3.2.64) yields

$$
\begin{aligned}
\left|\int \mathrm{d}s\,\gamma(s+\mathrm{i}t)\varphi(s)\right| \leq & \left|\int \mathrm{d}s\,\tilde{\gamma}_\epsilon(s+\mathrm{i}t)\varphi(s)\right| + \left|\int \mathrm{d}s\,(\gamma_\epsilon(s+\mathrm{i}t)-g_\epsilon(s))\varphi(s)\right| \\
& + \left|\int \mathrm{d}s\, g_\epsilon(s)\varphi(s)\right| < \tilde{\epsilon}\ \forall\, 0<t\leq\Im(z^*)
\end{aligned}
$$

and since $\tilde{\epsilon}>0$ was arbitrary, this proves eq. (3.2.53). □

Remark 14. By [SW64][Theorem 2-17] the result of Lemma 15 still holds true if $\mathbb{R}$ is replaced by an open set. This is used in Lemma 16, where test functions in $C_0^\infty(\mathbb{R}^+)$ are considered instead.

3.2.6 Analysis of the singularities

Lemma 16 (Limit of the holomorphic Fourier transform)**.** *Let* $q_0 = (p_0, k_0, \tilde{p}_0, \tilde{k}_0), q_1 = (p_1, k_1, \tilde{p}_1, \tilde{k}_1) \in Q$. *Once* q_0 *has been chosen arbitrarily,* q_1 *is then obtained from* q_0 *and a rotation* $\lambda \in \mathrm{SO}(2)$ *(cf. eq. (2.1.24)) by the assumptions*

$$p_1 = p_0,\ \tilde{p}_1 = \tilde{p}_0,\ \textit{but}\ k_1 = k_0\lambda\ \textit{and}\ \tilde{k}_1 = \tilde{k}_0\lambda^{-1}.$$

Define the functions

$$\hat{\gamma}_\epsilon(z) := \hat{\gamma}_{q_1,\epsilon}(z) - P_\lambda(z, q_0)\hat{\gamma}_{q_0,\epsilon}(z)$$

and

$$P_\lambda(p_-, q) := \exp\left(-\mathrm{i}k\lambda \cdot \left(\overline{p} - \overline{\tilde{p}}\frac{p_-}{\tilde{p}_-}\right)^{-1}\right) \exp\left(+\mathrm{i}\tilde{k}\lambda^{-1} \cdot \left(\overline{\tilde{p}} - \overline{p}\frac{\tilde{p}_-}{p_-}\right)^{-1}\right)$$
$$\exp\left(+\mathrm{i}k \cdot \left(\overline{p} - \overline{\tilde{p}}\frac{p_-}{\tilde{p}_-}\right)^{-1}\right) \exp\left(-\mathrm{i}\tilde{k} \cdot \left(\overline{\tilde{p}} - \overline{p}\frac{\tilde{p}_-}{p_-}\right)^{-1}\right).$$

Then

$$\lim_{\epsilon \to 0} \hat{\gamma}_\epsilon(z) = 0,\ \textit{i.e.}\ \hat{\gamma}_{q_1}(z) = P_\lambda(z, q_0)\hat{\gamma}_{q_0}(z)\ \forall\, z \in H^+.$$

Remark 15. The third part of the proof of Lemma 16 which concerns the limit of the function $\hat{\gamma}_\epsilon$ on the real boundary can be simplified by making the additional assumption of smoothness for the coefficient functions in eq. (3.1.1) introduced previously in Remark 13: All points in the domain of the function K which enters in the proof in eq. (3.2.70) become Lebesgue points and one can therefore proceed directly to (3.2.77). The final estimate has to be considered only partially in this case as well, since the integrability and smoothness of K already yields the desired result once eq. (3.2.75) has been used.

Proof of Lemma 16. It is sufficient to establish the assumptions of Lemma 15:

1. Showing the existence and compact convergence of the limit is a result of Lemma 14.

2. The boundary value of the first term is discussed in Lemma 10, while the prefactor in the second term has to be discussed separately: Let $\varphi \in C_0^\infty(\mathbb{R}^+)$. For $t > 0$ sufficiently small, define the function

$$\varphi_t :\ \mathbb{R} \to \mathbb{C},\ s \mapsto \varphi(s)P(s + \mathrm{i}t). \tag{3.2.65}$$

Let $\epsilon > 0$. φ_t is an integrable function and $\gamma_{q_0,\epsilon}$ is bounded, hence Fubini's Theorem may be applied to obtain

$$\begin{aligned} &\int \mathrm{d}s\, P(s + \mathrm{i}t)\hat{\gamma}_{q_0,\epsilon}(s + \mathrm{i}t)\varphi(s) \qquad (3.2.66)\\ \overset{(3.2.33)}{=} &\int \mathrm{d}s\, \varphi(s)P(s + \mathrm{i}t) \int \mathrm{d}a\, \mathrm{e}^{-\mathrm{i}a(s+\mathrm{i}t)}\tilde{\gamma}_{q_0,\epsilon}(a)\\ = &\int \mathrm{d}a\, \mathrm{e}^{at}\tilde{\gamma}_{q_0,\epsilon}(a) \int \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}as}\varphi(s)P(s + \mathrm{i}t) \overset{(3.2.65)}{=} \int \mathrm{d}a\, \mathrm{e}^{at}\tilde{\gamma}_{q_0,\epsilon}(a)\hat{\varphi}_t(a). \end{aligned}$$

P is an analytic function in a neighborhood of the real axis, which implies

$$\lim_{t\searrow 0} P^{(n)}(s+\mathrm{i}t) = P^{(n)}(s) \;\forall\, n\in\mathbb{N}, s\in\mathbb{R}.$$

But the limit is uniform for $s\in\operatorname{supp}\varphi$, since this is a compact set. Therefore $\lim_{t\searrow 0}\varphi_t=\varphi_0$ in the sense of $\mathcal{S}(\mathbb{R})$ and consequently, by the continuity properties of the Fourier transform, $\lim_{t\searrow}\hat{\varphi}_t=\hat{\varphi}_0$ (cf. eq. (3.2.7)) in the sense of $\mathcal{S}(\mathbb{R})$. In particular, considering a suitable seminorm on $\mathcal{S}(\mathbb{R})$, this means that for $\epsilon'>0$ the function

$$a\mapsto \sup_{a_1\in\mathbb{R}}|\gamma_{q_0,\epsilon,N_{q_0,\epsilon}}(a_1)|\left(\sup_{a_2\in\mathbb{R}}\left|(1+{a_2}^2)\hat{\varphi}_0(a_2)\right|+\epsilon'\right)\frac{1}{1+a^2}$$

dominates the integrand on the right hand side of eq. (3.2.66) for $t>0$ sufficiently small and the Dominated Convergence Theorem implies

$$\begin{aligned}\lim_{t\searrow 0}\int \mathrm{d}a\, \mathrm{e}^{at}\tilde{\gamma}_{q_0,\epsilon}(a)\hat{\varphi}_t\,(a) &= \int \mathrm{d}a\,\lim_{t\searrow 0}\tilde{\gamma}_{q_0,\epsilon}(a)\hat{\varphi}_t\,(a)\\ &= \int \mathrm{d}a\,\tilde{\gamma}_{q_0,\epsilon}(a)\hat{\varphi}_0(a),\end{aligned}\tag{3.2.67}$$

where the last step exploits the fact that the convergence in $\mathcal{S}(\mathbb{R}^+)$ also implies pointwise convergence via $\lim_{t\searrow 0}||\hat{\varphi}_t-\hat{\varphi}_0||_\infty=0$. But the boundedness of $\hat{\varphi}_0$ together with $\tilde{K}_{q_0,\epsilon}\in L^1(\mathbb{R})$ implies that, again by Fubini's Theorem,

$$\begin{aligned}&\int \mathrm{d}a\,\tilde{\gamma}_{q_0,\epsilon}(a)\hat{\varphi}_0(a) \overset{(3.2.32)}{=} \int \mathrm{d}a\int \mathrm{d}p_-\,\mathrm{e}^{\mathrm{i}p_-a}\tilde{K}_{q_0,\epsilon}(p_-)\hat{\varphi}_0(a)\\ &=\int \mathrm{d}p_-\,\tilde{K}_{q_0,\epsilon}(p_-)\int \mathrm{d}a\,\mathrm{e}^{\mathrm{i}p_-a}\hat{\varphi}_0(a) \overset{(3.2.7)}{=} 2\pi\int \mathrm{d}p_-\,\tilde{K}_{q_0,\epsilon}(p_-)\varphi_0(p_-)\\ &\overset{(3.2.65)}{=} 2\pi\int \mathrm{d}p_-\,\tilde{K}_{q_0,\epsilon}(p_-)\varphi(p_-)P(p_-).\end{aligned}\tag{3.2.68}$$

Substitution of eq. (3.2.66) into eq. (3.2.67) and combining the result with eq. (3.2.68) yields

$$\lim_{t\searrow 0}\int \mathrm{d}s\,P(s+\mathrm{i}t)\hat{\gamma}_{q_0,\epsilon}(s+\mathrm{i}t)\varphi(s) = 2\pi\int \mathrm{d}p_-\,K_{q_0,\epsilon}(p_-)\varphi(p_-)P(p_-).$$

This argument may be repeated in a slightly simpler form when replacing q_0 by q_1 and the function P by a constant, resulting in

$$\lim_{t\searrow 0}\int \mathrm{d}s\,\hat{\gamma}_{q_1,\epsilon}(s+\mathrm{i}t)\varphi(s) = 2\pi\int \mathrm{d}p_-\,K_{q_1,\epsilon}(p_-)\varphi(p_-),$$

which can subsequently be combined with the result of the original argument for q_0 to obtain the boundary value

$$\begin{aligned}&\lim_{t\searrow 0}\int \mathrm{d}s\,\hat{\gamma}_{\epsilon}(s+\mathrm{i}t)\varphi(s) = \int \mathrm{d}p_-\,g_\epsilon(p_-)\varphi(p_-)\\ &\qquad\text{with } g_\epsilon(p_-) := 2\pi(K_{q_1,\epsilon}(p_-)-P(p_-)K_{q_0,\epsilon}(p_-)).\end{aligned}\tag{3.2.69}$$

Since P is bounded and $K_{q,\epsilon}\in L^1(\mathbb{R})$ for all $q\in Q$, $g_\epsilon\in L^1(\mathbb{R})$.

3. Let $\tilde{\epsilon} > 0$. The first step consists in using eq. (3.2.30) to show the estimate

$$\begin{aligned}||g_\epsilon||_1 \overset{(3.2.69)}{=} & 2\pi \int \mathrm{d}p_- \left|\tilde{K}_{q_1,\epsilon}(p_-) - P_{q_0,\lambda}(p_-)\tilde{K}_{q_0,\epsilon}(p_-)\right| \\ \leq & 2\pi \Bigg(\int \mathrm{d}p_- \left|K_{q_1,\epsilon}(p_-) - P_{q_0,\lambda}(p_-)K_{q_0,\epsilon}(p_-)\right| \\ & + \int \mathrm{d}p_- \left|\tilde{K}_{q_1,\epsilon}(p_-) - K_{q_1,\epsilon}(p_-)\right| + \int \mathrm{d}p_- \left|\tilde{K}_{q_0,\epsilon}(p_-) - K_{q_0,\epsilon}(p_-)\right| \Bigg) \\ \overset{(3.2.30)}{\leq} & 2\pi \int \mathrm{d}p_- \left|K_{q_1,\epsilon}(p_-) - P_{q_0,\lambda}(p_-)K_{q_0,\epsilon}(p_-)\right| + \frac{2}{3}\tilde{\epsilon}\ \forall\, 0 < \epsilon < \tilde{\epsilon}/6\pi.\end{aligned} \tag{3.2.70}$$

Define $R_n := [-(n+1), -n] \cup [n, n+1]$ for all $n \in \mathbb{N}$. Since $K \in L^1(\mathbb{R} \times Q)$, the set

$$M := \left\{ q \in Q \,\middle|\, \int \mathrm{d}p_- \, |K(p_-, q)| < \infty \right\}$$

contains μ-almost all $q \in Q$, i.e. $\mu(Q \backslash M) = 0$, hence for $q \in M$ and $I \in \mathbb{I} := \{\mathbb{R}\} \cup \{R_n | n \in \mathbb{N}\}$ the assignment

$$\tilde{K}_I(q) := \int_I \mathrm{d}p_- \, |K(p_-, q)| \leq \int \mathrm{d}p_- \, |K(p_-, q)| < \infty \tag{3.2.71}$$

yields a well-defined function $\tilde{K}_I \in L^1(Q)$ for each $I \in \mathbb{I}$. By Lemma 11, the sets of Lebesgue points $L(\tilde{K}_I) \subseteq M$ of these functions contain again μ-almost all $q \in Q$ each:

$$\mu\left(Q \backslash L(\tilde{K}_I)\right) = 0 \ \forall\, I \in \mathbb{I}$$

Since $\mathbb{I}$ is countable, the intersection of these sets consequently still contains μ-almost all $q \in Q$:

$$L := \bigcap_{I \in \mathbb{I}} L(\tilde{K}_I) \Rightarrow \mu(Q \backslash L) = \mu\left(\bigcup_{I \in \mathbb{I}} \left(Q \backslash L(\tilde{K}_I)\right)\right) \leq \sum_{I \in \mathbb{I}} \mu\left(Q \backslash L(\tilde{K}_I)\right) = 0$$

Therefore $q_0 \in L$ can be assumed. Since the sequence of functions $(\chi_n)_{n \in \mathbb{N}}$ defined by

$$\chi_n(p_-) := \begin{cases} 1 & \text{for } |p_-| > n \\ 0 & \text{otherwise} \end{cases} \tag{3.2.72}$$

converges to zero pointwise and since $\tilde{K}_{\mathbb{R}}(q_0)$ is finite by construction of $L \subseteq L(\tilde{K}_{\mathbb{R}}) \subseteq M$, the Dominated Convergence Theorem yields

$$\lim_{n \to \infty} \int \mathrm{d}p_- \, \chi_n(p_-)|K(p_-, q_0)| \overset{(3.2.72)}{=} 0.$$

Using the notation introduced in eq. (3.2.71), this implies that for $\epsilon' > 0$, there is $N \in \mathbb{N}$ such that

$$\left| \tilde{K}_{\mathbb{R}}(q_0) - \sum_{n=0}^{N-1} \tilde{K}_{R_n}(q_0) \right| < \epsilon'. \tag{3.2.73}$$

The function

$$P_{q_0,\lambda} : \mathbb{R} \times Q \to \mathbb{C},\ (p_-, q) \mapsto P_\lambda(p_-, q) - P_\lambda(p_-, q_0) \tag{3.2.74}$$

vanishes on $\mathbb{R}\times\{q_0\}$ and is uniformly continuous on the compact subset $[-n,n]\times\{q_0\}$, i.e. for any $\epsilon''>0$ there is $\epsilon_0>0$ such that

$$|P_{q_0,\lambda}(p_-,q)|<\epsilon''\ \forall\, p_-\in[-n,n], q\in B_{\epsilon_0}(q_0). \tag{3.2.75}$$

Moreover, $||P_{q_0,\lambda}||_\infty=2$. Let $\epsilon'''>0$ and $I\in\mathbb{I}_N:=\{\mathbb{R}\}\cup\{R_n|N>n\in\mathbb{N}\}$. By construction of L and using Lemma 11, in particular eq. (3.2.13), there is $\epsilon_I>0$ such that

$$\int \mathrm{d}\mu(q)\,\delta_{q_0,\epsilon}(q)\left|\tilde{K}_I(q)-\tilde{K}_I(q_0)\right|<\epsilon'''\ \forall\, 0<\epsilon<\epsilon_I. \tag{3.2.76}$$

Since $\mathbb{I}_N$ is a finite set it is possible to pick the smallest value ϵ_I,

$$0<\epsilon_{\min}:=\min_{I\in\mathbb{I}_N}\epsilon_I,$$

such that eq. (3.2.76) implies

$$\int \mathrm{d}q\,\delta_{q_0,\epsilon}(q)\left|\tilde{K}_I(q)-\tilde{K}_I(q_0)\right|<\epsilon'''\ \forall\, 0<\epsilon<\epsilon_{\min},\ I\in\mathbb{I}_N. \tag{3.2.77}$$

Combining these preparatory steps and the observation

$$\delta_{q_1,\epsilon}(\mathrm{p},\tilde{p},k\lambda,\tilde{k}\lambda)=\delta_{q_0,\epsilon}(\mathrm{p},\tilde{p},k,\tilde{k})\text{ and }c_{k_1}(\mathrm{p},k\lambda)=c_{k_0}(\mathrm{p},k) \tag{3.2.78}$$

yields the following estimate: For all $0<\epsilon<\epsilon_{\min}$,

$$\begin{aligned}
&\int \mathrm{d}p_-\,|K_{q_1,\epsilon}(p_-)-P_\lambda(p_-,q_0)K_{q_0,\epsilon}(p_-)|\\
&\overset{(3.2.26)}{=}\int \mathrm{d}p_-\left|\int \mathrm{d}\mu(q)\,(\delta_{q_1,\epsilon}(q)c_{k_1}(\mathrm{p},k)-\delta_{q_0,\epsilon}(q)P_\lambda(p_-,q_0)c_{k_0}(\mathrm{p},k))K(p_-,q)\right|\\
&\overset{(3.2.78)}{=}\int \mathrm{d}p_-\left|\int \mathrm{d}\mu(q)\,\delta_{q_0,\epsilon}(q)(P_\lambda(p_-,q)-P_\lambda(p_-,q_0))c_{k_0}(\mathrm{p},k)K(p_-,q)\right|\\
&\overset{(3.2.74)}{\leq}\int \mathrm{d}q\delta_{q_0,\epsilon}(q)\int \mathrm{d}p_-\,|P_{q_0,\lambda}(p_-,q)||K(p_-,q)|\\
&\overset{(3.2.75)}{<}\int \mathrm{d}q\delta_{q_0,\epsilon}(q)\left(\epsilon''\int_{[-N,N]}\mathrm{d}p_-\,|K(p_-,q)|+2\int_{\mathbb{R}\setminus[-N,N]}\mathrm{d}p_-\,|K(p_-,q)|\right)\\
&\overset{(3.2.71)}{=}\int \mathrm{d}q\delta_{q_0,\epsilon}(q)\left(\epsilon''\sum_{n=0}^{N-1}\tilde{K}_{R_n}(q)+2\left(\tilde{K}_{\mathbb{R}}(q)-\sum_{n=0}^{N-1}\tilde{K}_{R_n}(q)\right)\right)\\
&\leq\epsilon''\sum_{n=0}^{N-1}\left(\tilde{K}_{R_n}(q_0)+\underbrace{\int \mathrm{d}q\delta_{q_0,\epsilon}(q)\left|\tilde{K}_{R_n}(q)-\tilde{K}_{R_n}(q_0)\right|}_{\overset{(3.2.77)}{<}\epsilon'''}\right)\\
&\quad+2\Bigg(\underbrace{\left|\tilde{K}_{\mathbb{R}}(q_0)-\sum_{n=0}^{N-1}\tilde{K}_{R_n}(q_0)\right|}_{\overset{(3.2.73)}{<}\epsilon'}+\underbrace{\int \mathrm{d}q\,\delta_{q_0,\epsilon}(q)\left|\tilde{K}_{\mathbb{R}}(q)-\tilde{K}_{\mathbb{R}}(q_0)\right|}_{\overset{(3.2.77)}{<}\epsilon'''}\\
&\qquad+\sum_{n=0}^{N-1}\int \mathrm{d}q\,\delta_{q_0,\epsilon}(q)\left|\tilde{K}_{R_n}(q)-\tilde{K}_{R_n}(q_0)\right|\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&<\epsilon'' \sum_{n=0}^{N-1} \left(\tilde{K}_{R_n}(q_0) + \epsilon''' \right) + 2 \left(\epsilon' + \epsilon''' + \sum_{n=0}^{N-1} \epsilon''' \right) \\
&= \epsilon'' \sum_{n=0}^{N-1} \tilde{K}_{R_n}(q_0) + (N\epsilon'' + 2(N+1))\epsilon''' + 2\epsilon' < \frac{\tilde{\epsilon}}{6\pi},
\end{aligned}
\tag{3.2.79}
$$

if ϵ' (on which N depends, cf. eq. (3.2.73)) and subsequently ϵ'' and ϵ''' are chosen sufficiently small w.r.t. $\tilde{\epsilon}$. In the second step the substitutions $k \mapsto k\lambda$ and $\tilde{k} \mapsto \tilde{k}\lambda^{-1}$ have been made for the first term, which is possible while leaving K unchanged by the symmetry of the function I in eq. (3.2.11). This estimate can be used on the first term in eq. (3.2.70) which finally yields

$$
||g_\epsilon||_1 < \tilde{\epsilon} \;\forall\, 0 < \epsilon < \min \left\{ \frac{\tilde{\epsilon}}{6\pi}, \epsilon_{\min} \right\}.
$$

$\square$

Remark 16. The fact that in the proof of Lemma 16 the support of the function φ is contained in $\mathbb{R}^+$ shows that the singular behaviour of the function K, whose influence can be seen in particular in eq. (3.2.69), is only relevant to the proof for $p_- > 0$. This indicates the possibility to discuss the operators $B(g)$ only in terms of the two-particle states they create from the vacuum and not regard them as dependent on g in the specific form stated in Definition 3, which was necessary to extend the intertwiner equation and therefore the occurrence of the singular prefactors to the functions u_0, u_{0c}, which correspond to $p_- < 0$, in Lemma 8.

The essential singularities which appear in the resulting equation of Lemma 16 are incompatible with the analyticity of the limit which has been shown in Lemma 14. Therefore it is possible to show in the following lemma that the limit has to vanish on H^+.

Lemma 17 (Triviality of the holomorphic Fourier transform)**.** *For almost all $q_0 \in Q$, the function $\hat{\gamma}_{q_0}$, as defined in eq.* (3.2.34)*, vanishes on H^+.*

Proof. The conclusion of Lemma 14 holds for almost all $q_0 \in Q$. Applying a rotation $\mathbf{1} \neq \lambda \in \mathrm{SO}(2)$ to this set yields a corresponding set of almost all $q_1 \in Q$ (cf. Lemma 16 for the relation between each q_0 and q_1). Because the union of two null sets is again a null set, further restricting the choice of $q_0 \in Q$ to those points, where the conclusion of Lemma 14 applies to the corresponding q_1 as well, leaves almost all points of Q.

In conclusion, for $\mathbf{1} \neq \lambda \in \mathrm{SO}(2)$, Lemma 14 can be applied for almost all q_0 and gives functions $\hat{\gamma}_{q_0}$ and $\hat{\gamma}_{q_1}$ which are analytic on H^+. Again with the exception of a null set in Q, Lemma 16 can be applied for these points q_0. The result is the following:

$$
\hat{\gamma}_{q_1}(z) = P_\lambda(z, q_0) \hat{\gamma}_{q_0}(z) \;\forall\, z \in H^+, \text{almost all } q_0 \in Q \tag{3.2.80}
$$

At some point $z^* \in H^+$, for example z_{q_0} or z_{q_1}, the function

$$
P : \, H^+ \to \mathbb{C}, \, z \mapsto P_\lambda(z, q_0)
$$

has an essential singularity, i.e.

$$\forall\, n \in \mathbb{N}, \epsilon > 0, C > 0 \;\exists\, z^* \neq z \in D_\epsilon(z^*) :\; (z - z^*)^n P(z) > C.$$

In other words, for all $n \in \mathbb{N}$, there is a sequence $(z_m^{(n)})_{m\in\mathbb{N}}$ in $H^+ \backslash \{z^*\}$ which converges to z^* and such, that

$$\lim_{m\to\infty} \frac{1}{(z_m^{(n)} - z^*)^n P(z_m)} = 0. \tag{3.2.81}$$

The function $\hat{\gamma}_{q0}$ is analytic on H^+, in particular at the point z^*, and is therefore given by a power series expansion

$$\hat{\gamma}_{q0}(z) = \sum_{n=0}^{\infty} c_n (z - z^*)^n$$

in some open neighborhood $U \ni z^*$. Suppose that $\hat{\gamma}_{q0}$ does not vanish on all of H^+, i.e. there is some $n \in \mathbb{N}$ such that $c_n \neq 0$, and define

$$N := \min\{n \in \mathbb{N} : c_n \neq 0\}.$$

Then $\hat{\gamma}_{q0}$ has a root of order N at z^* and the function R defined by

$$\frac{\hat{\gamma}_{q0}(z)}{(z - z^*)^N} = c_N + R(z) \tag{3.2.82}$$

is analytic on U and vanishes at z^*. In particular, R is continuous at z^*, as is $\hat{\gamma}_{q1}$, hence

$$c_N = \lim_{m\to\infty} c_N + R(z_m^{(N)}) \overset{(3.2.82)}{=} \lim_{z\to z^*} \frac{\hat{\gamma}_{q0}(z)}{(z - z^*)^n} \overset{(3.2.80)}{=} \lim_{z\to z^*} \frac{\hat{\gamma}_{q1}(z)}{(z - z^*)^n P(z)} \overset{(3.2.81)}{=} 0$$

which is a contradiction to $c_N \neq 0$, hence $\hat{\gamma}_{q0}$ does in fact vanish on H^+. □

3.2.7 Extension to the real boundary

Finally, the result of the vanishing limit on H^+ shown in Lemma 17 can be extended to the real boundary to conclude that the integral kernel for $B(g)$ vanishes as well.

Lemma 18 (Triviality of the integral kernel)**.** *The function K (cf. Equation* (3.2.11)*) vanishes almost everywhere.*

Remark 17. As before in Remark 13 and Remark 15, assuming smoothness of the function K removes the restriction of only showing the result of Lemma 18 almost everywhere. In this case the central remaining step of the proof is the derivation of eq. (3.2.91) as well as the final estimate.

Proof of Lemma 18. For each $q \in Q$, define the function

$$K_q : \mathbb{R} \to \mathbb{C},\; p_- \mapsto K(p_-, q). \tag{3.2.83}$$

The fact that $K \in L^1(\mathbb{R} \times Q)$, may be stated as

$$\int \mathrm{d}\mu(q) \int \mathrm{d}p_- \,|h_q(p_-)| \overset{(3.2.83)}{=} ||K||_1 < \infty, \tag{3.2.84}$$

which means the p_- integral is finite, i.e. $h_q \in L^1(\mathbb{R})$, for almost all $q \in Q$. For such q and $n \in \mathbb{Z}$, $r \in \mathbb{N}$ the nth Fourier mode of h_q, restricted to an interval of length $2r$

$$\tilde{K}^{(r,n)}(q) := \int_{-r}^{r} \mathrm{d}p_- \, \mathrm{e}^{-\frac{\pi \mathrm{i}}{r} n p_-} K_q(p_-) \tag{3.2.85}$$

can be defined and is finite because if $h_q \in L^1(\mathbb{R})$,

$$\left|\tilde{K}^{(r,n)}\right| \overset{(3.2.85)}{\leq} \int_{-r}^{r} \mathrm{d}p_- |h_q(p_-)| \leq ||h_q||_1 < \infty.$$

This assignment is sufficient to define a countable set of functions $\tilde{K}^{(r,n)} \in L^1(Q)$, because

$$\int \mathrm{d}\mu(q) \left|\tilde{K}^{(r,n)}(q)\right| \overset{(3.2.85)}{\leq} \int \mathrm{d}\mu(q) \int_{-r}^{r} \mathrm{d}p_- \, |h_q(p_-)| \overset{(3.2.83)}{\leq} ||h||_1 \overset{(3.2.84)}{<} \infty \; \forall \, r \in \mathbb{N}, n \in \mathbb{Z},$$

hence the intersection of the sets

$$L := \bigcap_{\substack{r \in \mathbb{N} \\ n \in \mathbb{Z}}} L\left(\tilde{K}^{(r,n)}\right)$$

contains almost all $q \in Q$. Let $\tilde{\epsilon} > 0$. With

$$\tilde{K}^{(r,n)}_{q_0,\epsilon} := \int \mathrm{d}\mu(q) \, \delta_{q_0,\epsilon}(q) c_{k_0}(\mathrm{p}, k) \tilde{K}^{(r,n)}(q) \tag{3.2.86}$$

this definition means that for $q_0 \in L$, there is $\epsilon_0 > 0$ such that

$$\left|\tilde{K}^{(r,n)}_{q_0,\epsilon} - \tilde{K}^{(r,n)}(q_0)\right| < \frac{\tilde{\epsilon}}{5} \; \forall \, 0 < \epsilon < \epsilon_0. \tag{3.2.87}$$

The definition (3.2.86) can be reformulated using Fubini's Theorem:

$$\begin{aligned}
\tilde{K}^{(r,n)}_{q_0,\epsilon} &\overset{(3.2.85)}{=} \int \mathrm{d}\mu(q) \, \delta_{q_0,\epsilon}(q) c_{k_0}(\mathrm{p}, k) \int_{-r}^{r} \mathrm{d}p_- \, \mathrm{e}^{-\frac{\pi \mathrm{i}}{r} n p_-} h_q(p_-) \\
&\overset{(3.2.83)}{=} \int_{-r}^{r} \mathrm{d}p_- \, \mathrm{e}^{-\frac{\pi \mathrm{i}}{r} n p_-} \int \mathrm{d}\mu(q) \, \delta_{q_0,\epsilon}(q) c_{k_0}(\mathrm{p}, k) h_q(p_-) \\
&\overset{(3.2.26)}{=} \int_{-r}^{r} \mathrm{d}p_- \, \mathrm{e}^{-\frac{\pi \mathrm{i}}{r} n p_-} \int \mathrm{d}\mu(q) \, h_{q_0,\epsilon}(p_-)
\end{aligned} \tag{3.2.88}$$

For r, n given as in eq. (3.2.85) and $\Delta > 0$, define the function

$$\varphi^{(r,n)}_{\Delta}(p_-) := \mathrm{e}^{-\frac{\pi \mathrm{i}}{r} n p_-} \int_{-\infty}^{p_-} \mathrm{d}s \left(\delta_{-(r+\Delta),\Delta}(s) - \delta_{r+\Delta,\Delta}(s)\right).$$

When restricted to the interval $[-r, r]$, the function $\varphi^{(r,n)}_{\Delta}$ becomes the phase factor in eq. (3.2.85) and consequently

$$\begin{aligned}
\int \mathrm{d}p_- \, \tilde{K}_{q_0,\epsilon}(p_-) \varphi^{(r,n)}_{\Delta}(p_-) = &\int_{-r}^{r} \mathrm{d}p_- \, \mathrm{e}^{-\frac{\pi \mathrm{i}}{r} n p_-} \tilde{K}_{q_0,\epsilon}(p_-) \\
&+ \int_{\mathbb{R} \setminus [-r,r]} \mathrm{d}p_- \, \tilde{K}_{q_0,\epsilon}(p_-) \varphi^{(r,n)}_{\Delta}(p_-)
\end{aligned} \tag{3.2.89}$$

Since $K_{q_0,\epsilon} \in L^1(\mathbb{R})$, the Dominated Convergence Theorem may be applied to obtain

$$\lim_{\Delta\to 0}\int_{\pm(r+[0,2\Delta])} \mathrm{d}s\ |K_{q_0,\epsilon}(s)| = \int \mathrm{d}s\ |K_{q_0,\epsilon}(s)| \lim_{\Delta\to 0}\chi_{\pm(r+[0,2\Delta])}(s) = 0, \text{ i.e.}$$
$$\exists\,\epsilon_1 > 0 : \int_{\pm(r+[0,2\Delta])} \mathrm{d}s\ |K_{q_0,\epsilon}(s)| < \frac{\tilde{\epsilon}}{5}\ \forall\, 0 < \Delta < \epsilon_1. \tag{3.2.90}$$

Pick Δ such that the preceding inequalities are satisfied. In a similar way to the proof of Lemma 16 it can be shown that $\tilde{K}_{q_0,\epsilon}$ is the distributional boundary value of $\hat{\gamma}_{q_0,\epsilon}$ up to a prefactor,

$$\begin{aligned}
&\lim_{t\searrow 0}\int \mathrm{d}s\,\hat{\gamma}_{q_0,\epsilon}(s+\mathrm{i}t)\varphi(s) \overset{(3.2.4)}{=} \lim_{t\searrow 0}\int \mathrm{d}s \int \mathrm{d}a\,\mathrm{e}^{-\mathrm{i}(s+\mathrm{i}t)a}\tilde{\gamma}_{q_0,\epsilon}(a)\varphi(s)\\
&= \lim_{t\searrow 0}\int \mathrm{d}a\,\mathrm{e}^{ta}\tilde{\gamma}_{q_0,\epsilon}(a)\underbrace{\int \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}sa}\varphi(s)}_{\overset{(3.2.7)}{=}\hat{\varphi}(a)} = \int \mathrm{d}a\ \lim_{t\searrow 0}\mathrm{e}^{ta}\tilde{\gamma}_{q_0,\epsilon}(a)\hat{\varphi}(a)\\
&= \int \mathrm{d}a\,\tilde{\gamma}_{q_0,\epsilon}(a)\hat{\varphi}(a) \overset{(3.2.32)}{=} \int \mathrm{d}a \int \mathrm{d}p_-\ \mathrm{e}^{\mathrm{i}p_- a}\tilde{K}_{q_0,\epsilon}(p_-)\hat{\varphi}(a)\\
&= \int \mathrm{d}p_-\ \tilde{K}_{q_0,\epsilon}(p_-)\int \mathrm{d}a\,\mathrm{e}^{\mathrm{i}p_- a}\hat{\varphi}(a) = 2\pi\int \mathrm{d}p_-\tilde{K}_{q_0,\epsilon}(p_-)\varphi(p_-),
\end{aligned}$$

i.e. there is $\epsilon_2 > 0$ such that

$$\left|\int \mathrm{d}s\ \left(\hat{\gamma}_{q_0,\epsilon}(s+\mathrm{i}t) - 2\pi\tilde{K}_{q_0,\epsilon}(s)\right)\varphi_\Delta^{(r,n)}(s)\right| < \frac{2\pi}{5}\tilde{\epsilon}\ \forall\, 0 < t < \epsilon_2. \tag{3.2.91}$$

Let t be fixed in accordance this criterion. Using Lemma 17, for almost all $q_0 \in Q$ the limit in eq. (3.2.34) becomes

$$\lim_{\epsilon\to 0}\hat{\gamma}_{q_0,\epsilon}(z) = 0\ \forall\, z \in H^+.$$

Moreover, by the compact convergence shown in Lemma 14 for this limit it follows that there is $\epsilon_3 > 0$ such, that

$$\sup_{|s|\le r+2\Delta}|\hat{\gamma}_{q_0,\epsilon}(s+\mathrm{i}t)| < \frac{2\pi}{5}\tilde{\epsilon}\ \forall\, 0 < \epsilon < \epsilon_3. \tag{3.2.92}$$

Using eq. (3.2.88) and eq. (3.2.89), applying the triangle inequality and finally the previous estimates yield

$$
\begin{aligned}
\left|\tilde{K}^{(r,n)}(q_0)\right| \overset{(3.2.89)}{\leq}\; & \underbrace{\left|\tilde{K}^{(r,n)}(q_0) - \tilde{K}^{(r,n)}_{q_0,\epsilon}\right|}_{\overset{(3.2.87)}{<}\frac{\tilde{\epsilon}}{5}} + \left|\int_{-r}^{r} \mathrm{d}p_-\, \mathrm{e}^{-\frac{\pi \mathrm{i}}{r} n p_-} \left(K_{q_0,\epsilon}(p_-) - \tilde{K}_{q_0,\epsilon}(p_-)\right)\right| \\
& + \left|\int_{\mathbb{R}\setminus[-r,r]} \mathrm{d}s\, K_{q_0,\epsilon}(s) \varphi_\Delta^{(r,n)}(s)\right| + \left|\int \mathrm{d}s\, \hat{\gamma}_{q_0,\epsilon}(s+\mathrm{i}t)\varphi_\Delta^{(r,n)}(s)\right| \\
& + \left|\int \mathrm{d}s\, \left(K_{q_0,\epsilon}(s) - \frac{1}{2\pi}\hat{\gamma}_{q_0,\epsilon}(s+\mathrm{i}t)\right)\varphi_\Delta^{(r,n)}(s)\right| \\
& < \frac{\tilde{\epsilon}}{5} + \underbrace{\int_{-r}^{r} \mathrm{d}p_- \left|K_{q_0,\epsilon}(p_-) - \tilde{K}_{q_0,\epsilon}(p_-)\right|}_{\overset{(3.2.30)}{<}\frac{\tilde{\epsilon}}{5}} + \underbrace{\int_{\pm(r+[0,2\Delta])} \mathrm{d}s\, \left|K_{q_0,\epsilon}(s)\right|}_{\overset{(3.2.90)}{<}\frac{\tilde{\epsilon}}{5}} \\
& + \underbrace{\left|\int \mathrm{d}s \left(K_{q_0,\epsilon}(s) - \frac{1}{2\pi}\hat{\gamma}_{q_0,\epsilon}(s+\mathrm{i}t)\right)\varphi_\Delta^{(r,n)}(s)\right|}_{\overset{(3.2.91)}{<}\frac{\tilde{\epsilon}}{5}} \\
& + \underbrace{\frac{1}{2\pi}\sup_{|s|\leq r+2\Delta} \left|\hat{\gamma}_{q_0,\epsilon}(s+\mathrm{i}t)\right|}_{\overset{(3.2.92)}{<}\frac{\tilde{\epsilon}}{5}} < \tilde{\epsilon}\ \forall\, 0 < \epsilon < \min\left\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \frac{\tilde{\epsilon}}{5}\right\}.
\end{aligned}
$$

In summary, it has been shown up to this point that for almost all $q_0 \in Q$

$$\tilde{K}^{(r,n)}(q_0) = 0\ \forall\, r \in \mathbb{N}, n \in \mathbb{Z}.$$

But these coefficients occur in the Fejér-approximation of $K^{(r)}_{q_0} := K_{q_0} \upharpoonright [-r,r]$:

$$\int_{-r}^{r} \mathrm{d}s\, \mathcal{F}_{s_0,N}(s) K_{q_0}(s) = \sum_{n=-N}^{N} \mathcal{F}_N^{(n)} \int_{-r}^{r} \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}s_0 r} K_{q_0}(s) = \sum_{n=-N}^{N} \mathcal{F}_N^{(n)} \tilde{K}^{(r,n)}(q_0) = 0$$

By Lemma 11, part 3., this function converges to $K_q(s_0)$ for almost all $s_0 \in [-r,r]$. Consequently, $K_{q_0} \upharpoonright [-r,r] = 0$ as a function in $L^1([-r,r])$. Since $r \in \mathbb{N}$ was arbitrary, K_{q_0} vanishes even as a function in $L^1(\mathbb{R})$. But this can be shown for almost all $q_0 \in Q$, therefore $K = 0$ by eq. (3.2.83). □

Remark 18. A possible alternative to the proof technique which is applied in the proof of Lemma 18 would be to replace the Fejér kernel by the box kernels discussed in part 2 of Lemma 11.

3.3 Outlook and Generalizations

3.3.1 Relaxing the square-integrability assumptions

It should be emphasized that the assumption of local square-integrability for the coefficient functions and u_0, u_{0c} as they are exploited in the proof of Lemma 14 is a technical one and not necessarily intrinsic to the class of operators for which locality fails.

A partially alternative proof strategy which does not rely on the assumption goes as follows: Applying the Schwartz Kernel Theorem yields a distribution instead of an L^2-function as the "integral kernel" for $B(g)$. The specialized Reeh-Schlieder Theorem presented in Lemma 12 allows for a restriction of this kernel in q to a compact set (with an error controlled by L^2-bounds) such that the technique from [RS75] for representing cone-localized distributions as finite-order derivatives of continuous functions cited as Lemma 4 applies. Instead of the complications associated with the application of the Lebesgue Differentiation Theorem the admissible points q_0 are now determined by this continuous function.

3.3.2 Classification of modular localized n-particle states

Apart from the fact that Definition 3 concerns only two-particle generation from the vacuum, the main restriction is contained in eq. (3.1.3), namely that the two-particle states that the observable $B(g)$ creates from the vacuum should be of the form given in Lemma 2. This assumption is motivated by the statement in [MSY06][Section 6.3] that if any observable $A = A^*$ is localized in a region $\mathcal{O} \subset \mathbb{M}$, then the vector it creates from the vacuum should satisfy $A\Omega \in \mathcal{K}(\mathcal{O})$, i.e. the localization of an observable implies the corresponding modular localization of the resulting vector.

If there are any infinite spin observables localized in $\mathcal{O}$ which are not covered by theorem 4 and generate nontrivial two-particle vectors $\psi \in \mathcal{H}_2$, these vectors therefore have to satisfy $\psi \in \mathcal{K}_2(\mathcal{O})$ and their form has to differ from eq. (2.2.23). Thus a natural way to extend the scope of this theorem is to show that compact modular localization for two-particle states can only occur with the corresponding vector being of the specified form.

G. Lechner and R. Longo have characterized the standard subspaces in the case of one-dimensional massless and two-dimensional massive free theories for modular localization in an interval and in a double cone, respectively [LL14], see also [LRT78]. Consequently, one idea to show that the two-particle vectors defined in eq. (2.2.23) do in fact span the subspace $\mathcal{K}_2(\mathcal{O})$ would be to extend these results to the present case of massless infinite spin theories in four spacetime dimensions and to all subspaces $\mathcal{H}_n$ of the Fock space $\mathcal{H}$ with fixed particle number. One strategy to show that this extension is possible is outlined in the following.

For the one-particle space[6] $\mathcal{H}_1 = L^2(\partial V^+) \otimes L^2(\kappa S^1)$ decomposing the momentum variable into one lightray-component and the transversal components as $p = (p_-, \mathrm{p})$ and

[6]The structure of the one-particle subspaces for double cones is also currently being investigated by R. Longo, V. Morinelli and K.-H. Rehren [K.-H. Rehren, personal communication, 04/2015].

combining the latter with the infinite spin variable k allows for a decomposition of the form $\mathcal{H}_1 = L^2(\mathbb{R}^+) \otimes \mathcal{H}_\perp$, where the first factor accounts for the coordinate p_- and the Hilbert space $\mathcal{H}_\perp$ captures the dependency on p and k. Since this decomposition is compatible with the standard wedge W_0, the Tomita operator S_{W_0} (cf. eq. (2.2.2)) acts only on the first factor which reproduces the Hilbert spaces considered in [LL14] for fixed p, k such that the discussion therein characterizes the real space $\mathcal{K}(W_0)$ by analytic dependence on the variable p_-.

On the other hand, if even $\psi \in \mathcal{K}(\mathcal{O})$ is assumed for a compact set $\mathcal{O} \subset W_0$, defining $u \in \mathcal{H}_1$ by

$$\psi(p,k) = \mathrm{e}^{\mathrm{i}\varphi(p,k,e)} u(p_-, \mathrm{p}, k) \text{ with } \varphi(p,k,e) = \frac{k \cdot \left(\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right) - \kappa}{2p \cdot e}$$

with $e \in H \cap W_0$ presents ψ as a function on which the representation U_1 acts via Lorentz transformations on e (cf. Lemma 6) and via the rotation part of $\widetilde{E(2)}$ on k, according to the representation eq. (2.1.24). Decomposing u into Fourier modes $u_n(p)$ and applying a Lorentz-boost in the direction of the edge of W_0 then introduces singularities in p for all modes $n \neq 0$ which cannot be removed by corresponding roots of u_n since their position depends continuously on the boost parameter which contradicts the analyticity which was implied by modular localization in W_0. But if only the $n = 0$ component remains the dependence of u on k has effectively vanished and ψ is in fact a string-localized one-particle state.

Concerning the space $\mathcal{H}_n$ for fixed $2 \leq n \in \mathbb{N}$ an analogous splitting $\mathcal{H}_1 = L^2(\mathbb{R}^+) \otimes \mathcal{H}_\perp$ allows for an arbitrary n-particle vector $\psi \in \mathcal{K}_n(\mathcal{O})$ to be rewritten similarly as

$$\begin{aligned}\psi(p_1, \ldots, p_n, k_1, \ldots, k_n) =& \mathrm{e}^{\mathrm{i}\alpha_1} \cdots \mathrm{e}^{\mathrm{i}\alpha_n} u(p_-, \mathrm{p}_1, \ldots, \mathrm{p}_n, k_1, \beta_1, \ldots, \beta_{n-1}) \\ \text{with } \alpha_i :=& - k_i \cdot \frac{1}{\overline{\mathrm{p}_i} - \overline{\mathrm{p}_{i+1}} \frac{p_{i-}}{(p_{i+1})_-}} \text{ for } i = 1, \ldots, n \\ \text{and } \beta_i :=& k_i k_{i+1} \frac{1}{\overline{\mathrm{p}_i} - \overline{\mathrm{p}_{i+1}} \frac{p_{i-}}{(p_{i+1})_-}} \frac{1}{\overline{\mathrm{p}_{i+1}} - \overline{\mathrm{p}_i} \frac{(p_{i+1})_-}{p_{i-}}}, \; i = 1, \ldots, n-1\end{aligned}$$

for a suitable function $u \in \mathcal{H}_n$ and $p_{n+1} := p_1$. This assignment is well-defined since the variables $p_1, \ldots, p_n$ can be determined from their transversal components and the values of $\beta_1, \ldots, \beta_{n-1}$. The variables α_i and β_i are invariant under the action of the second-quantized representation U_n, therefore an analogous argument to the one-particle case shows that the dependence on k_1 is in fact trivial.

For $n = 2$ the result states that ψ is of the form which was determined in Lemma 7 by the intertwiner equation eq. (3.1.3). However, the result for $n > 2$ is useful as well, because the dependency on products of the variables $k_1, \ldots, k_n$ instead of the variables themselves enters crucially in Lemma 17 by rotating all k_i with alternating $\mathbf{1} \neq \lambda \in \mathrm{SO}(2)$, such that the variables β_i are not changed, while the rest of the proof of Theorem 4 generalizes to the n-particle case in a straightforward way.

Chapter 4

Further Constructions Related to Infinite Spin Theories

4.1 Unified Description of Massive and Infinite Spin Representations

4.1.1 Construction of Wigner representations

Wigner construction

A brief summary of the Wigner construction for massive and massless representations follows: For any value of the mass $m \geq 0$, given by $P^2 = m^2\mathbf{1}$ for an irreducible representation of the translations, one fixes the reference momentum $q \in H_m^+$ and constructs the Wigner boost B_p with the property $q\Lambda(B_p) = p$, as well as a representation D of the little group $G_q := \operatorname{stab} q$ on the space $\mathcal{H}_q$. The action of the Wigner representation of any $(A, a) \in \mathcal{P}^c$ on the wave function $\psi \in L^2(H_m^+) \otimes \mathcal{H}_q$ is given by

$$(U(A,a)\psi)(p) = \mathrm{e}^{\mathrm{i}pa} D(R(A,p))\psi(p\Lambda(A)),$$

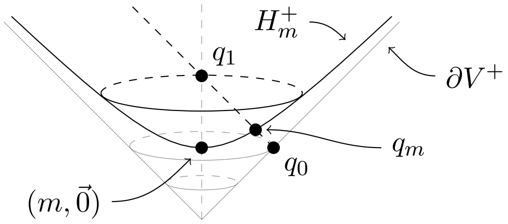
where $R(A,P) = B_p A B_{p\Lambda(A)}^{-1} \in G_q$ is the Wigner rotation.

Conventional approach $m = 1$, $m = 0$

The momentum p is usually described in light cone coordinates $p_\pm \in \mathbb{R}$, $\mathrm{p} \in \mathbb{C}$ by $p_0 = (p_+ + p_-)/2$, $p_1 = \Re(\mathrm{p})$, $p_2 = \Im(\mathrm{p})$, $p_3 = (p_+ - p_-)/2$, i.e. $p_\pm = p_0 \pm p_3$, $\mathrm{p} = p_1 + \mathrm{i}p_2$. For $m = 1$ and $m = 0$ the following choices are made for the reference momentum, Wigner Boost and the little group as well as its representation:

- $m = 1$: $\widehat{q} = m\mathbf{1}$, $B_p = \sqrt{\widehat{p}/m}$, $G_q = \mathrm{SU}(2)$, $\mathcal{H}_q = \operatorname{Sym}\mathbb{C}^{2s}$, $D(R) = R^{\otimes 2s}$
- $m = 0$: $q = \frac{\mathbf{1}+\sigma_3}{2}$, $B_p = \frac{1}{\sqrt{p_-}}\begin{pmatrix} p_- & \overline{\mathrm{p}} \\ & 1 \end{pmatrix}$, $G_q = \widetilde{E(2)}$, $\mathcal{H}_q = L^2(\kappa S^1)$, $[D(\varphi,a)v](k) = \mathrm{e}^{-\mathrm{i}k\cdot\overline{a}} v(k\lambda(-\varphi))$, where the covering $\lambda(\varphi)$ is a rotation by $\varphi/2$.

A priori, the representation spaces look entirely different, with (integer or half-integer) spin s-Hilbert spaces on the one hand and the space of plane waves whose momenta k satisfy $k^2 = \kappa^2$ on the other hand. The aim of the following discussion to show how these



mass m	parametrized	rest frame	relation
$m=1$	q_1	$(1,\vec{0})$	coincidence
$0<m\leq 1$	q_m	$(m,\vec{0})$	$q_m\Lambda(B_m)=(m,\vec{0})$
$m=0$	$q_0=\frac{1}{2}(1,\vec{e}_3)$	-	no rest frame

Figure 4.1.1: Parametrized versus rest-frame reference momenta

representation spaces can be related by the stereographic projection. In particular, the limit $m\to 0$, $s\to\infty$ from massive to massless representations (keeping the modulus of the Pauli-Lubanski vector constant by fixing $\kappa^2=m^2s(s+1)$) can be given a geometrical meaning in the picture constructed here.

m^2-parametrized approach

Proceeding to the general Wigner construction, which contains m only as a parameter, it is possible to avoid a splitting of the discussion into separate cases. The reference vector $q_m\in H_m^+$ is defined by

$$q_{m+}=m^2,\ q_{m-}=1,\ \mathrm{q}_m=0\Leftrightarrow \breve{q}_m=\begin{pmatrix}1&\\&m^2\end{pmatrix},$$

in particular $\breve{q}_1=\mathbf{1}$ and $\breve{q}_0=(\mathbf{1}+\sigma_3)/2$, in accordance with the usual choices given in the previous paragraph. The definition is illustrated in Figure 4.1.1. The Wigner boost is chosen as

$$B_p=\frac{1}{\sqrt{p_-}}\begin{pmatrix}p_-&\overline{\mathrm{p}}\\&1\end{pmatrix}\Rightarrow B_p^{-1}=\frac{1}{\sqrt{p_-}}\begin{pmatrix}1&-\overline{\mathrm{p}}\\&p_-\end{pmatrix}.$$

This formula looks like the Wigner boost for $m=0$, but one observes that it can be used for any m^2. Also, $B_{q_m}=\mathbf{1}\ \forall m$. It follows

$$\begin{aligned}m^2=p^2=p_0^2-p_3^2-(p_1^2+p_2^2)&=\frac{1}{4}(p_++p_-)^2-\frac{1}{4}(p_+-p_-)^2-(p_1-\mathrm{i}p_2)(p_1+\mathrm{i}p_2)\\&=p_+p_--|\mathrm{p}|^2\end{aligned}$$

and hence, applying B_p to the corresponding q_m:

$$\begin{aligned}(q_m\Lambda(B_p))^{\smile}=B_p^\dagger\breve{q}_mB_p&=\frac{1}{p_-}\begin{pmatrix}p_-&\\\mathrm{p}&1\end{pmatrix}\begin{pmatrix}1&\\&m^2\end{pmatrix}\begin{pmatrix}p_-&\overline{\mathrm{p}}\\&1\end{pmatrix}\\&=\begin{pmatrix}p_-&\overline{\mathrm{p}}\\\mathrm{p}&\frac{|\mathrm{p}|^2+m^2}{p_-}\end{pmatrix}=\breve{p}\end{aligned}$$

However, if $p^2 \neq m^2$, the matrix representation $\breve{p}$ shows that the p_+ component does not agree, but still $(q_m\Lambda(B_p))^2 = m^2$. This in principle also allows for a parametrization of p by m in the following way: Writing any p as $p = q_{\sqrt{p^2}}\Lambda(B_p)$, one can fix B_p and substitute p^2 by an arbitrary m^2. The resulting momentum p is consequently shifted along the p_+-direction until it coincides with H_m^+. Now it is possible to study the m-dependence of an arbitrary Wigner rotation

$$\begin{aligned} R(A,p) &= B_pAB^{-1}_{p\Lambda(A)} = B_pAB^{-1}_{q_m\Lambda(B_p)\Lambda(A)} = \underbrace{B_pA}_{=:C}B^{-1}_{q_m\Lambda(B_pA)} = \underbrace{B_{q_m}}_{=1}CB^{-\mathbf{1}}_{q_m\Lambda(C)} \\ &= R(C,q_m) \end{aligned}$$

In this expression, C is independent of m, because in the chosen parametrization only p_+ depends on m, while B_p is independent of p_+. This Wigner rotation can be described more concretely, if one starts with an explicit form of the m-independent part C:

$$\begin{aligned} C :&= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } 1 = \det C = ad - bc \\ \Rightarrow (q_m\Lambda(C))^{\smile} &= C^\dagger \breve{q}_m C = \begin{pmatrix} |a|^2 + m^2|c|^2 & \overline{a}b + m^2\overline{c}d \\ a\overline{b} + m^2c\overline{d} & |b|^2 + m^2|d|^2 \end{pmatrix} \\ \Rightarrow (q_m\Lambda(C))_- &= |a|^2 + m^2|c|^2 \text{ and } \underline{q_m\Lambda(C)} = a\overline{b} + m^2c\overline{d} \\ \Rightarrow B^{-1}_{q_m\Lambda(C)} &= \frac{1}{\sqrt{|a|^2+m^2|c|^2}} \begin{pmatrix} 1 & -\overline{a}b + m^2\overline{c}d \\ & |a|^2 + m^2|c|^2 \end{pmatrix} \\ \Rightarrow R(C,q_m) = CB^{-1}_{q_m\Lambda(C)} &= \frac{1}{\sqrt{|a|^2+m^2|c|^2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\overline{a}b + m^2\overline{c}d \\ & |a|^2 + m^2|c|^2 \end{pmatrix} \\ &= \frac{1}{\sqrt{|a|^2+m^2|c|^2}} \begin{pmatrix} a & -(ad-bc)m^2\overline{c} \\ c & (ad-bc)\overline{a} \end{pmatrix} \\ &= \frac{1}{\sqrt{|a|^2+m^2|c|^2}} \begin{pmatrix} a & -m^2\overline{c} \\ c & \overline{a} \end{pmatrix} \end{aligned}$$

Hence, only one component of $R(A,p)$ depends on m, relatively to the others, with an overall normalization. The usual little groups are recovered for $m = 1$ and $m = 0$. For all functions $f: \mathbb{C} \to \mathbb{C}$, a representation D of all R is given by

$$[D(R)f](z) = f(R^{-1}.z) \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+c}{bz+d}.$$

4.1.2 Properties of the generic representation

Stereographic projection

The projection from a sphere of diameter d to the complex plane is used. In polar coordinates, the azimuthal angle φ is mapped to itself, while $r = d\tan\frac{\vartheta}{2}$ for the radial coordinate r on the plane and the latitude ϑ. It is also convenient to use the coordinates

$$\zeta := \cos\vartheta = \frac{d^2 - r^2}{d^2 + r^2} \text{ and } \rho := \sin\vartheta = \frac{2rd}{d^2 + r^2} \text{ with } z^2 + \rho^2 = 1.$$

The basis vectors of the representation space are the spherical harmonics

$$Y^l_n(\vartheta,\varphi) = P^l_n(\zeta)\mathrm{e}^{\mathrm{i}n\varphi},$$

where the Legendre polynomials P^l_n are defined by the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}\zeta}(1-\zeta^2)\frac{\mathrm{d}}{\mathrm{d}\zeta} + l(l+1) - \frac{n^2}{1-\zeta^2}\right) P^l_m(\zeta) = 0.$$

The function ${}_dP^l_n(r)$ is defined as the pullback of the function $P^l_m(\zeta)$ via stereographic projection for the sphere with diameter d.

Relation to known representations

For the massive representations, define

$$B_m := B_{(m,\vec{0})} = \begin{pmatrix} \sqrt{m} & \\ & \sqrt{m}^{-1} \end{pmatrix}$$

as the Wigner boost which maps q_m to $(m,\vec{0})$, the conventional choice for the reference momentum. Adjoining the Wigner rotation R with these boosts gives an element of the correspondingly transformed little group SU(2):

$$\begin{aligned} B_m^{-1} R B_m &= \frac{1}{\sqrt{|a|^2+m^2|c|^2}} \begin{pmatrix} \sqrt{m}^{-1} & \\ & \sqrt{m} \end{pmatrix} \begin{pmatrix} a & -m^2\bar{c} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} \sqrt{m} & \\ & \sqrt{m}^{-1} \end{pmatrix} \\ &= \frac{1}{\sqrt{|a|^2+m^2|c|^2}} \begin{pmatrix} a & -m\bar{c} \\ mc & \bar{a} \end{pmatrix} =: R_m \in \mathrm{SU}(2) \end{aligned}$$

These intertwined Wigner rotations naturally arise when comparing the generic representation with the conventional one for $m > 0$. In the following, let the mass m and the diameter d be inversely proportional, i.e. $md = 1$. Using

$$R_m.(mz) = \begin{pmatrix} a & -m\bar{c} \\ mc & \bar{a} \end{pmatrix}.(mz) = \frac{amz+mc}{-m^2\bar{c}z+\bar{a}} = m \begin{pmatrix} a & -m^2\bar{c} \\ c & \bar{a} \end{pmatrix}.z = m(R.z)$$

and since ${}_dP^l_m(r) = {}_1P^l_m(r/d)$, one obtains for z a point in the complex plane and ${}_dY^l_n(z)$ defined by pullback from $Y^l_n(\vartheta,\varphi)$ via stereographic projection

$$\begin{aligned} [D(R)\,{}_{\frac{1}{m}}Y^l_n](z) &= {}_{\frac{1}{m}}\,Y^l_n(R^{-1}.z) = {}_1\,Y^l_n(mR^{-1}.z) = {}_1\,Y^l_n(R_m^{-1}.(mz)) \\ &= D^l(R_m)^{n'}_n Y^l_{n'}(mz) = D^l(R_m)^{n'}_n \,{}_{\frac{1}{m}}Y^l_{n'}(z), \end{aligned}$$

where D^l is the usual spin l-representation of SU(2). Hence, for all $m > 0$, the generic representation is in a natural way related to the conventional representation, i.e. by a Wigner boost to the correct reference momentum and adjusting diameter $d = m^{-1}$ of the sphere.

Infinite spin limit

The stereographic projection in the coordinate ζ yields

$$r^2 = d^2\frac{1-\zeta}{1+\zeta} \Rightarrow 2r\mathrm{d}r = d^2\left(-\frac{1}{1+\zeta} - \frac{1-\zeta}{(1+\zeta)^2}\right)\mathrm{d}\zeta = -\frac{2d^2}{(1+\zeta)^2}\mathrm{d}\zeta$$

$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\zeta} = -\frac{d^2}{r(1+\zeta)^2} = -\frac{d^2}{r\left(1+\frac{d^2-r^2}{d^2+r^2}\right)^2} = -r\left(\frac{d^2+r^2}{2rd}\right)^2 = -\frac{r}{\zeta^2}.$$

Substituting into the Legendre differential equation then results in

$$\left(\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 + \underbrace{\left(\frac{2rd}{d^2+r^2}\right)^2 l(l+1)}_{=\frac{1}{\left(1+\left(\frac{r}{d}\right)^2\right)^2}\underbrace{4\frac{l(l+1)}{d^2}}_{=:\kappa^2}r^2} -n^2\right){}_dP^l_n(r) = 0.$$

For κ constant and $m = \frac{1}{d} \to 0$, the equation is solved by the Bessel function ${}_kJ_n(r) = J_n(kr)$, where

$$J_n(\kappa r) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{-\mathrm{i}n\varphi}\mathrm{e}^{-\mathrm{i}\kappa r\sin\varphi}.$$

This integral representation can be considered as the 2d Fourier transform of the nth Fourier mode on the circle:

$$J_{\kappa,n}(z) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{-\mathrm{i}n\varphi}\mathrm{e}^{-\mathrm{i}k(\varphi)\cdot z},$$

where κ, φ are polar coordinates for the vector $k(\varphi)$. Now for

$$R = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & \\ a & \mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix}$$

the resulting transformation is $R^{-1}.z = \mathrm{e}^{-2\mathrm{i}\varphi}z - \mathrm{e}^{-\mathrm{i}\varphi}a$, which gives the usual rotation/shift phases.

Helicity representations

The presented unified description of the representation spaces in terms of functions on $\mathbb{C}$ breaks down for $k=0$, which can be seen in the following way:

- All previous solutions degenerate to $J_n(0)$, with $J_0(0) = 1$ and $J_{n\neq 0}(0) = 0$. Therefore only the $n=0$ case can be described by a function on $\mathbb{C}$, the constant function.
- The Bessel equation becomes ($r = \mathrm{e}^s \Rightarrow \mathrm{d}r = \mathrm{e}^s\mathrm{d}s = r\mathrm{d}s$)

$$\left(\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 - n^2\right){}_0J_n(r) = 0 \Rightarrow \left(\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^2 - n^2\right)\tilde{J}_n(s) = 0$$

$$\Rightarrow \tilde{J}_n(s) = \mathrm{e}^{\pm ns} \Rightarrow J_n(r) = r^{\pm n}.$$

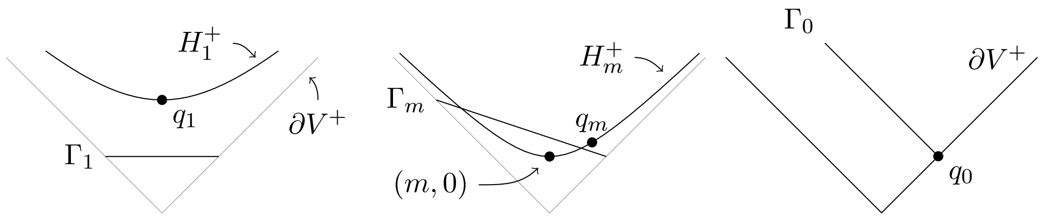


Figure 4.1.2: Reference momenta and little group orbits for $0 \leq m \leq 1$

However, these solutions are not translation-invariant for $n \neq 0$ and are therefore not suitable for the helicity representations. The reason is that their arguments are not correctly scaled due to circumventing the limit: $(\lambda z)^n = \lambda^n z^n$, hence the scaling can be absorbed into a rescaling of the representation space vector.

A much simpler description of the helicity representations can be obtained by considering the limit $m \to 0$ of the representations with fixed l:

- $D(R)_{\frac{1}{m}} f =_{\frac{1}{m}} (D(R_m)f)$, but the off-diagonal elements of R_m vanish for $m \to 0$.
- Considering the function f on S^2 directly, no scaling of the argument is needed and the fact that R_0 is a diagonal matrix directly decouples the spin multiplets.

4.1.3 Parametrization of the little group orbit

In the construction of fields which transform covariantly under these respective representations, one considers the G_q-orbit $\Gamma_q := \{p \in H_0^+ \,|\, pq = 1\}$, which is isometrically mapped to the little group in the following way:

- $m = 1$: $\xi\,:\,\mathrm{SU}(2) \to \Gamma_1$, where $(\xi(A))^{\backsim} = (\mathbf{1} + A)/2$
- $m = 0$: $\xi\,:\,\widetilde{E(2)} \to \Gamma_0$, where $(\xi(z))^{\backsim} = \begin{pmatrix} |z|^2 & \overline{z} \\ z & 1 \end{pmatrix}$

The generic parametrization is defined on $\mathbb{R}^2$ by the stereographic projection to S^2 and visualized in Figure 4.1.2:

$$n_1 = \frac{2d\Re(z)}{d^2 + |z|^2},\ n_2 = \frac{2d\Im(z)}{d^2 + |z|^2},\ n_3 = -\frac{d^2 - |z|^2}{d^2 + |z|^2}$$

$$[\xi_d(z)]^{\backsim} := [\xi(\vec{\sigma} \cdot \vec{n}(z))\Lambda(B_m^{-1})]^{\backsim} = \frac{d^2}{d^2 + |z|^2} \begin{pmatrix} |z|^2 & \overline{z} \\ z & 1 \end{pmatrix}$$

The original representation for $m = 1$ has been used and transformed to Γ_{q_m} via $\Lambda(B_m^{-1})$.

The definition of a G_{q_m}-invariant measure is also needed for the construction of an intertwiner. It is sufficient to consider the θ-dependent part, since the φ-part is identical in both coordinate systems:

$$\begin{aligned}\mathrm{d}\theta\,\sin\theta &= -\mathrm{d}\cos\theta = -\mathrm{d}\frac{d^2-r^2}{d^2+r^2} = -2r\mathrm{d}r\left(-\frac{1}{d^2+r^2}-\frac{d^2-r^2}{(d^2+r^2)^2}\right)\\ &= 4r\mathrm{d}r\frac{d^2}{(d^2+r^2)^2} = 4\frac{r}{d}\frac{\mathrm{d}r}{d}\frac{1}{\left(1+\left(\frac{r}{d}\right)^2\right)^2}\end{aligned}$$

Since the original measure is rotation invariant on S^2, the resulting measure is SU(2)-invariant with respect to the coordinate $\frac{z}{d}$ and hence by $R_m.\frac{z}{d} = \frac{1}{d}(R.z)$ it is $\mathrm{SL}(2,\mathbb{C})$-invariant for z.

An extra factor of d^2 is necessary to give the correct surface area of the sphere on scale d. The resulting measure converges to $4r\mathrm{d}r$ which is just the radial part of the measure on $\widetilde{E(2)}$.

Using these ingredients, it is possible to define an m-dependent string-intertwiner u_1 as well as a two-particle intertwiner u_2:

$$\begin{aligned}u_1(p,e)_n &:= \int \mathrm{d}^2z\left(\frac{d^2}{d^2+|z|^2}\right)^2 Y_n^l(z)F(\xi_m(z)\Lambda(B_p)e)\\ u_2(p,\tilde{p})_{n\tilde{n}} &:= \int \mathrm{d}^2z\left(\frac{d^2}{d^2+|z|^2}\right)^2 Y_n^l(z)\int \mathrm{d}^2\tilde{z}\left(\frac{d^2}{d^2+|\tilde{z}|^2}\right)^2 Y_{\tilde{n}}^l(\tilde{z})\\ &\qquad F(\xi_m(z)\Lambda(B_p)\cdot\xi_m(\tilde{z})\Lambda(B_{\tilde{p}}))\end{aligned}$$

(using the abbreviations $d = 1/m$ and $l(l+1) = (\kappa/2m)^2$ implicitly) The resulting intertwiner equations now involve the corresponding matrix representations of SU(2) on scale d:

$$\begin{aligned}D^l(R(A,p)_m)_n^{n'}u_1(p\Lambda(A),\Lambda(A)e)_{n'} &= u_1(p,e)_n\\ D^l(R(A,p)_m)_n^{n'}D^l(R(A,\tilde{p}))_{\tilde{n}}^{\tilde{n}'}u_2(p,\tilde{p})_{n'\tilde{n}'} &= u_2(p,\tilde{p})_{n\tilde{n}}\end{aligned}$$

This form of the intertwiners illustrates the difficulties that are encountered when attempting to construct localized wave functions or operators: For finite d, the measure which occurs under the integral ensures the convergence of the integrals and the functions ξ_m are bounded in z for $m > 0$. Consequently, the functions F can be chosen as polynomials, for example. This ensures the possibility of analytic continuation in e and p. However, for $d \to \infty$ or equivalently $m \to 0$, the measure becomes translation invariant and the convergence of the integral relies on the function F: Choosing F as a polynomial yields the derivative of a δ-distribution supported at the origin in the limit, while choosing a decreasing function for F is incompatible with analytic continuation, due to Liouville's Theorem in complex analysis.

The more general construction of similar limiting procedures for Lie groups can be found in [IW53] and is applied to the de Sitter group in [MN72].

A procedure to obtain the Wigner equations for infinite spin as a limit of the field equations for positive mass is constructed in [BM06].

4.2 Deformations from String- to Wedge-Local Infinite Spin Fields

The construction of wedge-localized fields by applying a certain deformation to the canonical commutation relations by H. Grosse and G. Lechner is motivated by the idea to construct Quantum Field Theories on noncommutative Minkowski space [GL07][GL08], see also [BS08][BLS11][Lec08][Lec12].

The procedure is applicable to the free string-localized fields defined in eq. (2.2.17) as well, simply by introducing the twisted creation and annihilation operators

$$a_Q^\dagger(p,k) := \mathrm{e}^{-\frac{\mathrm{i}}{2}pQP} a^\dagger(p,k) \text{ and } a_Q(p,k) := \mathrm{e}^{\frac{\mathrm{i}}{2}pQP} a(p,k). \tag{4.2.1}$$

with Q an antisymmetric $4x4$-matrix. Substituting these operators into the string-field yields the corresponding expression

$$\Phi_Q(x,e) = \int \widetilde{\mathrm{d}p} \int \mathrm{d}\nu(k) \, \left(\mathrm{e}^{\mathrm{i}px} u_1(p,e)(k) a_Q^\dagger(p,k) + \mathrm{e}^{-\mathrm{i}px} \overline{u_{1c}(p,e)(k)} a_Q(p,k) \right).$$

The momentum operator P which appears in eq. (4.2.1) ensures a covariant transformation behaviour of the form

$$U(A,a)\Phi_Q(x,e)U(A,a)^\dagger = \Phi_{\Lambda(A)Q\Lambda(A)^{-1}}(\Lambda x + a, \Lambda(A)e) \tag{4.2.2}$$

presented here without smearing functions. In [GL07] the matrix Q can be associated to a wedge W, which is obtained from the standard wedge W_0 (cf. eq. (2.2.1)) by applying a Lorentz transformation, and by eq. (4.2.2) this mapping is compatible with the representation U and invariant with respect to the translation subgroup. Consequently, the statement $e \in W$ is Lorentz-invariant as well. Combining the analyticity properties for the string-field (cf. [MSY04]) with those of the translation operators in eq. (4.2.1) yields

$$[\Phi_Q(x,e), \Phi_{Q'}(x',e')] = 0$$

whenever Q and Q' are associated to the wedge $W \ni e$ and its causal complement $W' \ni e'$ respectively and $W + x$ is spacelike separated from $W' + x'$ because in this case the momentum integral in the commutator can by continued analytically as shown in [GL07][Proposition 3.4].

There may occur difficulties when trying to discuss the relativistic scattering theory (cf. [BS06][BS08]) for these fields since this is to be expected in general for the scattering theory of massless particles, see for example [Dyb05].

Chapter 5

Summary & Outlook

The present thesis constitutes a step towards the understanding of the possible localization properties for observables in massless infinite spin representations of the Poincaré group.

Following the definition of two-particle states with modular localization in a compact region due to J. Mund, B. Schroer and J. Yngvason it was shown that the corresponding two-particle observables, which generate these states from the Fock vacuum, cannot be relatively local to the known string-localized fields.

Key elements of this investigation have been to establish the general form of the solutions of the intertwiner equation and to analyze the momentum-space singularities which are implied by this equation in the relative commutator with a string-field. One related result is an alternative characterization of the string-localized one-particle intertwiners.

Another driving motivation for this work has been to understand qualitatively how the apparent incompatibilities between infinite spin and compact localization arise in comparison to a massive theory with finite spin. This has lead to the unified description of massive and infinite spin representations presented here, which shows in a geometrically intuitive way how the representations behave when approaching zero mass at fixed Pauli-Lubanski parameter.

It has also been mentioned how the string-localization of fields is compatible with the wedge-localization resulting from a deformed CCR-algebra, following the construction by H. Grosse and G. Lechner.

There are at least two possible ways to generalize the results presented in this thesis: One idea would be to strengthen the results by removing some of the more technical assumptions, like the required integrability requirements for the coefficient functions; a method to potentially achieve this has been introduced.

Another approach to generalize these results would be to remove the assumptions on the form of states with compact localization, for example by showing that all such states are of the form of intertwiners with smearing functions, in which case one could hope to extend the present results to arbitrary particle numbers. One strategy which builds on the characterization of standard subspaces in one- and two-dimensional theories due to G. Lechner and R. Longo has been presented here.

A different possibility would be the discovery of different types of states with compact localization properties that are currently unknown, which would potentially require new proof methods.

Appendix A

Auxiliary theorems and proofs

Proof of Lemma 10. A summary of the arguments found in the remarks before [RS75, Thm. IX.15] (for 1. and 3.) and in the proof of [RS75, Thm. IX.16] (for 2.), slightly adapted to the case at hand, is sufficient:

1. Analyticity: For $z = s + \mathrm{i}t \in H^+ \Rightarrow t > 0$, using $\operatorname{supp}\gamma \subseteq (-\infty, -b]$, eq. (3.2.4) can be written as
$$\hat{\gamma}(s+\mathrm{i}t) = \int_{-\infty}^{-b} \mathrm{d}a\, \mathrm{e}^{-\mathrm{i}sa}\mathrm{e}^{ta}\gamma(a),$$
and since γ is polynomially bounded, the integral converges absolutely and therefore complex differentiation with respect to z may be performed under the integral up to arbitrarily high orders.

2. Boundedness: The properties of γ imply the existence of constants $C, L > 0, N \in \mathbb{N}$ such that
$$|\gamma(a)| \leq C\left(\frac{1}{L}\chi_{[-L,0]-b}(a) + \frac{1}{\Gamma(N)}|a+b|^{N-1}\right)\Theta(b-a)\ \forall a \in \mathbb{R}$$
which is a way of stating the support and polynomial boundedness property of γ: The first term bounds the behaviour close to the upper boundary $-b$ of the support of γ by continuity, while the second term suffices for $a \to -\infty$, due to the polynomial boundedness of γ. It yields the following bound for the analytic function $\hat{\gamma}$:
$$\begin{aligned}|\hat{\gamma}(s+\mathrm{i}t)| &\leq \int_{-\infty}^{-b}\mathrm{d}a\,\mathrm{e}^{ta}|\gamma(a)| \leq C\int_{-\infty}^{-b}\mathrm{d}a\,\mathrm{e}^{ta}\left(\frac{1}{L}\chi_{[-L,0]-b}(a) + \frac{1}{\Gamma(N)}|a+b|^{N-1}\right)\\ &= C\mathrm{e}^{-bt}\int_{-\infty}^{0}\mathrm{d}a\,\mathrm{e}^{ta}\left(\frac{1}{L}\chi_{[-L,0]}(a) + \frac{1}{\Gamma(N)}|a|^{N-1}\right)\\ &= C\mathrm{e}^{-bt}\left(\frac{1}{L}\int_{-L}^{0}\mathrm{d}a\,\mathrm{e}^{ta} + \frac{1}{\Gamma(N)}\int_{-\infty}^{0}\mathrm{d}a\,\mathrm{e}^{ta}|a|^{N-1}\right)\\ &= C\mathrm{e}^{-bt}\left(\frac{1-\mathrm{e}^{-Lt}}{Lt} + t^{-N}\right) \leq C\mathrm{e}^{-bt}\left(1 + t^{-N}\right)\end{aligned}$$

3. Distributional boundary value: Let $f \in \mathcal{S}(\mathbb{R})$. For $t > 0$ the following integral converges absolutely and by Fubini's Theorem the order of integrations may be exchanged:
$$\begin{aligned}\hat{\gamma}_t(f) &\overset{(3.2.5)}{=} \int \mathrm{d}s\, f(s)\hat{\gamma}(s+\mathrm{i}t) \overset{(3.2.4)}{=} \int \mathrm{d}s\, f(s)\int \mathrm{d}a\,\mathrm{e}^{-\mathrm{i}(s+\mathrm{i}t)a}\gamma(a)\\ &= \int \mathrm{d}a\,\mathrm{e}^{ta}\gamma(a)\int \mathrm{d}s\,\mathrm{e}^{-\mathrm{i}sa}f(s) \overset{(3.2.7)}{=} \int \mathrm{d}a\,\mathrm{e}^{ta}\gamma(a)\hat{f}(a)\end{aligned} \tag{A.0.1}$$

Since $\operatorname{supp}\gamma(a) \subseteq (-\infty, -b]$, the integrand is majorized by the value at $t = 0$. This is an integrable function of a, because γ is polynomially bounded and $\hat{f} \in \mathcal{S}(\mathbb{R})$. By the Dominated Convergence Theorem, the limit $t \to$ may therefore be exchanged with the integral:

$$\Rightarrow \lim_{t\to 0} \hat{\gamma}_t(f) \overset{\text{(A.0.1)}}{=} \lim_{t\to 0} \int \mathrm{d}a\, \mathrm{e}^{ta}\gamma(a)\hat{f}(a) = \int \mathrm{d}a \lim_{t\to 0} \mathrm{e}^{ta}\gamma(a)\hat{f}(a)$$
$$= \int \mathrm{d}a\, \gamma(a)\hat{f}(a) \overset{\text{(3.2.6)}}{=} \hat{\gamma}(f)$$

Because the choice of $f \in \mathcal{S}(\mathbb{R})$ was arbitrary, the claim follows.

□

Definition 6. Let f be a $\mathbb{C}$-valued locally square-integrable function on $\mathbb{R}^2$ and denote the equivalence class of functions which are almost everywhere equal to f by $\underline{f}$, i.e. $\underline{f} \in L^2_{\text{loc}}(\mathbb{R}^2)$ and

$$\underline{f} = \{\tilde{f} : \mathbb{R}^2 \to \mathbb{C} \,|\, \mu(\{x \in \mathbb{R}^2 | \tilde{f}(x) \neq f(x)\}) = 0\}.$$

The space $L^2_{\text{loc}}(\mathbb{R}^2)$ is equipped with a set of seminorms defined as

$$||\underline{f}||^2_K := \int_K \mathrm{d}\mu(x)\, |f(x)|^2$$

for all compact $K \subset \mathbb{R}^2$. Elements in $\mathbb{R}^2$ are denoted as $x = (x_1, x_2)$.

For all $a \in \mathbb{R}$ define the translation $\underline{T}_a$ in $L^2_{\text{loc}}(\mathbb{R}^2)$ by

$$\underline{T}_a\underline{f} = \underline{T_a f}, \text{ where } (T_a f)(x_1, x_2) = f(x_1 - a, x_2).$$

This mapping is well defined: For any $\tilde{f}$ with $\underline{\tilde{f}} = \underline{f}$, the function $n := \tilde{f} - f$ has the property $\mu(\operatorname{supp} n) = 0$. Therefore, $\underline{T_a\tilde{f}} = \underline{T_a f + T_a n} = \underline{T_a f}$, since $\mu(T_a n) = 0$ as well.

Remark 19. Definition 6 has been stated for $\mathbb{R}^2$ only for the sake of simplicity. The construction and the proof of Lemma 19 can be done analogously for $\mathbb{R}^n$, $2 \leq n \in \mathbb{N}$.

Lemma 19 (Representatives for translation invariant integrable functions)**.** *Let $\underline{f}$ have the invariance property $\underline{T}_a\underline{f} = \underline{f}$ for all $a \in \mathbb{R}$. Pick an interval $I \subset \mathbb{R}$ and define*

$$c(x) := \frac{1}{|I|}\int_I \mathrm{d}a\, (T_a f)(x).$$

This integral exists for all $x \in \mathbb{R}^2$, is independent of x_1 and is equivalent to f.

Proof. In the subsequent discussion of the properties of the function c the Cauchy-Schwarz inequality is used in the following form several times:

$$\left(\int_I \mathrm{d}\mu\, |f|\right)^2 = \left(\int \mathrm{d}\mu\, |f\chi_I|\right)^2 = ||f\chi_I||_1^2 \overset{CS}{\leq} ||f||_I^2 ||\chi_I||_2^2 = |I| \int \mathrm{d}\mu\, |f|^2 \tag{A.0.2}$$

χ_I is the indicator function of the interval I.
Existence: Let $K \subset \mathbb{R}^2$ be compact. Then there are intervals I_1, I_2 such that $I_1 \times I_2 \supseteq K$.

It follows

$$
\begin{aligned}
||c||_K^2 \leq ||c||_{I_1\times I_2}^2 = & \int_{I_1\times I_2} \mathrm{d}\mu(x) \left| \frac{1}{|I|} \int_I \mathrm{d}a\, (T_a f)(x) \right|^2 \\
= & \frac{1}{|I|^2} \int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2 \left| \int_I \mathrm{d}a\, f(x_1 - a, x_2) \right|^2 = \frac{1}{|I|^2} \int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2 \left| \int_{I-x_1} \mathrm{d}a\, f(-a, x_2) \right|^2 \\
\leq & \frac{1}{|I|^2} \int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2 \left(\int_{I-x_1} \mathrm{d}a\, |f(-a,x_2)| \right)^2 \\
\overset{(A.0.2)}{\leq} & \frac{1}{|I|} \int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2 \int_{I-x_1} \mathrm{d}a\, |f(-a,x_2)|^2 = \frac{1}{|I|} \int_{I_1} \mathrm{d}x_1\, ||\underline{f}||_{(x_1-I)\times I_2}^2 \\
\leq & \frac{1}{|I|} \int_{I_1} \mathrm{d}x_1\, ||\underline{f}||_{(I_1-I)\times I_2}^2 = \frac{|I_1|}{|I|} ||\underline{f}||_{(I_1-I)\times I_2}^2 < \infty.
\end{aligned}
$$

Invariance: Let I_2 be an interval and $x_1 \in \mathbb{R}$. Then the estimate

$$
\begin{aligned}
0 \leq \int_{I_2} \mathrm{d}x_2\, |c(x_1,x_2) - c(0,x_2)|^2 = & \int_{I_2} \mathrm{d}x_2 \left| \frac{1}{|I|} \int_I \mathrm{d}a\, (f(x_1-a,x_2) - f(-a,x_2)) \right|^2 \\
\frac{1}{|I|^2} \int_{I_2} \mathrm{d}x_2 \int_I \mathrm{d}a\, |f(x_1-a,x_2) - f(-a,x_2)|^2 = & \frac{1}{|I|^2} ||\underline{T}_{x_1}\underline{f} - \underline{f}||_{(-I)\times I_2}^2 = 0
\end{aligned}
$$

turns out to be an equality and it follows that $c(x_1,x_2) = c(0,x_2)$ for almost all x_2. *Equivalence:* Let $b_n(a,x_2)$ a sequence of step functions which converges to $(a,x_2) \mapsto (T_a f)(0,x_2)$ in the sense of $L^2_{\mathrm{loc}}(\mathbb{R}^2)$, in particular

$$
\lim_{n\to\infty} \int_I \mathrm{d}a \int_{I_2} \mathrm{d}x_2\, |(T_a f)(0,x_2) - b_n(a,x_2)|^2 = 0\ \forall\, I, I_2 \text{ intervals,}
$$

and define $b_n(a)(x) := b_n(a - x_1, x_2)$. The strategy for the remaining part of the proof is as follows:

1. $\underline{f}$ is equivalent to a version of c where not the value of $T_a f$ at x, but the class $\underline{T_a f}$ itself is averaged over a as a Bochner integral.

$$
\frac{1}{|I|} \int \mathrm{d}a\, \underline{T_a f} = \frac{1}{|I|} \int \mathrm{d}a\, \underline{f} = \underline{f}
$$

2. This integral is approximated by averaging the step function $\underline{b_n}$, for which the Bochner integral is just a sum and therefore commutes with the evaluation.

$$
\begin{aligned}
& \left|\left| \frac{1}{|I|} \int_I \mathrm{d}a\, (\underline{b_n(a)} - \underline{T_a f}) \right|\right|_K^2 \leq \frac{1}{|I|^2} \left(\int_I \mathrm{d}a\, ||\underline{b_n(a)} - \underline{T_a f})||_K \right)^2 \\
= & \overset{(A.0.2)}{\leq} \frac{1}{|I|} \int_I \mathrm{d}a\, ||\underline{b_n(a)} - \underline{T_a f})||_K^2 \\
= & \frac{1}{|I|} \int_I \mathrm{d}a \int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2\, |b_n(a)(x_1,x_2) - (T_a f)(x_1,x_2)|^2 \qquad \text{(A.0.3)} \\
= & \frac{1}{|I|} \int_I \mathrm{d}a \int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2\, |b_n(a - x_1, x_2) - (T_{a-x_1} f)(0,x_2)|^2 \\
= & \frac{1}{|I|} \int_{I_1} \mathrm{d}x_1 \int_{I-x_1} \mathrm{d}a \int_{I_2} \mathrm{d}x_2\, |b_n(a,x_2) - (T_a f)(0,x_2)|^2 \\
\leq & \frac{1}{|I|} \int_{I_1} \mathrm{d}x_1 \int_{I-I_1} \mathrm{d}a \int_{I_2} \mathrm{d}x_2\, |b_n(a,x_2) - (T_a f)(0,x_2)|^2 \overset{n\to 0}{0}
\end{aligned}
$$

3. But the step function approximates c as well:

$$\left|\left|\underline{c} - \frac{1}{|I|}\int_I \mathrm{d}a\, \underline{b_n(a)}\right|\right|_K^2 = \int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2 \left|c(x) - \frac{1}{|I|}\int_I \mathrm{d}a\, b_n(a)(x)\right|^2$$

$$\leq \frac{1}{|I|^2}\int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2 \left(\int_I \mathrm{d}a\, |(T_a f)(x) - b_n(a)(x)|\right)^2$$

$$\overset{\text{(A.0.2)}}{\leq} \frac{1}{|I|}\int_{I_1} \mathrm{d}x_1 \int_{I_2} \mathrm{d}x_2 \int_I \mathrm{d}a\, |(T_a f)(x) - b_n(a)(x)|^2 \overset{\text{cf. (A.0.3)}}{\to} 0$$

Therefore $\underline{f}$ and $\underline{c}$ have to be arbitrarily close, hence equal: Combining the previous inequalities yields

$$||\underline{c} - \underline{f}||_K \leq \left|\left|\underline{c} - \frac{1}{|I|}\int_I \mathrm{d}a\, \underline{b_n(a)}\right|\right|_K + \left|\left|\frac{1}{|I|}\int_I \mathrm{d}a\, (\underline{b_n(a)} - \underline{T_a f})\right|\right|_K \overset{n\to\infty}{\to} 0,$$

hence $\underline{c} = \underline{f}$. □

Lemma 20 (Higher derivatives of reciprocal functions)**.** *Let f be a quadratic polynomial. For $x \in \mathbb{R}$ such that $f(x) \neq 0$ and $n \in \mathbb{N}$, there are constants $b_{n,i} \in \mathbb{Z}$, $i = 0, ..., \lfloor n/2 \rfloor$ such that*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{1}{f(x)} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_{n,i}(f(x))^{-1-n+i}(f'(x))^{n-2i}(f''(x))^i \tag{A.0.4}$$

Proof. For $n = 0$, the sum in eq. (A.0.4) consists of only one term and becomes correct with $b_{0,0} := 1$. One may proceed inductively by assuming the lemma to be true for some arbitrary $n \in \mathbb{N}$ and differentiating once more. The result is

$$\begin{aligned}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\frac{1}{f(x)} &\overset{\text{(A.0.4)}}{=} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_{n,i}(-1-n+i)(f(x))^{-1-(n+1)+i}(f'(x))^{n+1-2i}(f''(x))^i \\ &+ \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} b_{n,i}(n-2i)(f(x))^{-1-n+i}(f'(x))^{n-2i-1}(f''(x))^{i+1},\end{aligned} \tag{A.0.5}$$

because if n is even, the term with $i = n/2$ in the second sum is not present and the last term is the one with $i = n/2 - 1 = \lfloor (n-1)/2 \rfloor$. This does not happen if n is odd, but in that case $\lfloor n/2 \rfloor = \lfloor (n-1)/2 \rfloor$. Therefore in both cases, the sum runs up to $i = \lfloor (n-1)/2 \rfloor$. Shifting the summation index $i \mapsto i+1$ in the second sum, eq. (A.0.5) becomes

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\frac{1}{f(x)} = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} b_{n+1,i}(f(x))^{-1-(n+1)+i}(f'(x))^{n+1-2i}(f''(x))^i \tag{A.0.6}$$

with the following new constants:

$$\begin{aligned} b_{n+1,0} :=& b_{n,0}(-1-n),\ b_{n+1,\lfloor \frac{n+1}{2} \rfloor} := b_{n,\lfloor \frac{n-1}{2} \rfloor} \\ \text{and } b_{n+1,i} :=& b_{n,i}(-1-n+i) + b_{n,i-1}(n-2(i-1)) \text{ for } i = 1, \ldots, \lfloor (n-1)/2 \rfloor \end{aligned}$$

Now eq. (A.0.6) has the same form as eq. (A.0.5), with n replaced by $n+1$. □

Proof of Lemma 12. If $\delta_{k_0,\epsilon}$ is chosen symmetric w.r.t. reflection about k_0, then there is a function $\tilde{\delta} : [-\kappa, \kappa] \to \mathbb{C}$ such that $\delta_{k_0,\epsilon}(k) = \tilde{\delta}(k_0 \cdot k/\kappa)$ for all $k \in \kappa S^1$. On the one hand, by the Stone-Weierstrass Theorem, $\tilde{\delta}$ can be approximated uniformly by any involutive point-separating subalgebra of $C([-\kappa, \kappa])$, which is taken to be the algebra of polynomials: There is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that for $\epsilon' > 0$, there is $N' \in \mathbb{N}$ with

$$|\delta_{k_0,\epsilon}(k) - p_n(k_0 \cdot k/\kappa)| < \epsilon' \;\forall\, k \in \kappa S^1,\; N' \leq n \in \mathbb{N}, \tag{A.0.7}$$

with p_n being of degree at most n. The coefficients are denoted by $p_{n,h}$, i.e.

$$p_n(\eta) = \sum_{h=0}^{n} p_{n,h}\eta^h \tag{A.0.8}$$

On the other hand, these polynomials may in turn be approximated by suitable linear combinations of higher derivatives of $\tilde{u}_1$. Let $v \in S^1$. Denoting by $\mathcal{Z}(j)$ the set of ordered partitions of $j \in \mathbb{N}$, i.e.

$$\mathcal{Z}(j) = \left\{ z = (z_1, \ldots, z_{l(z)}) \in \mathbb{N}^{l(z)},\, l(z) \in \mathbb{N} \;\middle|\; \sum_{i=1}^{l(z)} z_i = j,\, i < j \Rightarrow z_i \leq z_j \right\}, \tag{A.0.9}$$

where $l(z)$ is the *length* of the partition z, the derivatives of $\tilde{u}_1$ have the form

$$\begin{aligned}\left(e_- \frac{|\mathrm{p}|^2}{p_-} v \cdot \nabla_\mathrm{e}\right)^j \tilde{u}_1(p,e)(k) &= \left(e_- \frac{|\mathrm{p}|^2}{p_-} v \cdot \nabla_\mathrm{e}\right)^j \exp\left(\mathrm{i}\frac{k \cdot \left(\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right) - \kappa}{2e \cdot p}\right) \\ = \left(e_- \frac{|\mathrm{p}|^2}{p_-}\right)^j \exp\left(\mathrm{i}\frac{k \cdot \left(\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right) - \kappa}{2e \cdot p}\right) &\sum_{z\in\mathcal{Z}(j)} \prod_{i=1}^{l(z)} (v \cdot \nabla_\mathrm{e})^{z_i} \mathrm{i}\frac{k \cdot \left(\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right) - \kappa}{2e \cdot p},\end{aligned} \tag{A.0.10}$$

for $v \in \mathbb{R}^2$, because the recursive calculation of derivatives of an exponentiated function follows the same rule as the recursive enumeration of all ordered partitions of a natural number: Either one factor is differentiated, which corresponds to the incrementation of one summand in the partition, or the chain rule yields another factor when applied to the exponential, corresponding to adding one term to an existing partition. Each factor yields

$$\begin{aligned}(v \cdot \nabla_\mathrm{e})^{z_i} \mathrm{i}\frac{k \cdot \left(\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right) - \kappa}{2e \cdot p} &= \sum_{l=0}^{z_i} (v \cdot \nabla_\mathrm{e})^{z_i} \mathrm{i}\frac{k \cdot \left(\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right) - \kappa}{2e \cdot p} \\ = \sum_{l=0}^{z_i} \binom{z_i}{l} \left[(v \cdot \nabla_\mathrm{e})^l \left(k \cdot \left[\mathrm{e} - \frac{e_-}{p_-}\mathrm{p}\right] - \kappa\right)\right] &\left[(v \cdot \nabla_\mathrm{e})^{z_i - l}(2e \cdot p)^{-1}\right].\end{aligned} \tag{A.0.11}$$

The first factor here is linear in e, hence the index l only runs from 0 to 1, while Lemma 20, due to $2e \cdot p$ being quadratic in e, is applicable to the second factor:

$$\begin{aligned}(v \cdot \nabla_\mathrm{e})^{z_i - l}(2e \cdot p)^{-1} = \sum_{m=0}^{\left\lfloor \frac{z_i - l}{2} \right\rfloor} & b_{z_i - l, m}(2e \cdot p)^{-1 - z_i + l + m} \\ & \cdot ((v \cdot \nabla_\mathrm{e})2e \cdot p)^{z_i - l - 2m}((v \cdot \nabla_\mathrm{e})^2 2e \cdot p)^m\end{aligned} \tag{A.0.12}$$

with the constants $b_{z_i-l,m} \in \mathbb{Z}$, which appear in eq. (A.0.4). The various derivatives showing up in eq. (A.0.11) and eq. (A.0.12) can be computed explicitly:

$$
\begin{aligned}
(v\cdot\nabla_{\mathrm{e}})\left(k\cdot\left[\mathrm{e}-\frac{e_-}{p_-}\mathrm{p}\right]-\kappa\right) &= v\cdot k\\
2e\cdot p &\overset{(2.1.10)}{=} e_-\frac{|\mathrm{p}|^2}{p_-}+\frac{|\mathrm{e}|^2-1}{e_-}p_- - 2\mathrm{e}\cdot\mathrm{p}\ (\text{cf. } f_+ \text{ in eq. (3.1.26)})\\
\Rightarrow (v\cdot\nabla_{\mathrm{e}})2e\cdot p &= 2v\cdot\left(\mathrm{e}\frac{p_-}{e_-}-\mathrm{p}\right)\\
\Rightarrow (v\cdot\nabla_{\mathrm{e}})^2 2e\cdot p &= 2|v|^2\frac{p_-}{e_-}
\end{aligned}
$$

Combining these with eqs. (A.0.10)–(A.0.12) yields (suppressing the explicit dependencies)

$$
\begin{aligned}
P_j :=& \left(e_-\frac{|\mathrm{p}|^2}{p_-}v\cdot\nabla_{\mathrm{e}}\right)^j \tilde{u}_1(p,e)(k) = \left(e_-\frac{|\mathrm{p}|^2}{p_-}\right)^j \exp\left(\mathrm{i}\frac{k\cdot\left(\mathrm{e}-\frac{e_-}{p_-}\mathrm{p}\right)-\kappa}{2e\cdot p}\right)\\
&\cdot\sum_{z\in\mathcal{Z}(j)}\prod_{i=1}^{l(z)}\sum_{l=0}^{1}\binom{z_i}{l}\left(k\cdot\left[\mathrm{e}-\frac{e_-}{p_-}\mathrm{p}\right]-\kappa\right)^{1-l}(v\cdot k)^l \qquad \text{(A.0.13)}\\
&\cdot\sum_{m=0}^{\left\lfloor\frac{z_i-l}{2}\right\rfloor} b_{z_i-l,m}\,(2e\cdot p)^{-1-z_i+l+m}\left(2v\cdot\left[\mathrm{e}\frac{p_-}{e_-}-\mathrm{p}\right]\right)^{z_i-l-2m}\left(2|v|^2\frac{p_-}{e_-}\right)^m .
\end{aligned}
$$

For each term of the sum over $z\in\mathcal{Z}(j)$, the exponent j in $(e_-/p_-)^j$ from the global prefactor may be split as in eq. (A.0.9) w.r.t. the current partition z and the resulting factors $(e_-/p_-)^{z_i}$ can be distributed within the product over $i=1,\ldots,l(z)$. The result is

$$
\begin{aligned}
P_j :=& \frac{|\mathrm{p}|^{2j}}{2}\exp\left(\mathrm{i}\frac{k\cdot\left(\mathrm{e}-\frac{e_-}{p_-}\mathrm{p}\right)-\kappa}{2e\cdot p}\right) \qquad \text{(A.0.14)}\\
&\cdot\sum_{z\in\mathcal{Z}(j)}\prod_{i=1}^{l(z)}\sum_{l=0}^{1}\binom{z_i}{l}\left(k\cdot\left[\mathrm{e}-\frac{e_-}{p_-}\mathrm{p}\right]-\kappa\right)^{1-l}(v\cdot k)^l\\
&\cdot\sum_{m=0}^{\left\lfloor\frac{z_i-l}{2}\right\rfloor} b_{z_i-l,m}\left(\frac{e_-}{p_-}\right)^{l+m}(e\cdot p)^{-1-z_i+l+m}\left(v\cdot\left[\mathrm{e}-\frac{e_-}{p_-}\mathrm{p}\right]\right)^{z_i-l-2m}\left(|v|^2\right)^m .
\end{aligned}
$$

For $\epsilon''>0$, varying

$$
\mathrm{p}^*\in\overline{B^1_\epsilon(\mathrm{p}_0)}, \qquad \text{(A.0.15)}
$$

the open sets $B^1_{\epsilon''}(\mathrm{p}^*)$ provide an open cover of the set $\overline{B^1_\epsilon(\mathrm{p}_0)}$. By compactness, finitely many of these sets are sufficient, i.e. there are $\mathrm{p}^*_1,\ldots,\mathrm{p}^*_{N''}\in\overline{B^1_\epsilon(\mathrm{p}_0)}$ with $N''<\infty$ such that

$$
\bigcup_{i=1}^{N''} B^1_{\epsilon''}(\mathrm{p}^*_i)\supset\overline{B^1_\epsilon(\mathrm{p}_0)}.
$$

Let $b^{(i)}_{\mathrm{p}_0,\epsilon}$ be a corresponding partition of unity, i.e.

$$\begin{aligned} &b^{(i)}_{\mathrm{p}_0,\epsilon} \in C_0^\infty(\mathbb{R}^2),\ \operatorname{supp} b^{(i)}_{\mathrm{p}_0,\epsilon} \subset B^1_{\epsilon''}(\mathrm{p}^*_i)\ \forall\, i = 1,\dots,N'' \\ &\text{and } \sum_{i=1}^{n} b^{(i)}_{\mathrm{p}_0,\epsilon}(\mathrm{p}) = 1\ \forall\, \mathrm{p} \in \overline{B^1_\epsilon(\mathrm{p}_0)}. \end{aligned} \tag{A.0.16}$$

If the components of p are restricted by the requirements

$$\Delta^{-1} < |p_-| < \Delta \tag{A.0.17}$$

$$\mathrm{p} \in B^1_{\epsilon''}(\mathrm{p}^*) \cap B^1_\epsilon(\mathrm{p}_0), \tag{A.0.18}$$

where $\Delta > 0$, while for the components of e and a point $\mathrm{e}^* \in \mathbb{R}^2$ the restrictions

$$0 < e_- < \frac{\epsilon''}{\Delta} \min\{(|\mathrm{p}_0| + \epsilon)^{-2}, (|\mathrm{p}_0| + \epsilon)^{-1}, 1\} \tag{A.0.19}$$

$$\mathrm{e}, \mathrm{e}^* \in B^1_\epsilon(k_0/\kappa) \tag{A.0.20}$$

$$|\mathrm{e}|, |\mathrm{e}^*| \in]1/2, 1[\cap]1 - \epsilon'' e_- \Delta/2, 1[, \tag{A.0.21}$$

$$|\mathrm{e} - \mathrm{e}^*| < \epsilon''' \tag{A.0.22}$$

with $\epsilon''' > 0$ are assumed. The admissible set for e^*, as defined by eq. (A.0.20) and (A.0.21), is open. Let ϵ''' be chosen small enough such that the validity of eq. (A.0.22) already implies the former two conditions on e.
These restrictions imply the following estimates, which will be useful later on:

$$\begin{aligned} \left|\mathrm{e}^* \cdot \mathrm{p}^* - \frac{k_0}{\kappa} \cdot \mathrm{p}_0\right| &\le \left|\left(\mathrm{e}^* - \frac{k_0}{\kappa}\right) \cdot \mathrm{p}^*\right| + \left|\frac{k_0}{\kappa} \cdot (\mathrm{p}^* - \mathrm{p}_0)\right| \\ &< \left|\mathrm{e}^* - \frac{k_0}{\kappa}\right| |\mathrm{p}^*| + \left|\frac{k_0}{\kappa} \cdot (\mathrm{p}^* - \mathrm{p}_0)\right| \\ &\overset{\text{(A.0.20)(A.0.15)}}{<} \left(|\mathrm{p}^*| + \left|\frac{k_0}{\kappa}\right|\right) \epsilon \overset{\text{(A.0.15)}}{<} (1 + |\mathrm{p}_0| + \epsilon)\epsilon \\ \Rightarrow |\mathrm{e}^* \cdot \mathrm{p}^*| &> \left|\frac{k_0}{\kappa} \cdot \mathrm{p}_0\right| - (1 + |\mathrm{p}_0| + \epsilon)\epsilon \end{aligned} \tag{A.0.23}$$

$$\left|\frac{e_-}{p_-}\right| \overset{\text{(A.0.19)}}{<} \frac{\epsilon''}{|p_-|\Delta} \overset{\text{(A.0.17)}}{<} \epsilon'' \tag{A.0.24}$$

$$\left|k \cdot \mathrm{p}\frac{e_-}{p_-}\right| \le \kappa |\mathrm{p}| \frac{e_-}{|p_-|} \overset{\text{(A.0.19)}}{<} \kappa \frac{|\mathrm{p}|}{|\mathrm{p}_0| + \epsilon} \frac{\epsilon''}{|p_-|\Delta} \overset{\text{(A.0.17)(A.0.18)}}{<} \kappa\epsilon'' \tag{A.0.25}$$

$$\left|v \cdot \frac{e_-}{p_-}\mathrm{p}\right| \le |v||\mathrm{p}| \frac{e_-}{|p_-|} < \text{ (cf. eq. (A.0.25)) } < |v|\epsilon'' \tag{A.0.26}$$

$$|k \cdot \mathrm{e} - \kappa| \le |k \cdot \mathrm{e}| + \kappa \le \kappa|\mathrm{e}| + \kappa \overset{\text{(A.0.21)}}{<} 2\kappa \tag{A.0.27}$$

$$\left||\mathrm{p}|^2 \frac{e_-}{p_-}\right| = |\mathrm{p}|^2 \frac{e_-}{|p_-|} \overset{\text{(A.0.19)}}{<} \frac{|\mathrm{p}|^2}{(|\mathrm{p}_0| + \epsilon)^2} \frac{\epsilon''}{|p_-|\Delta} \overset{\text{(A.0.17)(A.0.18)}}{<} \epsilon'' \tag{A.0.28}$$

$$\begin{aligned} \left|\frac{|\mathrm{e}|^2 - 1}{e_-} p_-\right| &\overset{\text{(A.0.21)}}{=} \frac{1 - |\mathrm{e}|^2}{e_-} |p_-| = (1 + |\mathrm{e}|)(1 - |\mathrm{e}|) \frac{|p_-|}{e_-} \\ &\overset{\text{(A.0.21)}}{<} 2(1 - |\mathrm{e}|) \frac{|p_-|}{e_-} \overset{\text{(A.0.21)}}{<} |p_-|\Delta\epsilon'' \overset{\text{(A.0.17)}}{<} \epsilon'' \end{aligned} \tag{A.0.29}$$

$$\begin{aligned}
\Rightarrow |2e\cdot p+2\mathrm{e}^*\cdot\mathrm{p}^*| =& \left|2\left(\mathrm{e}^*\cdot\mathrm{p}^*-\mathrm{e}\cdot\mathrm{p}\right)+e_-\frac{|\mathrm{p}|^2}{p_-}+\frac{|\mathrm{e}|^2-1}{e_-}p_-\right|\\
\leq& 2\left|\mathrm{e}^*\cdot\mathrm{p}^*-\mathrm{e}\cdot\mathrm{p}\right|+\left|e_-\frac{|\mathrm{p}|^2}{p_-}\right|+\left|\frac{|\mathrm{e}|^2-1}{e_-}p_-\right|\\
\overset{(A.0.28)(A.0.29)}{<}& 2\left|(\mathrm{e}^*-\mathrm{e})\cdot\mathrm{p}^*\right|+2\left|\mathrm{e}\cdot(\mathrm{p}^*-\mathrm{p})\right|+2\epsilon''\\
\leq& 2\left|\mathrm{e}^*-\mathrm{e}\right|\left|\mathrm{p}^*\right|+2\left|\mathrm{e}\right|\left|\mathrm{p}^*-\mathrm{p}\right|+2\epsilon''\\
\overset{(A.0.18)}{<}& 2(|\mathrm{p}_0|+\epsilon)\left|\mathrm{e}^*-\mathrm{e}\right|+2|\mathrm{e}|\epsilon''+2\epsilon''\\
\overset{(A.0.21)(A.0.22)}{<}& 2(|\mathrm{p}_0|+\epsilon)\epsilon'''+4\epsilon'' \qquad \text{(A.0.30)}
\end{aligned}$$

P_j is a continuous function in the expressions which have been estimated in eq. (A.0.24)–(A.0.27) and (A.0.30) and can therefore be approximated, for ϵ sufficiently small and $e\cdot p\neq 0$, up to an error controlled by ϵ'', ϵ''', by its value at $e_-/p_-=0$ (cf. eq. (A.0.24)–(A.0.27); hence only the terms with $l=m=0$ need to be considered), $\mathrm{e}=\mathrm{e}^*$ (cf. (A.0.22)) and $e\cdot p=-\mathrm{e}^*\cdot\mathrm{p}^*$ (cf. eq. (A.0.30)), where the latter does occur several times as an inverse power, but due to (A.0.23), P_j is evaluated only at $e\cdot p\neq 0$:

$$\begin{aligned}
P_j =& \frac{|\mathrm{p}^*|^{2j}}{2}\exp\left(-\mathrm{i}\frac{k\cdot\mathrm{e}^*-\kappa}{2\mathrm{e}^*\cdot\mathrm{p}^*}\right)\sum_{z\in\mathcal{Z}(j)}(k\cdot\mathrm{e}^*-\kappa)^{l(z)}\\
&\prod_{i=1}^{l(z)} b_{z_i,0}(-\mathrm{e}^*\cdot\mathrm{p}^*)^{-1-z_i}(v\cdot\mathrm{e}^*)^{z_i}+\mathcal{O}(\epsilon'',\epsilon''')
\end{aligned}$$

Here and in the following the symbol $\mathcal{O}(\epsilon'',\epsilon''')$ is used to collect terms which vanish at least linearly in ϵ'',ϵ'''.

Up to the exponential prefactor, P_j is a polynomial in $k\cdot\mathrm{e}^*$ of degree j (the highest occurring value of $l(z)$, corresponding to the partition $j=1+1+\ldots+1$) and can therefore be represented as

$$Q_j := \exp\left(\mathrm{i}\frac{k\cdot\mathrm{e}^*}{2\mathrm{e}^*\cdot\mathrm{p}^*}\right)P_j=\sum_{i=0}^{j}c_{i,j}(k\cdot\mathrm{e}^*)^i+\mathcal{O}(\epsilon'',\epsilon''') \qquad \text{(A.0.31)}$$

with $c_{i,j}\in\mathbb{R}$, $i=0,\ldots,j$. Here $c_{j,j}\neq 0$, by the recursion relations following eq. (A.0.6) in the proof of Lemma 20. Linear combinations of Q_j, $j=0,\ldots,n$, which approximate the monomials $(k\cdot\mathrm{e}^*)^i$, can be constructed inductively, starting with coefficients

$$d_{i,i} =\frac{1}{c_{i,i}}\ \forall\, i=0,\ldots,n \text{ and proceeding via} \qquad \text{(A.0.32)}$$

$$d_{i+1,j} =-\sum_{h=j}^{i}\frac{c_{h,i+1}}{c_{i+1,i+1}}d_{h,j}\ \forall\, j=0,\ldots,i\ \forall\, i=0,\ldots,n-1. \qquad \text{(A.0.33)}$$

Then the equation

$$\sum_{j=0}^{i}d_{i,j}Q_j=(k\cdot\mathrm{e}^*)^i+\mathcal{O}(\epsilon'',\epsilon''') \qquad \text{(A.0.34)}$$

is clear for $i = 0$. Assuming its validity for some $i \in \{0, \dots, n-1\}$, the case $i+1$ follows from

$$
\begin{aligned}
\sum_{j=0}^{i+1} d_{i+1,j} Q_j &\overset{\text{(A.0.31)}}{=} d_{i+1,i+1} \sum_{h=0}^{i+1} c_{h,i+1} (k \cdot \mathrm{e}^*)^h + \sum_{j=0}^{i} d_{i+1,j} Q_j + \mathcal{O}(\epsilon'', \epsilon''') \\
&\overset{\text{(A.0.32)(A.0.33)}}{=} (k \cdot \mathrm{e}^*)^{i+1} + \sum_{h=0}^{i} \frac{c_{h,i+1}}{c_{i+1,i+1}} (k \cdot \mathrm{e}^*)^h - \sum_{j=0}^{i} \sum_{h=j}^{i} \frac{c_{h,i+1}}{c_{i+1,i+1}} d_{h,j} Q_j + \mathcal{O}(\epsilon'', \epsilon''') \\
&\overset{(*)}{=} (k \cdot \mathrm{e}^*)^{i+1} + \frac{1}{c_{i+1,i+1}} \sum_{h=0}^{i} c_{h,i+1} \left((k \cdot \mathrm{e}^*)^h - \sum_{j=0}^{h} d_{h,j} Q_j \right) + \mathcal{O}(\epsilon'', \epsilon''') \\
&\overset{\text{(A.0.34)}}{=} (k \cdot \mathrm{e}^*)^{i+1} + \mathcal{O}(\epsilon'', \epsilon'''),
\end{aligned}
$$

where the step marked with (*) consists in rearranging the double sum over j, h. Therefore eq. (A.0.34) holds for all $i \in \mathbb{N}$ and will subsequently be used in the form

$$
\left| \sum_{j=0}^{i} d_{i,j} Q_j - (k \cdot \mathrm{e}^*)^i \right| < C_i (\epsilon'' + \epsilon''') \; \forall\, i = 0, \dots, N' \tag{A.0.35}
$$

with suitable constants $C_i > 0$.

The ingredients collected so far now allow for the construction of the functions which constitute the sequence in (3.2.23):

$$
\begin{aligned}
\hat{f}^{ij}_\epsilon(p) &:= \left(\frac{|\mathrm{p}|^2}{p_-} \right)^j \hat{f}\left(p_-, \frac{|\mathrm{p}|^2}{p_-} \right) \delta_{\mathrm{p}_0,\epsilon}(\mathrm{p}) b^i_{\mathrm{p}_0,\epsilon}(\mathrm{p}) && \text{(A.0.36)} \\
h^{ij}_\epsilon(e) &:= e^j_- \exp\left(\mathrm{i} \frac{k_0 \cdot \mathrm{e}^*}{2\mathrm{e}^* \cdot \mathrm{p}^*_i} \right) \delta_{e^*_-,\epsilon_-}(e_-) \sum_{h=j}^{N'} p_{N',h} d_{h,j} (v \cdot \nabla_\mathrm{e})^j \delta_{\mathrm{e}^*,\epsilon'''}(\mathrm{e}) && \text{(A.0.37)}
\end{aligned}
$$

for $i = 1, \dots, N''$ and $j = 1, \dots, N'$

with $e^*_- \in \mathbb{R}$ such that (A.0.19) holds and ϵ_- small enough, that this is the case for all $e_- \in \mathbb{R}$ with $|e_- - e^*_-| < \epsilon_-$ as well.

Together with the j-dependent prefactor from eq. (A.0.36), the intertwiner $\tilde{u}_1$, when smeared in e with h_ϵ^{ij}, approximates $\delta_{k_0,\epsilon}$ with uniform convergence in k up to a prefactor:

$$
\begin{aligned}
&\left|\sum_{j=1}^{N'}\left(\frac{|\mathrm{p}|^2}{p_-}\right)^j \tilde{u}_1(p,h_\epsilon^{ij}) - \exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}_i^*}\right)\delta_{k_0,\epsilon}(k)\right| \qquad \text{(A.0.38)}\\
\stackrel{(A.0.8)}{\leq}&\left|\sum_{j=1}^{N'}\left(\frac{|\mathrm{p}|^2}{p_-}\right)^j \tilde{u}_1(p,h_\epsilon^{ij}) - \exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}_i^*}\right)\sum_{h=1}^{N'}p_{N',h}\left(k\cdot\frac{k_0}{\kappa}\right)^h\right|\\
&+\underbrace{\left|\sum_{h=1}^{N'}p_{N',\epsilon}\left(k\cdot\frac{k_0}{\kappa}\right)-\delta_{k_0,\epsilon}(k)\right|}_{\stackrel{(A.0.7)}{<}\epsilon'}\\
\stackrel{(A.0.37)}{<}&\sum_{h=1}^{N'}|p_{N',h}|\left|\exp\left(\mathrm{i}\frac{k\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}_i^*}\right)\sum_{j=1}^{h}d_{h,j}\int \mathrm{d}\sigma(e)\,\delta_{e_-^*,\epsilon_-}(e_-)\right.\\
&\qquad\left.\left[\left(e_-\frac{|\mathrm{p}|^2}{p_-}v\cdot\nabla_\mathrm{e}\right)^j\delta_{\mathrm{e}^*,\epsilon'''}(\mathrm{e})\right]\tilde{u}_1(p,e)(k)-\left(k\cdot\frac{k_0}{\kappa}\right)^h\right|+\epsilon'\\
=&\sum_{h=1}^{N'}|p_{N',h}|\left|\int \mathrm{d}\sigma(e)\,\delta_{e_-^*,\epsilon_-}(e_-)\delta_{\mathrm{e}^*,\epsilon'''}(\mathrm{e})\right.\\
&\qquad\left.\sum_{j=1}^{h}d_{h,j}\underbrace{\exp\left(\mathrm{i}\frac{k\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}^*}\right)\left(e_-\frac{|\mathrm{p}|^2}{p_-}v\cdot\nabla_\mathrm{e}\right)^j\tilde{u}_1(p,e)(k)}_{\stackrel{(A.0.31)}{=}Q_j}-\left(k\cdot\frac{k_0}{\kappa}\right)^h\right|+\epsilon'\\
\stackrel{(A.0.35)}{\leq}&(\epsilon''+\epsilon''')\underbrace{\sum_{h=1}^{N'}|p_{N',h}|C_h}_{=:C'_{N'}}\underbrace{\int \mathrm{d}\sigma(e)\,\delta_{e_-^*,\epsilon_-}(e_-)\delta_{\mathrm{e}^*,\epsilon'''}(\mathrm{e})}_{=1}+\epsilon'=\epsilon'+C'_{N'}(\epsilon''+\epsilon''')
\end{aligned}
$$

This prefactor leads to the definition of the function c:

$$
c_{k_0}(\mathrm{p},k)=\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}}\right) \qquad \text{(A.0.39)}
$$

It is clear from this definition that eq. (3.2.25), i.e. $c_{k_0}(p,k_0)=1$, is satisfied. For ϵ sufficiently small,

$$
\begin{aligned}
&|\mathrm{e}^*\cdot \mathrm{p}_i^*|\stackrel{(A.0.23)}{>}\frac{|k_0\cdot \mathrm{p}_0|}{2\kappa} \qquad \text{(A.0.40)}\\
\Rightarrow&\frac{|k_0\cdot \mathrm{p}_0|}{2\kappa}-|\mathrm{e}^*\cdot \mathrm{p}|<|\mathrm{e}^*\cdot \mathrm{p}_i^*|-|\mathrm{e}^*\cdot \mathrm{p}|\leq|\mathrm{e}^*\cdot(\mathrm{p}_i^*-\mathrm{p})|\stackrel{(A.0.18)(A.0.21)}{<}\epsilon''\\
\Rightarrow&|\mathrm{e}^*\cdot \mathrm{p}|>\frac{|k_0\cdot \mathrm{p}_0|}{2\kappa}-\epsilon \qquad \text{(A.0.41)}
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\left|\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}_i^*}\right)-\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}}\right)\right| \\
=&\left|\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2}\left[\frac{1}{\mathrm{e}^*\cdot \mathrm{p}_i^*}-\frac{1}{\mathrm{e}^*\cdot \mathrm{p}}\right]\right)-1\right| = \left|\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2}\frac{\mathrm{e}^*\cdot(\mathrm{p}_i^*-\mathrm{p})}{(\mathrm{e}^*\cdot \mathrm{p}_i^*)(\mathrm{e}^*\cdot \mathrm{p})}\right)-1\right| \\
\leq&\left|\frac{(k_0-k)\cdot \mathrm{e}^*}{2}\frac{\mathrm{e}^*\cdot(\mathrm{p}_i^*-\mathrm{p})}{(\mathrm{e}^*\cdot \mathrm{p}_i^*)(\mathrm{e}^*\cdot \mathrm{p})}\right| \overset{(A.0.21)}{\leq} \frac{|k_0-k|}{2}\frac{|\mathrm{p}_i^*-\mathrm{p}|}{|\mathrm{e}^*\cdot \mathrm{p}_i^*||\mathrm{e}^*\cdot \mathrm{p}|} < \frac{\epsilon}{2}\frac{\epsilon''}{|\mathrm{e}^*\cdot \mathrm{p}_i^*||\mathrm{e}^*\cdot \mathrm{p}|} \\
&\overset{(A.0.40)(A.0.41)}{<} \frac{\epsilon}{2}\frac{\epsilon''}{\frac{k_0\cdot \mathrm{p}_0}{2\kappa}\left(\frac{k_0\cdot \mathrm{p}_0}{2\kappa}-\epsilon\right)}. \qquad\qquad \text{(A.0.42)}
\end{aligned}
$$

Consequently with $R(\Delta) := [-\Delta,-\Delta^{-1}]\cup[\Delta^{-1},\Delta]$ (cf. eq. (A.0.17)),

$$
\begin{aligned}
&\int_{R(\Delta)}\frac{\mathrm{d}p_-}{|p_-|}\mathrm{d}^2\mathrm{p}\int \mathrm{d}\nu(k)\left|\sum_{i=1}^{N''}\sum_{j=1}^{N'}\hat{f}_\epsilon^{ij}(p)\tilde{u}_1(p,h_\epsilon^{ij})(k)-c_{k_0}(\mathrm{p},k)\Psi_{\mathrm{p}_0,k_0,\epsilon}(p,k)\right|^2 \\
\overset{(A.0.16)}{=}&\int_{R(\Delta)}\frac{\mathrm{d}p_-}{|p_-|}\mathrm{d}^2\mathrm{p}\int \mathrm{d}\nu(k)\left|\sum_{i=1}^{N''}\left(\sum_{j=1}^{N'}\hat{f}_\epsilon^{ij}(p)\tilde{u}_1(p,h_\epsilon^{ij})(k)-b^i_{\mathrm{p}_0,\epsilon}(\mathrm{p})c_{k_0}(\mathrm{p},k)\Psi_{\mathrm{p}_0,k_0,\epsilon}(p,k)\right)\right|^2 \\
\overset{(A.0.39)}{\leq}&\int_{R(\Delta)}\frac{\mathrm{d}p_-}{|p_-|}\mathrm{d}^2\mathrm{p}\int \mathrm{d}\nu(k)\left(\sum_{i=1}^{N''}\left|\sum_{j=1}^{N'}\hat{f}_\epsilon^{ij}(p)\tilde{u}_1(p,h_\epsilon^{ij})(k)\right.\right. \\
&\qquad\qquad\left.\left.-b^i_{\mathrm{p}_0,\epsilon}(\mathrm{p})\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}}\right)\Psi_{\mathrm{p}_0,k_0,\epsilon}(p,k)\right|\right)^2 \\
=&\int_{R(\Delta)}\frac{\mathrm{d}p_-}{|p_-|}\mathrm{d}^2\mathrm{p}\int \mathrm{d}\nu(k)\left(\left|\hat{f}\left(p_-,\frac{|\mathrm{p}|^2}{p_-}\right)\right|\sum_{i=1}^{N''}b^i_{\mathrm{p}_0,\epsilon}(\mathrm{p})\delta_{\mathrm{p}_0,\epsilon}(\mathrm{p})\right. \\
&\qquad\qquad\left.\left|\sum_{j=1}^{N'}\left(\frac{|\mathrm{p}|^2}{p_-}\right)^j\tilde{u}_1(p,h_\epsilon^{ij})(k)-\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}}\right)\delta_{k_0,\epsilon}(k)\right|\right)^2
\end{aligned}
$$

(using the definitions given by eq. (A.0.36), (A.0.37) and (3.2.22))

$$
\begin{aligned}
\leq&\int_{R(\Delta)}\frac{\mathrm{d}p_-}{|p_-|}\mathrm{d}^2\mathrm{p}\int \mathrm{d}\nu(k)\left(\left|\hat{f}\left(p_-,\frac{|\mathrm{p}|^2}{p_-}\right)\right|\sum_{i=1}^{N''}b^i_{\mathrm{p}_0,\epsilon}(\mathrm{p})\delta_{\mathrm{p}_0,\epsilon}(\mathrm{p})\right. \\
&\qquad\left[\left|\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}_i^*}\right)-\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}}\right)\right|\delta_{k_0,\epsilon}(k)\right. \\
&\qquad\left.\left.+\left|\sum_{j=1}^{N'}\left(\frac{|\mathrm{p}|^2}{p_-}\right)^j\tilde{u}_1(p,h_\epsilon^{ij})(k)-\exp\left(\mathrm{i}\frac{(k_0-k)\cdot \mathrm{e}^*}{2\mathrm{e}^*\cdot \mathrm{p}_i^*}\right)\delta_{k_0,\epsilon}(k)\right|\right]\right)^2 \\
<&\int_{R(\Delta)}\frac{\mathrm{d}p_-}{|p_-|}\mathrm{d}^2\mathrm{p}\int \mathrm{d}\nu(k)\left(\left|\hat{f}\left(p_-,\frac{|\mathrm{p}|^2}{p_-}\right)\right|\delta_{\mathrm{p}_0,\epsilon}(\mathrm{p})\right. \qquad\qquad \text{(A.0.43)} \\
&\qquad\left.\left(\frac{\epsilon}{2}\frac{\epsilon''}{\frac{k_0\cdot \mathrm{p}_0}{2\kappa}\left(\frac{k_0\cdot \mathrm{p}_0}{2\kappa}-\epsilon\right)}\delta_{k_0,\epsilon}(k)+\epsilon'+C'_{N'}(\epsilon''+\epsilon''')\right)\right)^2
\end{aligned}
$$

(using the estimates (A.0.38) and (A.0.42)).

Based on this estimate, the sequence (3.2.23) can be constructed by picking a sequence $(\epsilon_N)_{N\in\mathbb{N}}$ with $\epsilon_N > 0\,\forall N \in \mathbb{N}$ and $\lim_{N\to\infty}\epsilon_N = 0$. For $N \in \mathbb{N}$, choose $\Delta > 0$ sufficiently big, such that the left hand side of eq. (A.0.43) differs from the integral over $\mathbb{R}$ instead of $R(\Delta)$ by at most $\epsilon_N/2$, and ϵ' and ϵ'',ϵ''' sufficiently small, such that the right hand side of (A.0.43) is smaller than $\epsilon_N/2$. Relabelling the functions given in eq. (A.0.36) and (A.0.37) as $\hat{f}^i_\epsilon$ and h^i_ϵ, where $i = 1, \ldots, M_{\epsilon,N} := N'N''$, yields

$$\int \frac{\mathrm{d}p_-}{|p_-|}\mathrm{d}^2\mathrm{p} \int \mathrm{d}\nu(k) \left| \sum_{i=1}^{M_{\epsilon,N}} \hat{f}^i_{\epsilon,N}(p)\tilde{u}_1(p, h^i_{\epsilon,N})(k) - c_{k_0}(\mathrm{p}, k)\Psi_{\mathrm{p}_0,k_0,\epsilon}(p, k) \right|^2 < \epsilon_N,$$

which proves eq. (3.2.24).

□

Remark 20. It might seem like an option to proceed to the smaller ϵ''-neighborhood of p_0 directly instead of introducing a partition of unity from these neighborhoods which ultimately covers an area with radius ϵ around p_0. The reason why the latter method has been necessary is within the proof of part 1 of Theorem 2, where *Lebesgue's Theorem* is used: Its assumptions contain the requirement for the sets on which the function in question is averaged to converge to a point with *bounded* eccentricity, i.e. they must be bounded from the inside and outside by parallelepipeds with ratios of the edge lengths that are bounded in the limit, which excludes the case where the averaging in p becomes sharper than the one in k, i.e. where ϵ'' approaches 0 faster than does ϵ. On the other hand, the strategy involving a partition of unity ensures that all relevant averages are controlled by ϵ.

List of Figures

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