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Special classes of inverse semigroups

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Abstract

In this diploma thesis in the class of inverse semigroups four particular types of semigroups are considered. Their consideration is motivated first by particular restrictions of the natural partial order of the inverse semigroups and also by the existence of certain representations of them. These four classes are the following:

1. Primitive inverse semigroups (any two non-zero idempotents are incomparable – orthogonal sums of Brandt semigroups)
2. Simple or bisimple inverse ω -semigroups (idempotents are order isomorphic to the set of natural numbers in the dual order – Bruck-Reilly semigroups)
3. E-unitary inverse semigroups (every element greater than an idempotent is again idempotent – McAlister's P-semigroups)
4. F-inverse semigroups (each class of the least group congruence contains a maximum element – P-semigroups over semilattices containing a particular order ideal, that is, particular E-unitary inverse semigroups)

In addition, several connections between these classes are established, for example: Simple or bisimple ω -semigroups versus E-unitary inverse semigroups respectively F-inverse semigroups; primitive inverse semigroups versus E-unitary inverse semigroups respectively F-inverse semigroups.

Zusammenfassung

In dieser Diplomarbeit wählen wir aus der Klasse der inversen Halbgruppen vier spezielle Unterklassen aus. Die Auswahl erfolgt einerseits aufgrund bestimmter Forderungen, die wir an die natürliche partielle Ordnung der inversen Halbgruppen stellen, andererseits existiert für jede der vier Klassen eine gewisse explizite Darstellung. Die vier Klassen sind die Folgenden:

1. Primitive inverse Halbgruppen (je zwei Idempotente ungleich Null sind unvergleichbar - orthogonale Summen von Brandt Halbgruppen)
2. Einfache oder bisimple inverse ω -Halbgruppen (Idempotente sind ordnungs-isomorph zur Menge der natürlichen Zahlen mit der dualen Ordnung – Bruck-Reilly-Halbgruppen)
3. E-unitäre inverse Halbgruppen (jedes Element größer als ein Idempotentes ist selbst idempotent – McAlisters P-Halbgruppen)
4. F-inverse Halbgruppen (jede Klasse der minimalen Gruppenkongruenz besitzt ein Maximum – P-Halbgruppen über Halbverbänden mit speziellem Ordnungsideal, also insbesondere E-unitär)

Darüber hinaus stellen wir Beziehungen zwischen diesen Klassen her, etwa zwischen einfachen oder bisimplen ω -Halbgruppen und E-unitären inversen Halbgruppen bzw. F-inversen Halbgruppen, sowie zwischen primitiven inversen Halbgruppen und E-unitären inversen Halbgruppen bzw. F-inversen Halbgruppen.

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1 Introduction

The subject of this diploma thesis is inverse semigroups or more precisely, special classes of them. Inverse semigroups emerged in the 1950s, when Wagner and Preston independently started to study them. In both cases the origin being the study of semigroups of partial one-one mappings of a set. Inverse semigroups are particularly useful when it comes to partial symmetries, i.e. symmetries between parts of a structure, much like the link between groups and symmetries. They have applications in many branches of mathematics including: algebra, differential geometry, C^* -algebras, tilings, quasicrystals and solid-state physics, combinatorial group theory, model theory and linear logic. The theory of inverse semigroups is a vast and steadily growing field. For this reason we confine ourselves to four special classes of inverse semigroups. The selection has been made in view of the so-called natural partial order which can be defined for every inverse semigroup. Each class is restricted by a particular form or condition of the natural partial order. Moreover, we are able to present a general method of construction for each of our classes. This is something that cannot be taken for granted in general.

The core of the thesis consists of the chapters 4 to 7 where we investigate the special classes. In the preliminaries we provide basic concepts of semigroup theory that will be needed later on. Many of the presented concepts will be familiar to the reader. A special focus lies on the Rees theorem which characterises completely 0-simple semigroups. The result is required in section 4.1 when we describe Brandt semigroups as Rees matrix semigroups of a particular easy form. The third chapter introduces inverse semigroups. Although they do have many idempotents in general they can be seen as generalised groups. Every element in an inverse semigroup possesses a uniquely determined generalised inverse. We establish several equivalent definitions for inverse semigroups. In particular we describe them in terms of regularity and Green's relations. Idempotents are of great importance. In the case of an inverse semigroup they do form a semilattice and we often speak of the 'semilattice of idempotents'. Section 3.1 presents important examples, including the symmetric inverse monoid, the bicyclic semigroup and Clifford semigroups. In section 3.2 we define the natural partial order relation that can be defined on each inverse semigroup. It extends the partial order of idempotents in an arbitrary semigroup. The natural partial order is compatible with multiplication and taking inverses. Section 3.3 presents the Wagner-Preston representation theorem 3.3.1 which states that any inverse semigroup can be embedded in a symmetric inverse monoid. The remaining two sections in chapter 3 deal with the compatibility relation, the minimum group congruence and the maximum idempotent-separating congruence.

The first special class that we cover in this thesis are primitive inverse semigroups. They resemble groups insofar as the natural partial order relation is equality when

restricted to non-zero elements. Brandt semigroups are defined as completely 0-simple inverse semigroups. In 4.0.13 we prove that they are primitive. In 4.0.15 we show that every primitive inverse semigroup is an orthogonal sum of Brandt semigroups, which is why we focus on the study of Brandt semigroups in the remainder of the chapter. The Rees theorem states that every completely 0-simple semigroup is isomorphic to some Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$. Since a Brandt semigroup is also inverse we obtain an easy characterisation of Brandt semigroups. To establish another characterisation of Brandt semigroups we define a particular semigroup obeying two axioms and then show that the semigroups constructed in this way are exactly the Brandt semigroups.

For primitive inverse semigroups all non-zero idempotents are incomparable. With inverse ω -semigroups we consider the other extreme in chapter 5. Here the idempotents do form a particular chain. The aim of the chapter is to present construction methods for simple and bisimple ω -inverse semigroups. In both cases so called Bruck-Reilly extensions play an important part. These semigroups help us to tackle the non-fundamental cases wherein the result for bisimple inverse ω -semigroups is easier to obtain.

Chapter 6 is devoted to E-unitary inverse semigroups. For every inverse semigroup the semilattice of idempotents E is a lower ideal with respect to the natural partial order. E-unitary semigroups have the property that E is an upper ideal as well, i.e. every element greater than an idempotent is itself an idempotent. This is only one of many definitions we present throughout the chapter. McAlister's covering theorem gives a first hint of why E-unitary inverse semigroups are important. It states that every inverse semigroup is a particular morphic image of an E-unitary inverse semigroup. Another reason for the importance of E-unitary inverse semigroups is that many 'naturally' occurring semigroups are E-unitary. In section 6.1 we list various examples such as the bicyclic semigroup, free semigroups and semidirect products of semilattices by groups. We also examine when a bisimple inverse ω -semigroup is E-unitary. The same is done for simple inverse ω -semigroups and Clifford semigroups. In section 6.2 we develop a construction due to McAlister which yields all E-unitary inverse semigroups. The final chapter deals with F-inverse semigroups. These are E-unitary inverse semigroups in which each σ -class has a maximum element. Every E-unitary inverse semigroup can be embedded in an F-inverse semigroup. This shows that F-inverse semigroups are in a sense as big as E-unitary inverse ones. In the first part of the chapter we present examples and characterise for familiar semigroups when they are F-inverse. Perhaps surprisingly we can show that bisimple as well as simple E-unitary inverse ω -semigroups are always F-inverse. We also mention a modification of the McAlister covering theorem in 7.1.6. In section 7.2 we set about establishing a result analogous to 6.2.5. The particular structure of F-inverse semigroups admits considerable simplification in its McAlister representation. In order to achieve this we need to introduce R-inverse semigroups. A construction method is given in 7.2.7 and we prove in 7.2.12 that every R-inverse semigroup is obtained in such a way. This enables us to give a construction method for any F-inverse semigroups in 7.2.16.

2 Preliminaries

This chapter serves as a collection of basic concepts that will be used throughout the thesis. The intention is to create a manual that can be used to look up the respective definitions when they do occur later on in the main text. In particular we present regular semigroups and (0-)simple ones. A reader familiar with the fundamentals of semigroup theory might want to skip this chapter for the time being and come back to it if necessary. An exception of this is section 2.7 in which we prove Rees' theorem.

Definition 2.0.1 *A semigroup is a non-empty set S equipped with an associative binary operation \cdot .*

Definition 2.0.2 *Let (S, \cdot) be a semigroup. A subset A of S is a subsemigroup if A is itself a semigroup with respect to \cdot .*

A special case occurs if the subsemigroup of S is also a group. We then call A a *subgroup* of S . A semigroup S might or might not contain an identity element. In the latter case we can make S into a semigroup having an identity element by formally adding an element 1 with the properties

$$\begin{aligned} 1 \cdot 1 &= 1, \\ \forall s \in S: 1 \cdot s &= s = s \cdot 1. \end{aligned}$$

We thus get a new semigroup denoted by S^1 . If the semigroup S had an identity element in the first place we put $S^1 = S$.

A similar thing can be done for a semigroup without a zero element. We define $S^0 = S \cup \{0\}$, where 0 denotes a formal element such that

$$\forall s \in S: 0 \cdot s = s \cdot 0 = 0 \cdot 0 = 0.$$

For a semigroup with zero we put $S^0 = S$.

Definition 2.0.3 *Let G be a group. We call the semigroup G^0 a 0-group.*

Definition 2.0.4 *A semigroup with an identity element is called a monoid.*

Definition 2.0.5 *Let S be a monoid. An element u is called unit if there exists an element u' such that $uu' = 1 = u'u$.*

It is easy to see that the set of all units forms a subgroup, the so-called *group of units* of S .

Definition 2.0.6 A semigroup S with zero is called null semigroup if $S^2 = \{0\}$, i.e. if $\forall a, b \in S: ab = 0$.

Definition 2.0.7 Let (S, \cdot) and (T, \otimes) be semigroups. A mapping $\phi: S \rightarrow T$ is called a (semigroup) morphism if

$$\forall a, b \in S: \phi(a \cdot b) = \phi(a) \otimes \phi(b).$$

Definition 2.0.8 Let S be a semigroup. An idempotent is an element e of S such that $e^2 = e$. We denote the set of all idempotents of S by E_S .

Definition 2.0.9 Let S be a semigroup. An equivalence relation ρ on S is called a congruence relation if $a \rho b$ and $c \rho d$ implies $ac \rho bd$.

It is called a left congruence if $a \rho b$ implies $ca \rho cb$ for all $c \in S$. Right congruences are defined analogously.

Remark 2.0.10 It is not difficult to check that an equivalence relation is a congruence relation if and only if it is both a left and a right congruence.

A congruence ρ on a semigroup S partitions the semigroup in its equivalence classes. We denote the set of equivalence classes by S/ρ and define a multiplication \otimes on it by

$$\rho(a) \otimes \rho(b) = \rho(ab).$$

The following proposition is well known and we will therefore omit the proof.

Proposition 2.0.11 The set S/ρ equipped with the multiplication \otimes is a semigroup, the so-called factor semigroup (or quotient semigroup) of S modulo ρ .

In particular the resulting class $\rho(ab)$ does not depend on the chosen representatives a and b .

Definition 2.0.12 Let S be a semigroup with zero and let there be given a system of subsemigroups $\{S_\alpha\}_{\alpha \in A}$ such that $S_\alpha \cap S_\beta = S_\alpha S_\beta = 0$ if $\alpha \neq \beta$ and $S = \bigcup_{\alpha \in A} S_\alpha$. In such a case, S is called an orthogonal sum¹ of semigroups S_α to be denoted by $S = \sum_{\alpha \in A} S_\alpha$.

2.1 Partially ordered sets and semilattices

A partially ordered set (or poset) is a set P endowed with an order relation \leq . The order relation satisfies the following axioms:

$\forall a, b, c \in P$:

(O1) $a \leq a$

(O2) $a \leq b, b \leq a \implies a = b$

(O3) $a \leq b, b \leq c \implies a \leq c$

¹In the literature orthogonal sums are often called 0-direct unions

We will write (P, \leq) for the poset or—if no misunderstandings are to be expected—merely P .

Proposition 2.1.1 *Let S be a semigroup. Then the relation*

$$e \rho f \iff ef = fe = e \tag{2.1.1}$$

is an order relation on E_S .

Proof *(O1) and (O2) are obvious. To show transitivity suppose that $e \rho f, f \rho g$. Then $eg = (ef)g = e(fg) = ef = e$ and $ge = g(fe) = (gf)e = fe = e$.*

□

After we have shown that ρ is an order relation we prefer to write \leq instead of ρ as usual.

If $l \leq a, b$ for some $l, a, b \in P$ we call l a lower bound of a and b . If in addition $l = \max\{x \in P : x \text{ is a lower bound of } a \text{ and } b\}$ we say that l is the greatest lower bound of a and b . Notice that the greatest lower bound of two elements does not necessarily exist. For example the set $\{a, b, c, d\}$ together with the order relation \leq determined by $a \leq c, d$ and $b \leq c, d$ is a poset. Both a and b are lower bounds of c and d . But a and b are not comparable with respect to \leq and so there is no greatest lower bound of c and d . If there exists a greatest lower bound however it is unique.

We are now ready to introduce semilattices. We will see that one can think of them as particular posets or as particular semigroups.

We start with an order theoretical definition:

Definition 2.1.2 *We call a poset (P, \leq) semilattice if for each pair (a, b) the greatest lower bound of a and b exists.*

Now we characterise semilattices in algebraic terms.

Proposition 2.1.3 *Let (P, \leq) be a semilattice. Define a binary operation \wedge on P by the rule that $a \wedge b$ is the greatest lower bound of a and b . Then (P, \wedge) is a commutative idempotent semigroup.*

Proof The following properties for a, b, c in P are easily verified:

$$(A) \quad (a \wedge b) \wedge c = a \wedge (b \wedge c),$$

$$(I) \quad a \wedge a = a,$$

$$(C) \quad a \wedge b = b \wedge a.$$

Property (A) states that \wedge is associative. By (I) all elements of P are idempotents and by (C) the binary relation is commutative. Hence (P, \wedge) is a commutative idempotent semigroup.

□

The converse of 2.1.3 is also true. This is the content of the following proposition.

Proposition 2.1.4 *Let (S, \cdot) be a commutative idempotent semigroup. Define a relation \leq on S by the rule*

$$a \leq b \iff a = a \cdot b. \quad (2.1.2)$$

Then \leq is an order relation and (S, \leq) is a semilattice.

Proof We will use juxtaposition. All elements in S are idempotents. Thus $a = aa$ and so $a \leq a$. This is (O1). Suppose that $a \leq b, b \leq a$. Hence $a = ab = ba = b$ since S is commutative. This is (O2). Finally suppose that $a \leq b, b \leq c$. Then $ac = (ab)c = a(bc) = ab = a$ and so $a \leq c$. This is (O3). We have proved that (S, \cdot) is a poset. It remains to show that for each pair (a, b) the greatest lower bound of a and b exists. We claim that ab is this greatest lower bound. Certainly $ab \leq a, b$ since S is commutative. Now assume that $c \leq a, b$ for some c . Then $ca = c = cb$ by assumption. Hence $c(ab) = (ca)b = cb = c$ and so $c \leq ab$.

□

Remark 2.1.5 *We have seen that semilattices and commutative idempotent semigroups are essentially the same thing. Thus we shall not distinguish rigorously between the ‘semilattices’ (P, \leq) and (P, \wedge) . If we emphasise on the algebraic aspect we often call $a \wedge b$ the meet of a and b and say that (P, \wedge) is a meet semilattice.*

Remark 2.1.6 *We defined a semilattice via greatest lower bounds. This definition is somewhat arbitrary. We could have used least upper bounds as well in which case we would have gotten the same algebraic structure.*

We explained earlier how a congruence partitions a semigroup into its congruence classes. A special case occurs when the congruence classes are subsemigroups and the resulting factor semigroup S/ρ is a semilattice.

Definition 2.1.7 *Let S be a semigroup and let ρ be a congruence such that S/ρ (regarded as a set) consists of subsemigroups of S and such that the resulting factor semigroup is a semilattice. We then say that the semigroup S is the semilattice of its subsemigroups. We may likewise define semilattices of subgroups.*

We conclude this section with a lemma.

Lemma 2.1.8 *Let X be a semilattice and let $x, y \in X$. Then*

$$[x \wedge y] = [x] \cap [y],$$

where $[s] = \{t \in X : t \leq s\}$.

Proof This follows directly from the definition of the greatest lower bound.

□

2.2 Ideals

Definition 2.2.1 *Let S be a semigroup. A non-empty subset A is called a left ideal of S if $sA \subseteq A$ for all $s \in S$. It is called a right ideal if $As \subseteq A$ for all $s \in S$. The subset A is called an ideal if it is both a left and a right ideal.*

An ideal is thus a subsemigroup of S which is also closed under multiplication of elements in S . Of particular interest are those ideals which are generated by a single element. The proof of the following lemma is straightforward.

Lemma 2.2.2 *Let S be a semigroup and let $a \in S$. Then S^1a is the smallest left ideal, aS^1 is the smallest right ideal and S^1aS^1 is the smallest ideal containing a .*

Definition 2.2.3 *We call the ideals appearing in the previous lemma the principal left, principal right and principal ideal of a respectively.*

The following definition will be needed in chapter 7.

Definition 2.2.4 *Let A be a subsemigroup of a semigroup S . We call A a retract of S if there exists a morphism from S onto A which lets elements in A fixed. If in addition to that, A is an ideal of S , then A is a retract ideal of S .*

2.3 Green's relations

Definition 2.3.1 *Let S be a semigroup. We call two elements a and b \mathcal{L} -equivalent if $S^1a = S^1b$, that is if they constitute the same principal left ideals. We call them \mathcal{R} -equivalent if they constitute the same principal right ideals and we call them \mathcal{J} -equivalent if they constitute the same principal (two-sided) ideals. Moreover we say that a and b are \mathcal{H} -equivalent if they are both \mathcal{L} - and \mathcal{R} -equivalent. Finally a and b are said to be \mathcal{D} -equivalent if there exists some $x \in S$ such that $a \mathcal{L} x$ and $x \mathcal{R} b$. The relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ are called Green's relations.*

Remark 2.3.2 (i) *It is more or less easy to see that all Green's relations are equivalences².*

(ii) *\mathcal{D} is defined such that it is the smallest equivalence containing \mathcal{L} and \mathcal{R} .*

(iii) *If we regard relations as subsets of $S \times S$ we have the inclusions*

$$\mathcal{H} \subseteq \mathcal{L}, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}.$$

The last inclusion follows from (ii) and the fact that $\mathcal{L}, \mathcal{R} \subseteq \mathcal{J}$.

²A proof of this can be found for example in [3]

In contrast to other relations we write for example \mathcal{L}_a instead of $\mathcal{L}(a)$ to denote the \mathcal{L} -class containing the element a . We do this because we want to reserve the notation $\mathcal{L}(a)$ for the set S^1a . While it is true that $a \mathcal{L} b \iff \mathcal{L}(a) = \mathcal{L}(b)$ it is important to note that in general we only have $\mathcal{L}_a \subseteq \mathcal{L}(a)$. The notation $\mathcal{L}(a)$ enables us to define a partial order \leq for \mathcal{L} -classes:

$$\mathcal{L}_a \leq \mathcal{L}_b \iff \mathcal{L}(a) \subseteq \mathcal{L}(b)$$

By $\mathcal{L}_a < \mathcal{L}_b$ we shall mean that $\mathcal{L}(a) \subset \mathcal{L}(b)$. Analogously we define $\mathcal{R}(a), \mathcal{J}(a), \mathcal{D}(a), \mathcal{H}(a)$ and \leq for the other Green's relations.

Green's relations are very important in semigroup theory as we will soon see. The following results all belong to the requisite know-how of semigroup theory and we will therefore omit the proofs. The next proposition is usually known as Green's Lemma.

Proposition 2.3.3 ([3], Lemma 2.2.1) *Let a, b be elements of a semigroup S such that $a \mathcal{R} b$ and let $s, \hat{s} \in S^1$ be such that $a = bs, b = a\hat{s}$. Then the right translations $x \mapsto x\hat{s}$ and $y \mapsto ys$ are mutually inverse \mathcal{R} -class preserving bijections from \mathcal{L}_a onto \mathcal{L}_b and from \mathcal{L}_b onto \mathcal{L}_a respectively.*

There is an obvious counterpart of the previous lemma given by

Proposition 2.3.4 ([3], Lemma 2.2.2) *Let a, b be elements of a semigroup S such that $a \mathcal{L} b$ and let $s, \hat{s} \in S^1$ be such that $a = sb, b = \hat{s}a$. Then the left translations are mutually inverse \mathcal{L} -class preserving bijections from \mathcal{R}_a onto \mathcal{R}_b and from \mathcal{R}_b onto \mathcal{R}_a respectively.*

Proposition 2.3.5 ([3], Lemma 2.3.7) *Let a, b be elements in a \mathcal{D} -class D . Then $ab \in R_a \cap L_b \iff L_a \cap R_b$ contains an idempotent.*

The following result is usually called Green's Theorem.

Theorem 2.3.6 ([3], Theorem 2.2.5) *Let S be a semigroup and H be an \mathcal{H} -class. Then either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of S .*

Corollary 2.3.7 ([3], Corollary 2.2.6) *Every \mathcal{H} -class containing an idempotent is a group.*

2.4 Group actions on partially ordered sets

In this section our main goal is to define group actions on partially ordered sets. This is basically what we need in chapters 6 and 7. If not said otherwise, X shall denote a poset in the following.

Definition 2.4.1 *We shall say that a bijection $\alpha: X \rightarrow X$ is an order automorphism if, for all a, b in X , $a \leq b \iff \alpha(a) \leq \alpha(b)$.*

The set of all order automorphisms of X can easily be seen to form a group under composition of functions. We shall denote this group by $Aut(X)$.

Definition 2.4.2 Let G be a group and (X, \leq) be a poset. We shall say that G acts on X by order automorphisms if there exists a group morphism θ from G to $\text{Aut}(X)$.

For reasons of convenience we shall always write gx ($g \in G, x \in X$) instead of $\theta(g)(x)$ and think of G itself as acting on X (on the left). The fact that it does so by means of order automorphisms is expressed by saying that

$$\begin{aligned} gx = gy &\iff x = y \quad (g \in G, x, y \in X), \\ \forall g \in G \forall y \in X \exists x \in X : gx = y, \\ x \leq y &\iff gx \leq gy \quad (g \in G, x, y \in X). \end{aligned} \tag{2.4.1}$$

The first line states that the action of g on X is one-one, the second tells us that it is also onto and the third describes the order morphism property.

To avoid any misconceptions we point out that for instance by ghx we mean in fact $\theta(g)(\theta(h)(x))$. To get a feeling for group actions we prove two lemmas.

Lemma 2.4.3 Let G act on (X, \leq) by order automorphisms. Fix some $g \in G$ and suppose that $x \wedge y$ exists. Then $gx \wedge gy$ also exists and $gx \wedge gy = g(x \wedge y)$.

Proof Clearly $g(x \wedge y) \leq gx$ and $g(x \wedge y) \leq gy$. Let $z \leq gx, gy$ for some $z \in X$. Now $g^{-1}z \leq g^{-1}gx = 1x = x$ and similarly $g^{-1}z \leq y$. Hence $g^{-1}z \leq x \wedge y$ and consequently $z \leq g(x \wedge y)$. This means that $g(x \wedge y)$ is indeed the greatest lower bound of gx and gy .

□

For $A \subseteq X$ we define $gA := \{gx : x \in A\}$.

Lemma 2.4.4 Let G act on (X, \leq) by order automorphisms. Then

$$\forall g \in G \forall x \in X : g[x] = [gx].$$

Proof Let $y \in [gx]$, i.e. $y \leq gx$. Hence $g^{-1}y \leq x$ or to put it differently $g^{-1}y \in [x]$. Consequently $y = 1y = gg^{-1}y \in g[x]$.

Conversely, suppose that $y \in g[x]$. Then $y = gz$ for some $z \leq x$. But $z \leq x$ implies $gz \leq gx$ and so $y \in [gx]$ as required.

□

Group actions are not restricted to posets. If we have more structure we usually want the group action to respect this additional structure.

2.5 Direct and semidirect products

Direct products are a common concept throughout mathematics. They serve as a means of building new structures from given building blocks.

Let (S, \cdot) and (T, \otimes) be semigroups. We define a new semigroup $S \times T$ by the rule $(s, t)(s', t') = (s \cdot s', t \otimes t')$. It is immediate from the properties of the semigroups S and T that $S \times T$ is also a semigroup.

Definition 2.5.1 *For given semigroups S and T we call the semigroup $S \times T$ as defined above the direct product of S and T .*

Direct products can be seen as a special case of what are called semidirect products. We confine ourselves to the case where one factor is a group and one is a semilattice.

Let G be a group and (E, \wedge) be a semilattice. Further, let G act on E by group automorphisms. Consider the set $E \times G$ equipped with the multiplication

$$(e, g)(f, h) = (e \wedge (g \cdot f), gh).$$

We must show that this multiplication is associative:

$$\begin{aligned} ((e, g)(f, h))(i, k) &= (e \wedge gf, gh)(i, k) = (e \wedge gf \wedge (gh)i, ghk) \stackrel{2.4.3}{=} \\ &= (e \wedge g(f \wedge hi), ghk) = (e, g)(f \wedge hi, hk) = (e, g)((f, h)(i, k)). \end{aligned}$$

Definition 2.5.2 *The semigroup defined above is called the semidirect product of E by G and denoted by $E \rtimes G$.*

2.6 Regular semigroups

Regular semigroups are of course interesting on their own account. We need a little more than the definition to fulfil our later requirements.

Definition 2.6.1 *Let S be a semigroup and let $a \in S$. We say that a is a regular element if there exists an element x in S such that $axa = a$. S is called regular if all elements in S are regular.*

Proposition 2.6.2 ([3], Proposition 2.3.1) *Let S be a semigroup and D be a \mathcal{D} -class. If D contains a regular element, then all elements of D are regular.*

Proof We are first going to prove that the statement is true for \mathcal{L} -classes and \mathcal{R} -classes. Having achieved this the assertion becomes trivial. Let a be a regular element and let $b \in \mathcal{L}_a$. By assumption $\exists x: axa = a$. Since a and b are \mathcal{L} -equivalent there exist s, t in S^1 such that $a = sb, b = ta$. Now

$$b = ta = (ta)xa = bxa = b(xs)b$$

and so b is regular. A similar argument can be applied to the \mathcal{R} -class case.

Combining both cases completes the proof. □

We shall therefore call a \mathcal{D} -class regular if it contains a regular element.

Proposition 2.6.3 ([3], Proposition 2.3.2) *Let S be a semigroup. Every \mathcal{L} - and every \mathcal{R} -class lying in a regular \mathcal{D} -class contains an idempotent.*

Proof Let \mathcal{D}_a be a regular \mathcal{D} -class, then a is a regular element and so $\exists x \in S: axa = a$. In effect $xa \in E_S$ and $a \mathcal{L} xa$. Hence \mathcal{L}_a contains an idempotent. A similar argument shows that \mathcal{R}_a also contains an idempotent. Since a was arbitrarily chosen, this completes the proof. □

2.7 The Rees theorem

In this section we define simple and 0-simple semigroups. Very soon we focus on completely 0-simple semigroups which admit an elegant characterisation due to Rees. The corresponding result is known as the Rees theorem. The Rees theorem is crucial in one of our characterisations of Brandt semigroups featuring in section 4.1. This justifies the considerable amount of work we have to put in the proof of it.

Definition 2.7.1 *A semigroup without zero is called simple if it has no proper ideals. A semigroup S with zero is called 0-simple if*

- (i) S and $\{0\}$ are its only ideals,
- (ii) $S^2 \neq \{0\}$.

The second condition serves only to exclude the two-element null semigroup, since any larger null semigroup fails to qualify on the grounds of having proper ideals (every set A with $\{0\} \subset A \subset S$ is an ideal).

Lemma 2.7.2 (i) S is simple if and only if $\mathcal{J} = S \times S$.

(ii) S is 0-simple if and only if $S^2 \neq \{0\}$ and $\{0\}, S \setminus \{0\}$ are the only \mathcal{J} -classes.

Proof

(i): First assume that S is simple. Let $a, b \in S$. Both $S^1 a S^1$ and $S^1 b S^1$ are ideals and so we have $S^1 a S^1 = S = S^1 b S^1$ by assumption. Since a, b were arbitrary it follows $a \mathcal{J} b$ and consequently $\mathcal{J} = S \times S$.

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To show the converse suppose now that $A \subset S$ is a proper ideal. Then $b \neq S^1aS^1 \subseteq A$ for $a \in A, b \in S \setminus A$. Thus $S^1aS^1 \neq S^1bS^1$ which means that a and b lie in different \mathcal{J} -classes.

The proof of (ii) is similar.

□

A simple semigroup can be made into a 0-simple one by adjoining a zero, that is by considering S^0 instead of S . The converse is not true in general. More precisely, a 0-simple semigroup becomes a simple semigroup by removing the zero element if $S \setminus \{0\}$ is still a semigroup. This might not be the case since there might be $a, b \neq 0: ab = 0$. We have seen that not all 0-simple semigroups can be derived from simple ones. It is always possible however to deduce a theorem about simple semigroups from one about 0-simple semigroups by specialising to the case where the semigroup has no proper zero-divisors. The next proposition and the subsequent corollary provide an example for such a strategy. They give an important alternative characterisation for simplicity and 0-simplicity respectively.

Remark 2.7.3 *Notice that in contrast to group or ring theory, a simple semigroup can have proper morphic images. This is because not every congruence on a semigroup is associated with an ideal.*

Proposition 2.7.4 ([3], Proposition 3.1.1) *A semigroup S with zero is 0-simple if and only if*

$$\forall a \neq 0: SaS = S \text{ and } S \neq \{0\}.$$

Proof Suppose first that S is 0-simple. Then, by definition, $S^2 \neq \{0\}$ and thus $S^2 = S$ since S^2 is an ideal. Consequently $S \neq \{0\}$ and $S^3 = S^2S = SS = S$. Letting $a \neq 0$, SaS clearly is an ideal and hence either $SaS = \{0\}$ or $SaS = S$ by assumption. Assume, by way of contradiction, that $SaS = \{0\}$. It follows easily that the subset $\{x: SxS = \{0\}\}$ is then an ideal which equals S since it contains by construction the non-zero element a . This implies $S^3 = \{0\}$ in contrast to our observation at the beginning. We have shown that $SaS = S$ for every $a \neq 0$, as required.

Conversely, suppose that $SaS = S$ for every $a \neq 0$. For $S = \{0\}$ this holds trivially but then $S^2 = \{0\}$ and S is not 0-simple which is why we have to exclude this case. In our situation it is clear that $S^2 \neq \{0\}$. Assume that $A \neq \{0\}$ is an ideal of S . We can find an element $0 \neq a \in A$ and it follows

$$S = SaS \subseteq SAS \subseteq A.$$

Hence S does not contain proper ideals apart from $\{0\}$ and so S is indeed 0-simple.

□

Corollary 2.7.5 ([3], Corollary 3.1.2) *A semigroup S without zero is simple if and only if*

$$\forall a \in S: SaS = S.$$

Proof Suppose that S is simple. Then S^0 is 0-simple and hence $S^0aS^0 = S^0$ for all a in $S^0 \setminus \{0\} = S$ by 2.7.4. It follows that $SaS = S$ for all $a \in S$.

Conversely, suppose that $\forall a \in S: SaS = S$. Then $S \subset S^0 \neq \{0\}$ and $S^0aS^0 = S^0$ for all a in $S^0 \setminus \{0\} = S$. We conclude from 2.7.4 that S^0 is 0-simple. Hence S is simple since the removal of the zero-element does not destroy the semigroup structure.

□

We now want to prove Rees' theorem. It gives a characterisation of completely 0-simple semigroups in terms of so-called Rees matrix semigroups. We will need this result urgently in chapter 4 where we investigate Brandt semigroups, i.e. particular primitive inverse semigroups.

not only primitive inverse semigroups but also Brandt semigroups.

Definition 2.7.6 *A non-zero idempotent in a semigroup with zero is said to be primitive if it is minimal in $E_S \setminus \{0\}$ with respect to (2.1.1).*

Thus a primitive idempotent f has the property that

$$ee = e, ef = fe = e \neq 0 \implies e = f.$$

Definition 2.7.7 *A semigroup S is said to be completely 0-simple if it is 0-simple and has a primitive idempotent.*

Rees gives a fairly easy method for constructing such semigroups:

Construction 2.7.8 Let G be a group with identity element e and let I, Λ be non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ -Matrix with entries in the 0-group G^0 which is regular in the sense that each column and each row contains at least one element $\neq 0$. Define an operation on $S = (I \times G \times \Lambda) \cup \{0\}$ as follows

$$(i, a, \lambda)(j, b, \kappa) = \begin{cases} (i, ap_{\lambda j}b, \kappa) & p_{\lambda j} \neq 0 \\ 0 & p_{\lambda j} = 0 \end{cases}$$

$$0(i, a, \lambda) = (i, a, \lambda)0 = 0 \cdot 0 = 0. \quad (2.7.1)$$

With the above definition we have

Proposition 2.7.9 ([3], Lemma 3.2.2) *S is a completely 0-simple semigroup.*

Proof First we have to show that the multiplication defined in (2.7.1) is associative. To do this we establish a bijection between the set $S \setminus \{0\}$ and the set of $I \times \Lambda$ -matrices $(a)_{i\lambda}$ ($a \in G$), where $(a)_{i\lambda}$ denotes the $I \times \Lambda$ -matrix with entry a in the position (i, λ) and zeros elsewhere, in the obvious way. Since $(0)_{i\lambda}$ is independent of i and λ , the correspondence can be extended to one between S and the set

$$T := \{(a)_{i\lambda} : a \in G^0, i \in I, \lambda \in \Lambda\}.$$

A simple calculation shows that

$$(a)_{i\lambda} P(b)_{j\kappa} = (ap_{\lambda j} b)_{i\kappa},$$

where the juxtaposition on the left denotes matrix multiplication. Bearing in mind that we are calculation in a 0-group rather than a ring, the appearance of sums should not bother us, since owing to the high density of zero entries we have in fact only one summand. The composition (2.7.1) corresponds in T to the composition \circ given by

$$(a)_{i\lambda} \circ (b)_{j\kappa} = (a)_{i\lambda} P(b)_{j\kappa},$$

which is apparently associative.

Next we show that S is 0-simple. Using 2.7.4 what we have to show is that for any two non-zero elements (i, a, λ) and (j, b, κ) there are elements $x, y \in S$ such that $x(i, a, \lambda)y = (j, b, \kappa)$ holds. The regularity of P allows us to find non-zero elements $p_{\mu i}$ and $p_{j\nu}$ and it is easily shown that

$$(j, a^{-1}p_{\mu i}^{-1}, \mu) (i, a, \lambda) (\nu, p_{\lambda\nu}^{-1}b, \kappa) = (j, b, \kappa).$$

Hence S is 0-simple. To complete the proof we have to detect a primitive idempotent. We are going to prove a bit more. In fact every idempotent is primitive. To see this we must first identify the idempotents. The equation

$$(i, a, \lambda) = (i, a, \lambda)(i, a, \lambda) = (i, ap_{\lambda i}a, \lambda)$$

yields that $p_{\lambda i} = a^{-1}$. If we now take two non-zero idempotents $e = (i, p_{\lambda i}^{-1}, \lambda)$ and $f = (j, p_{\kappa j}^{-1}, \kappa)$ then

$$\begin{aligned} e \leq f &\iff ef = fe = e \iff (i, p_{\lambda i}^{-1}p_{\lambda j}p_{\kappa j}^{-1}, \kappa) = (j, p_{\kappa j}^{-1}p_{\kappa i}p_{\lambda i}^{-1}, \lambda) = (i, p_{\lambda i}^{-1}, \lambda) \\ &\iff i = j, \lambda = \kappa \iff e = f. \end{aligned}$$

In total we have shown that S is a completely 0-simple semigroup. □

Definition 2.7.10 *We call the semigroup constructed above the $I \times \Lambda$ Rees matrix semigroup over the 0-group G^0 with the regular sandwich matrix P and denote it by $\mathcal{M}^0[G; I, \Lambda; P]$.*

Lemma 2.7.11 For $(i, a, \lambda), (j, b, \kappa) \in \mathcal{M}^0[G; I, \Lambda; P] \setminus \{0\}$ we have the following equivalency:

$$(i, a, \lambda) \mathcal{L} (j, b, \kappa) \iff \lambda = \kappa.$$

Proof Suppose first that $(i, a, \lambda) \mathcal{L} (j, b, \kappa)$. Then either $(i, a, \lambda) = (j, b, \kappa)$, in which case there is nothing to prove, or there exists some element (k, c, μ) such that $(j, b, \kappa) = (k, c, \mu)(i, a, \lambda) = (k, cp_{\mu i}a, \lambda)$. Particularly we have $\kappa = \lambda$.

To show the converse suppose that $\kappa = \lambda$. The regularity of P ensures that we can find elements $\mu(i) \in \Lambda: p_{\mu(i)i} \neq 0$ and $\mu(j) \in \Lambda: p_{\mu(j)j} \neq 0$. Now a simple computation yields

$$(j, ba^{-1}p_{\mu(i)i}^{-1}, \mu(i))(i, a, \lambda) = (j, b, \lambda)$$

and

$$(i, ab^{-1}p_{\mu(j)j}^{-1}, \mu(j))(j, b, \lambda) = (i, a, \lambda).$$

Hence $(i, a, \lambda) \mathcal{L} (j, b, \kappa)$, as required. □

The construction given in 2.7.8 is of great importance since the converse of 2.7.9 also holds. Before we can prove this we need some preparation. In the following, S shall always denote a completely 0-simple semigroup.

Lemma 2.7.12 ([3], Lemma 3.2.4) Let $e \in S$ be a primitive idempotent. Then $\mathcal{R}_e = eS \setminus \{0\}$.

Proof The inclusion $\mathcal{R}_e \subseteq eS \setminus \{0\}$ is evident by the definition of the \mathcal{R} -relation and the fact that zero forms an \mathcal{R} -class of its own. To show the other inclusion let $a = es$ be a non-zero element in eS . By the 0-simplicity of S there exist u, v in S such that $e = uav$. Notice that

$$(eue)a(ve) = eu \underbrace{ea}_{=ees} ve = euave = e(uav)e = eee = e$$

so that we may use $\tilde{u} = eue$ and $\tilde{v} = ve$ instead of u and v . Clearly we have $e\tilde{u} = \tilde{u}e = \tilde{u}, \tilde{v}e = \tilde{v}$. Put $f = a\tilde{v}\tilde{u}$ and observe that it is an idempotent:

$$f^2 = a\tilde{v} \underbrace{\tilde{u}a\tilde{v}}_{=e} \tilde{u} = a\tilde{v}\tilde{u} = f.$$

The equations

$$ef = ea\tilde{v}\tilde{u} = a\tilde{v}\tilde{u} = f \text{ and } fe = a\tilde{v}\tilde{u}e = a\tilde{v}\tilde{u} = f$$

tell us that $f \leq e$. Moreover by

$$e = e^2 = \tilde{u}(a\tilde{v}\tilde{u})a\tilde{v} = \tilde{u}fa\tilde{v}$$

we see that $f \neq 0$ since we know by assumption that $e \neq 0$. We conclude that

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$e = f = a(\tilde{v}\tilde{u})$ and so $e \in aS$. Together with $a = es$ we have proved the desired $a\mathcal{R}e$.

□

The next lemma shows that the assertion is in fact also true for arbitrary a in $S \setminus \{0\}$.

Lemma 2.7.13 ([3], Lemma 3.2.5) *Let a be any element in $S \setminus \{0\}$. Then $\mathcal{R}_a = aS \setminus \{0\}$.*

Proof It is immediate that $\mathcal{R}_a \subseteq aS \setminus \{0\}$. Conversely, suppose that b is a non-zero element in aS . Fix some primitive element e . The 0-simplicity of S yields elements u and v such that $a = uev$. Combined with the assumption we get $b = uew$ for some $w \in S$. By virtue of the previous lemma we deduce $ev\mathcal{R}ew$ and therefore $\underbrace{uev}_a \mathcal{R} \underbrace{uew}_b$ since \mathcal{R} is apparently a left congruence. Hence we have also shown that $aS \setminus \{0\} \subseteq \mathcal{R}_a$.

□

Of course the right-handed nature of the previous lemmas is only apparent and we have the dual version as well:

Lemma 2.7.14 ([3], Lemma 3.2.6) *Let a be any element in $S \setminus \{0\}$. Then $\mathcal{L}_a = Sa \setminus \{0\}$.*

Proposition 2.7.15 ([3], Lemma 3.2.7) *S is regular and consists of just two \mathcal{D} -classes, namely $\{0\}$ and $S \setminus \{0\}$. If a, b are non-zero elements, then either $ab = 0$ or $ab \in \mathcal{R}_a \cap \mathcal{L}_b$ where the latter happens exactly when $\mathcal{L}_a \cap \mathcal{R}_b$ contains an idempotent.*

Proof Let $a, b \in S \setminus \{0\}$. By 2.7.4 then $SaS = S = SbS$. We claim that aSb contains a non-zero element. Suppose, by way of contradiction, that $aSb = \{0\}$. Then $S^2 = (SaS)(SbS) \subseteq S(aSb)S = \{0\}$. This is a contradiction to the definition of 0-simplicity. Hence there exists some element u such that $aub = c \neq 0$. By the previous two lemmas it follows that $c \in (aS \setminus \{0\}) \cap (Sb \setminus \{0\}) = \mathcal{R}_a \cap \mathcal{L}_b$. Thus $a\mathcal{D}b$ and so $D := S \setminus \{0\}$ is a \mathcal{D} -class. It is straightforward to confirm that $\{0\}$ forms a \mathcal{D} -class of its own. In particular, the \mathcal{J} - and the \mathcal{D} -relations are equal for completely 0-simple semigroups.

Next we show regularity. We know that D contains a primitive idempotent, so in particular, it contains a regular element. As a consequence of 2.6.2, every element of D is then regular and since 0 is also regular S is a regular semigroup. Let finally $a, b \neq 0$ such that $ab = c \neq 0$. Then $c \in aS \setminus \{0\} = \mathcal{R}_a$ and $c \in Sb \setminus \{0\} = \mathcal{L}_b$ by 2.7.13 and 2.7.14. Proposition 2.3.5 deals with the rest.

□

Now we have assembled all the ingredients necessary to shape S into the form of a Rees matrix semigroup.

Let $a, b \in H$, where $H \subseteq D$ is an \mathcal{H} -class and D is defined as in the proof of 2.7.15. Then either $ab = 0$ or $ab \in \mathcal{R}_a \cap \mathcal{L}_b = H$. Thus, in the latter case, $H^2 \cap H \neq \emptyset$. Using Green's theorem 2.3.6 we deduce that H is a group. Since the elements $a, b \in H$ were arbitrarily chosen we have $H^2 = \{0\}$, where H is not a group. We shall subsequently distinguish between group \mathcal{H} -classes and zero \mathcal{H} -classes. The non-zero \mathcal{L} - and \mathcal{R} -classes will play an important part in our endeavours to establish a Rees-like structure. We shall denote the set of non-zero \mathcal{R} -classes by I and the set of non-zero \mathcal{L} -classes by Λ but treat them as index sets rather than sets of classes. That is to say we will write the \mathcal{R} -classes as \mathcal{R}_i ($i \in I$) and the \mathcal{L} -classes as \mathcal{L}_λ ($\lambda \in \Lambda$) respectively by means of convenience. Accordingly, by $\mathcal{H}_{i\lambda}$ we mean $\mathcal{R}_i \cap \mathcal{L}_\lambda$.

By 2.6.3 and 2.3.7, for every \mathcal{L} -class \mathcal{L}_λ there exists some i in I such that $\mathcal{H}_{i\lambda}$ is a group and analogously for every \mathcal{R} -class \mathcal{R}_j there exists $\mu \in \Lambda$ such that $\mathcal{H}_{j\mu}$ is a group. Without loss of generality we suppose that there exists an element $I \ni 1 \in \Lambda$ such that \mathcal{H}_{11} is a group with identity element e . This group \mathcal{H}_{11} will feature as the group G in our construction of the Rees matrix semigroup.

For each $i \in I$ and each $\lambda \in \Lambda$ we choose in an arbitrary way an element $r_i \in \mathcal{H}_{i1}$ and an element $s_\lambda \in \mathcal{H}_{1\lambda}$. In view of $r_i \mathcal{L} e$ and $s_\lambda \mathcal{R} e$ we have $r_i e = r_i$ for all $i \in I$ and $e s_\lambda = s_\lambda$ for all $\lambda \in \Lambda$ since $e \in E_S$. Hence Green's Lemmas 2.3.4 and 2.3.3 imply that $x \mapsto r_i x$ maps \mathcal{H}_{11} onto \mathcal{H}_{i1} , that $y \mapsto y s_\lambda$ maps \mathcal{H}_{i1} onto $\mathcal{H}_{i\lambda}$ and that the mappings are one-one. Notice that

$$D = \bigcup \{ \mathcal{H}_{i\lambda} : i \in I, \lambda \in \Lambda \},$$

where D is again as in the proof of 2.7.15 and the union is disjoint. Together with the preceding considerations we have successfully established a bijection

$$\phi: (I \times \mathcal{H}_{11} \times \Lambda) \cup \{0\} \rightarrow S$$

given by

$$\phi(r_i, a, s_\lambda) = r_i a s_\lambda, \quad \phi(0) = 0.$$

It remains to set up a multiplication which makes $(I \times \mathcal{H}_{11} \times \Lambda) \cup \{0\}$ into a regular Rees matrix semigroup. The equation

$$(r_i a s_\lambda)(r_j b s_\mu) = r_i (a s_\lambda r_j b) s_\mu$$

encourages us to define the sandwich matrix P as $p_{\lambda i} = s_\lambda r_i$ ($i \in I, \lambda \in \Lambda$). At this point, 2.7.15 tells us that

$$p_{\lambda i} \in \mathcal{R}_{s_\lambda} \cap \mathcal{L}_{r_i} = \mathcal{H}_{11}$$

if and only if the \mathcal{H} -class

$$\mathcal{L}_{s_\lambda} \cap \mathcal{R}_{r_i} = \mathcal{L}_\lambda \cap \mathcal{R}_i = \mathcal{H}_{i\lambda}$$

contains an idempotent. But an \mathcal{H} -class contains an idempotent if and only if it is a group \mathcal{H} -class. If $\mathcal{H}_{i\lambda}$ is not a group however, then $p_{\lambda i} = 0$. Thus the

sandwich matrix P as defined above has indeed entries in \mathcal{H}_{11}^0 and it is moreover regular by virtue of 2.6.3. To prove that ϕ is an isomorphism from the Rees matrix semigroup $\mathcal{M}^0[\mathcal{H}_{11}; I, \Lambda; P]$ onto S is entirely straightforward. Eventually we have proved the prominent Rees theorem which will figure in one of our characterisations of Brandt semigroups.

Theorem 2.7.16 ([3], Theorem 3.2.3) *Let G^0 be a 0-group, let I, Λ be non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G^0 . Suppose that P is regular. Let $S = (I \times G \times \Lambda) \cup \{0\}$ and define a multiplication on S by (2.7.1). Then S is a completely 0-simple semigroup. Conversely, every completely 0-simple semigroup is isomorphic to one constructed in this way.*

To conclude this section we establish an isomorphism theorem for Rees matrix semigroups which tells us more about the relationship between a completely 0-simple semigroup and a representation of it as a Rees matrix semigroup. Indeed, our construction of such a representation given a completely 0-simple semigroup has been fairly arbitrary. Our choice of an \mathcal{H} -class group was arbitrary and so was our choice of elements r_i and s_λ . These choices amount in different sandwich matrices and so it is natural to ask when two regular Rees matrix semigroups are essentially the same, i.e. isomorphic. The next theorem answers this question. We will revert to it in section 4.1.

Theorem 2.7.17 ([3], Theorem 3.4.1) *Two regular Rees matrix semigroups*

$$S = \mathcal{M}^0[G; I, \Lambda; P] \text{ and } T = \mathcal{M}^0[K; J, M; Q]$$

are isomorphic if and only if there exist an isomorphism $\theta: G \rightarrow K$, bijections $\psi: I \rightarrow J$, $\chi: \Lambda \rightarrow M$ and elements $u_i (i \in I)$, $v_\lambda (\lambda \in \Lambda) \in K$ such that

$$\theta(p_{\lambda i}) = v_\lambda q_{\chi(\lambda)\psi(i)} u_i \tag{2.7.2}$$

whenever $p_{\lambda i} \neq 0$.

Proof Our needs only require us to prove the direct half. Suppose that we are given $\theta, \psi, \chi, \{u_i: i \in I\}, \{v_\lambda: \lambda \in \Lambda\}$ with the properties stated in the theorem. Consider the mapping

$$\phi: S \rightarrow T, \phi(i, a, \lambda) = (\psi(i), u_i \theta(a) v_\lambda, \chi(\lambda)).$$

Certainly, it is well-defined. The morphism property follows from the assumption as do injectivity and surjectivity. Hence ϕ is indeed the isomorphism we are looking for.

□

3 Basics of inverse semigroups

As pointed out in the introduction, the aim of this thesis is to investigate four special classes of inverse semigroups. Before we can do that however, we need to prepare ourselves by covering the fundamental concepts of general inverse semigroups. These include the natural partial ordering which can be defined for any inverse semigroup and the Wagner-Preston representation. Green's relations admit an easier characterisation for inverse semigroups. We also deal with the compatibility relation as well as the minimum group congruence and the maximum idempotent-separating congruence.

Definition 3.0.18 *Let S be a semigroup and let a in S . We call an element a' an inverse of a if $aa'a = a$ and $a'aa' = a'$.*

In an arbitrary semigroup there might exist no or more than one inverse for an element. Consider $(\mathbb{N}, +)$ as an example for the former case. For the latter, consider a left-zero semigroup, i.e. a semigroup such that $\forall a, b \in S: ab = a$. We have however

Remark 3.0.19 *Every regular element of a semigroup possesses an inverse.*

Proof Let a denote the regular element and let x be such that $axa = a$. Consider the element axa . Then

$$a(axa)a = (axa)xa = axa = a$$

and

$$(axa)a(axa) = x(axa)ax = x(axa)x = axa.$$

Hence axa is an inverse of a .

□

Definition 3.0.20 *We call a semigroup S an inverse semigroup if every element of S has a unique inverse.*

Examples of inverse semigroups will follow soon. We first give alternative characterisations of inverse semigroups.

Theorem 3.0.21 *([3], Theorem 5.1.1) Let S be a semigroup. Then the following statements are equivalent:*

- (i) S is an inverse semigroup.

3 Basics of inverse semigroups

(ii) For every element a in S there is a uniquely determined element a^{-1} such that $(a^{-1})^{-1} = a$ and $aa^{-1}a = a$ and the following equation holds for all $x, y \in S$:

$$xx^{-1}yy^{-1} = yy^{-1}xx^{-1}. \quad (3.0.1)$$

(iii) S is regular and its idempotents commute.

(iv) Every \mathcal{L} -class and every \mathcal{R} -class contains exactly one idempotent.

Proof

(i) \Rightarrow (ii): Denote the unique inverse of x by x^{-1} . This element x^{-1} obviously satisfies the first conditions in (ii). We merely have to prove that (3.0.1) holds for all $x, y \in S$. If we write e for xx^{-1} , f for yy^{-1} and define z to be the unique inverse of ef we get

$$(ef)(fze)(ef) = ef^2ze^2f = efzef = ef$$

and

$$(fze)(ef)(fze) = f(zeffz)e = fze,$$

so that fze is an inverse of ef as well. By uniqueness it follows that $z = fze$. In particular this yields that z is an idempotent since

$$z^2 = (fze)(fze) = f(zeffz)e = fze = z.$$

But then $z = ef$ since every idempotent is self-inverse. Furthermore, fe is also an idempotent by a similar argument. Since we now have

$$(ef)(fe)(ef) = (ef)(ef) = ef \text{ and } (fe)(ef)(fe) = (fe)(fe) = fe,$$

both fe and ef are inverses of ef . Hence, by uniqueness, $ef = fe$, i.e.

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

(ii) \Rightarrow (iii): Regularity follows from the assumption. We show that every idempotent in S can be expressed in the form xx^{-1} for some $x \in S$. Accordingly, let $e \in E_S$. By assumption there is an element e^{-1} such that $ee^{-1}e = e$ and $(e^{-1})^{-1} = e$. Hence

$$\begin{aligned} e^{-1} &= e^{-1}(e^{-1})^{-1}e^{-1} = e^{-1}ee^{-1} = e^{-1}(ee)e^{-1} = (e^{-1}e)(ee^{-1}) = \\ &= (e^{-1}(e^{-1})^{-1})(ee^{-1}) = (ee^{-1})(e^{-1}(e^{-1})^{-1}) = (ee^{-1})(e^{-1}e). \end{aligned}$$

It then follows that

$$e = ee^{-1}e = e(ee^{-1})(e^{-1}e)e = (e^2e^{-1})(e^{-1}e^2) = (ee^{-1})(e^{-1}e) = e^{-1}$$

and therefore $e = ee = ee^{-1}$. The same is true for any $f \in E_S$ and so we have

$$ef = (ee^{-1})(ff^{-1}) = (ff^{-1})(ee^{-1}) = fe$$

by assumption.

(iii) \Rightarrow (iv): Suppose that S is a regular semigroup in which idempotents commute and let L denote an \mathcal{L} -class. Now L contains at least one idempotent, since for an arbitrary $a \in L$ regularity gives us an element x such that $a(xa) = a$. Apparently $xa \mathcal{L} a$ and $xa \in E_S$. Next we show that there cannot be more than one idempotent in a single \mathcal{L} -class. Assume that e, f are idempotents, both lying in L . Because e and f are \mathcal{L} -equivalent, e can be written as af for some $a \in S$ and hence $ef = (af)f = a(ff) = af = e$. Similarly we get $fe = f$ and since by assumption idempotents commute this yields $e = f$. Thus each \mathcal{L} -class contains exactly one idempotent and the same goes for \mathcal{R} -classes as can be seen by a dual argument.

(iv) \Rightarrow (i): We start by showing that every element in S has at least one inverse. \mathcal{D} -classes consist of \mathcal{L} -classes. Every such \mathcal{L} -class contains an idempotent, i.e. a regular element. Hence, by 2.6.2, all \mathcal{D} -classes are regular and so also is S . But then there exist inverses for all elements due to 3.0.19.

Next we show that for every element there is at most one inverse. Let \hat{x} and \tilde{x} denote inverses of x . Clearly $x\hat{x}$ and $x\tilde{x}$ are idempotents \mathcal{R} -equivalent to x and thus $x\hat{x} = x\tilde{x}$ by assumption. Just as easily we can show $\hat{x}x = \tilde{x}x$ using the fact that \mathcal{L} -classes also contain exactly one idempotent. Now

$$\hat{x} = \hat{x}x\hat{x} = \hat{x}x\tilde{x} = \tilde{x}x\tilde{x} = \tilde{x}$$

and so every element in S has a unique inverse. □

Trying to decide whether a given semigroup is inverse the best way usually is to show (iii).

The idempotents of an inverse semigroup are of great importance. The following corollary gives a first hint why.

Corollary 3.0.22 *Let E_S denote the set of idempotents of S . Then E_S is a semilattice with respect to the operation inherited from S .*

Proof Clearly E_S is closed under multiplication since idempotents commute in inverse semigroups as we have seen in 3.0.21. Hence (E_S, \cdot) is a commutative idempotent semigroup and thus a semilattice by 2.1.4. □

The following proposition lists some basic properties of inverse semigroups.

Proposition 3.0.23 ([3], Proposition 5.1.2) *Let S be an inverse semigroup. Then*

$$(i) \quad \forall a \in S: aa^{-1}, a^{-1}a \in E_S.$$

$$(ii) \quad e \in E_S \implies e = e^{-1}.$$

$$(iii) \quad \forall a, b \in S: (ab)^{-1} = b^{-1}a^{-1}.$$

Proof

(i): $(aa^{-1})(aa^{-1}) = (aa^{-1}a)a^{-1} = aa^{-1}$ and similarly for $a^{-1}a$.

(ii): See the proof of 3.0.21.

(iii): Notice that by (i) both bb^{-1} and $a^{-1}a$ are idempotents and recall that idempotents commute in inverse semigroups. Thus

$$(ab)(b^{-1}a^{-1})(ab) = a(bb^{-1})(a^{-1}a)b = a(a^{-1}a)(bb^{-1})b = (aa^{-1}a)(bb^{-1}b) = ab$$

and

$$\begin{aligned} (b^{-1}a^{-1})(ab)(b^{-1}a^{-1}) &= b^{-1}(a^{-1}a)(bb^{-1})a^{-1} = b^{-1}(bb^{-1})(a^{-1}a)a^{-1} = \\ &= (b^{-1}bb^{-1})(a^{-1}aa^{-1}) = b^{-1}a^{-1} \end{aligned}$$

We have shown that $b^{-1}a^{-1}$ is the unique inverse of ab .

□

By induction property (iii) generalises in an obvious way:

Corollary 3.0.24 ([3], Corollary 5.1.3) *Let S be an inverse semigroup and let $a_1, \dots, a_n \in S$. Then*

$$(a_1 \cdot \dots \cdot a_n)^{-1} = a_n^{-1} \cdot \dots \cdot a_1^{-1}$$

In particular we have $(a^n)^{-1} = (a^{-1})^n$ for all a in S and so the notation a^{-n} can be used unambiguously. It is however important to realise that the index law $a^p a^q = a^{p+q}$ cannot be assumed for all p, q in \mathbb{Z} .

The next proposition states that a morphism on an inverse semigroup is automatically an ‘inverse semigroup morphism’. The preceding preparatory lemma remains true for just a regular semigroup S and is sometimes referred to as Lallement’s Lemma.

Lemma 3.0.25¹ *Let S be an inverse semigroup, let T be a semigroup and let ϕ be a morphism from S onto T . If f is an idempotent in T then there exists an idempotent e in S such that $\phi(e) = f$.*

Proof Since ϕ is surjective, f may be expressed as $\phi(s)$ for some (not necessarily idempotent) s in S . By assumption

$$\phi(s) = f = f^2 = \phi(s)^2 = \phi(s^2).$$

Now S is inverse and so there exists x in S such that $s^2 x s^2 = s^2$ and $x s^2 x = x$. Clearly then $sxs \in E_S$ and using the above,

$$\phi(sxs) = \phi(s)\phi(x)\phi(s) = \phi(s^2)\phi(x)\phi(s^2) = \phi(s^2 x s^2) = \phi(s^2) = \phi(s) = f.$$

□

¹Compare to ([3], Lemma 2.4.4)

Proposition 3.0.26 ([3], Theorem 5.1.4) *Let S be an inverse semigroup, let T be a semigroup and let ϕ be a morphism from S onto T . Then T is an inverse semigroup and we have*

$$\forall s \in S: \phi(s)^{-1} = \phi(s^{-1}). \quad (3.0.2)$$

Proof Every element t in T is (not necessarily uniquely) expressible as $\phi(s)$ for some s in S . Then

$$\phi(s)\phi(s^{-1})\phi(s) = \phi(ss^{-1}s) = \phi(s)$$

and

$$\phi(s^{-1})\phi(s)\phi(s^{-1}) = \phi(s^{-1}ss^{-1}) = \phi(s^{-1}).$$

Hence $\phi(s^{-1})$ is an inverse of $t = \phi(s)$ and T is thus regular. Now suppose that g and h are idempotents of T . By virtue of 3.0.25 there exist idempotents $e, f \in S$ such that $\phi(e) = g$, $\phi(f) = h$. Since in the inverse semigroup S idempotents commute, by

$$gh = \phi(e)\phi(f) = \phi(e f) = \phi(f e) = \phi(f)\phi(e) = hg,$$

they do as well in T . Hence T is an inverse semigroup. □

Corollary 3.0.27 *Let ρ be a congruence on an inverse semigroup S . Then S/ρ is also inverse and $\rho(s^{-1}) = \rho(s)^{-1}$.*

Proof Consider the natural morphism \natural which maps S onto S/ρ by the rule

$$s \mapsto \rho(s).$$

Now it is clear from the previous proposition that the factor semigroup is inverse. In addition to that (3.0.2) translates into $\rho(s^{-1}) = \rho(s)^{-1}$. □

The next few lemmas supply further properties of inverse semigroups that are easy to prove.

Lemma 3.0.28 ([3], Lemma 5.1.6) *Let S be an inverse semigroup. Let $e, f \in E_S, a \in S$. Then*

$$(i) \quad eS \cap fS = efS.$$

$$(ii) \quad aS = aa^{-1}S.$$

Proof

(i): Suppose that $a \in eS \cap fS$. Then there are elements s, t in S such that $es = a = ft$. Now $a = es = e(es) = ea = eft$ and hence $a \in efS$. Conversely, suppose that $a \in efS$. Then $a \in efS \subseteq eS$ and $a \in efS = feS \subseteq fS$ since idempotents commute.

(ii): Clearly we have $aa^{-1}S \subseteq aS$. The reverse inclusion is just as easy:

$$aS = aa^{-1}aS \subseteq aa^{-1}S.$$

□

We could have also proved the analogue for principal left ideals:

$$Se \cap Sf = Se f \text{ and } Sa = Sa^{-1}a.$$

Fairly often we want an idempotent to ‘switch sides’ with an element. The next lemma tells us how to do it.

Lemma 3.0.29 ([5], Lemma 1.4.2) *Let S be an inverse semigroup. Consider $x = ae$ with $e \in E_S$ and $a \in S$. Then $x = fa$ for some idempotent f .*

Proof It is straightforward to check that $f = aea^{-1}$ is an idempotent. Hence

$$ae = a \underbrace{a^{-1}a}_{\in E_S} e = (aea^{-1})a = fa.$$

□

We are already familiar with Green’s relations from section 2.3. They are easier to handle for inverse semigroups. First of all, principal ideals are of easier form here. Exemplarily consider the principal left ideal \mathcal{L}_a generated by a . We found earlier that it equals S^1a . In an inverse semigroup $(aa^{-1})a = a$ and thus $Sa = S^1a = \mathcal{L}_a$. Analogously, we do not need S^1 for right or two-sided ideals either.

Proposition 3.0.30 ([3], Proposition 5.1.2) *Let S be an inverse semigroup. Then*

$$(i) \ a \mathcal{L} b \iff a^{-1}a = b^{-1}b.$$

$$(ii) \ a \mathcal{R} b \iff aa^{-1} = bb^{-1}.$$

$$(iii) \ \text{For idempotents } e, f \text{ we have } e \mathcal{D} f \iff \exists a \in S: aa^{-1} = e \text{ and } a^{-1}a = f.$$

Proof

(i): Suppose that $a \mathcal{L} b$, i.e. $Sa = Sb$. Then $a = xb$ and $b = ya$ for some $x, y \in S$. Now $a^{-1}a = (xb)^{-1}(xb) = b^{-1}(x^{-1}x)b \leq b^{-1}b$. We may show analogously that $b^{-1}b \leq a^{-1}a$ which establishes equality.

Conversely, suppose that $a^{-1}a = b^{-1}b$. Then $a = aa^{-1}a = ab^{-1}b$ and thus $Sa = (Sab^{-1})b \subseteq Sb$. Similarly, $Sb \subseteq Sa$ and hence $a \mathcal{L} b$.

The proof of (ii) is similar.

(iii): Suppose that $e, f \in E_S$. Using (i) and (ii) we have

$$e \mathcal{D} f \iff \exists a: e \mathcal{R} a, a \mathcal{L} f \iff e = ee^{-1} = aa^{-1}, f = f^{-1}f = a^{-1}a.$$

□

3.1 Important examples

Of course groups are inverse semigroups having only one idempotent, namely the identity element. Groups are certainly important but it is hardly surprising that they appear in our list of inverse semigroups. Less trivial is that inverse semigroups having only one idempotent are in fact groups. This is why we referred to inverse semigroups as generalised groups in the introduction.

Proposition 3.1.1 ([5], Proposition 1.4.4) *Groups are precisely the inverse semigroups with exactly one idempotent.*

Proof Obviously a group is an inverse semigroup which contains exactly one idempotent.

Conversely, suppose that S is an inverse semigroup containing exactly one idempotent e . Then for every s in S we have

$$ss^{-1} = e = s^{-1}s \text{ and } es = (ss^{-1})s = s, se = s(s^{-1}s) = s.$$

Hence S is a group with identity e .

□

Semilattices are also inverse semigroups. They are commutative and consist entirely of idempotents. Moreover, they are regular since $aaa = a$ for all a . By 3.0.21 we deduce that semilattices are inverse semigroups. The next examples are more interesting and deserve to be dealt with in subsections.

The symmetric inverse monoid

Let X be a set. A partial bijection f of X is a bijection from some $A \subseteq X$ onto some $B \subseteq X$. The set A is called the domain of f and denoted by $\text{dom}(f)$, whereas B is called the image of f and denoted by $\text{im}(f)$.

The composite $g \circ f$ of f and g is defined as the partial bijection given by

$$\begin{aligned} g \circ f: f^{-1}(\text{im}(f) \cap \text{dom}(g)) &\rightarrow g(\text{im}(f) \cap \text{dom}(g)) \\ x &\mapsto g(f(x)). \end{aligned}$$

This means that $g \circ f$ is the familiar functional composition whenever defined, but we are not as restrictive as usual. This way we can define a composite for each pair of partial bijections regardless of the respective domains and images. It is usually the case that the composite is not defined on the whole domain of the first partial bijection. It might even be the case that we get in effect the ‘empty mapping’, i.e. the ‘mapping’ which maps the empty set on the empty set.

Proposition 3.1.2 ([3], Theorem 5.1.5) *For a set X let I_X denote the set of all partial bijections of X . Then I_X is an inverse monoid with respect to the composite \circ as defined above.*

Proof We shall show associativity first. It is convenient in this case to write the mappings as relations. For example we write $(x, y) \in f$ instead of $f(x) = y$. With this notation we have

$$\begin{aligned} (x, y) \in (h \circ g) \circ f &\iff \exists z \in X : (x, z) \in f, (z, y) \in h \circ g \iff \\ &\exists z, u \in X : (x, z) \in f, (z, u) \in g, (u, y) \in h \iff \\ \exists u \in X : (x, u) \in g \circ f, (u, y) \in h &\iff (x, y) \in h \circ (g \circ f). \end{aligned}$$

Hence \circ is associative and thus I_X is a semigroup.

Next we show that I_X is inverse. Each f in I_X is a bijection from $\text{dom}(f)$ onto $\text{im}(f)$ and so there is an inverse map f^{-1} , also an element of I_X , such that

$$\begin{aligned} \text{dom}(f^{-1}) &= \text{im}(f), \text{im}(f^{-1}) = \text{dom}(f), \\ f^{-1} \circ f &= \text{id}_{\text{dom}(f)}, f \circ f^{-1} = \text{id}_{\text{im}(f)} \end{aligned}$$

Certainly then $f \circ f^{-1} \circ f = f$ and $f^{-1} \circ f \circ f^{-1} = f^{-1}$ and thus I_X is regular. If we are able to verify that idempotents commute, then the proof is finished by virtue of 3.0.21. Of course we must first identify the idempotents. If f is an idempotent then

$$f^{-1}(\text{dom}(f) \cap \text{im}(f)) = \text{dom}(f^2) = \text{dom}(f) = f^{-1}(\text{im}(f)).$$

Since f^{-1} is one-one it follows that $\text{dom}(f) \cap \text{im}(f) = \text{im}(f)$. In particular we have $\text{im}(f) \subseteq \text{dom}(f)$. Similarly we can show

$$f(\text{dom}(f) \cap \text{im}(f)) = \text{im}(f^2) = \text{im}(f) = f(\text{dom}(f)),$$

which yields $\text{dom}(f) \subseteq \text{im}(f)$. Hence $\text{dom}(f) = \text{im}(f) = A \subseteq X$ (say) and $f(f(x)) = f^2(x) = f(x)$ for all x in A by assumption that f is an idempotent. Since f is one-one this implies $f(x) = x$ for all x in A and thus $f = \text{id}_A$. It is obvious that conversely, id_A is an idempotent for every subset A of X . It is easy to see that $\text{id}_B \circ \text{id}_A = \text{id}_{A \cap B}$ for all $A, B \subseteq X$. We deduce that idempotents commute and therefore I_X is an inverse semigroup. It is also a monoid with identity id_X . Hence I_X is an inverse monoid, as required.

□

Definition 3.1.3 *The inverse semigroup I_X is called the symmetric inverse monoid.*

As said at the beginning of this chapter, inverse semigroups were first investigated in connection with partial bijections, i.e. mathematicians like Wagner and Preston studied the symmetric inverse monoid I_X . The significance of this particular semigroup rests with an important embedding theorem for inverse semigroups, which we are going to prove in 3.3.1. It is called the ‘Wagner-Preston representation theorem’ in honour of the two previous mentioned.

The bicyclic semigroup

This subsection revolves around the bicyclic semigroup that will appear several times in this thesis. For example it serves us as an example for an E-unitary inverse semigroup. In a wider scale the bicyclic semigroup has important applications in ring theory, formal languages and λ -calculus. We will not cover this here and refer the interested reader to [5].

Proposition 3.1.4 *Let B be the set $\mathbb{N} \times \mathbb{N}$ equipped with a binary operation given by*

$$(m, n)(p, q) := (m - n + \max(n, p), q - p + \max(n, p)). \quad (3.1.1)$$

Then B forms an inverse monoid.

Proof We ought to show that the operation defined on B is associative but will postpone the proof of this for the moment and refer to 5.1.3 instead. The associativity will come naturally, whereas here the verification of it requires not particular difficult but tedious work. Let us thus assume we already know that B is a semigroup. We claim that it is also inverse. Indeed, it is certainly regular since for every $a, b \in \mathbb{N}$ one easily finds $(a, b)(b, a)(a, b) = (a, b)$. It is straightforward to check that the idempotents are precisely the elements of the form (a, a) . We have yet to show that idempotents commute. In view of that let $(a, a), (b, b) \in B$ and suppose without loss of generality that $a \leq b$. Then

$$(a, a)(b, b) = (b, b) = (b, b)(a, a).$$

Finally notice that $(0, 0)(a, b) = (a, b) = (a, b)(0, 0)$ for each element $(a, b) \in B$. Thus B is an inverse monoid with identity $(0, 0)$.

□

Definition 3.1.5 *The semigroup B appearing above is called the bicyclic semigroup.*

It is usually referred to as simply a semigroup although we have seen that it is in fact a monoid.

Clifford semigroups

Clifford semigroups can be defined in various ways. A taste of this is given in 3.1.11. We choose an approach that makes it easy to recognise Clifford semigroups as particular inverse semigroups. We will come across Clifford semigroups once more in sections 5.3 and 6.1.

Definition 3.1.6 *An element a of a semigroup S is called central if $ax = xa$ for all x in S .*

Definition 3.1.7 We shall call a semigroup a Clifford semigroup if it is regular and its idempotents are central.

Obviously, a Clifford semigroup is inverse, since in particular its idempotents commute. We will shortly see that a Clifford semigroup is always a semilattice of groups². Therefore we need the following construction:

Construction 3.1.8 Let E be a semilattice and let $\{G_e : e \in E\}$ be a family of disjoint groups indexed by the elements of E . We denote the identity element of G_e by 1_e . For each pair $e, f \in E$ such that $e \geq f$ let $\phi_{e,f} : G_e \rightarrow G_f$ be a group morphism such that the following two axioms hold:

(C1) $\phi_{e,e}$ is the identity morphism on G_e

(C2) If $e \geq f \geq g$ then $\phi_{f,g} \circ \phi_{e,f} = \phi_{e,g}$

We endow the set $\bigcup_{e \in E} G_e$ with a product \otimes defined by

$$x \otimes y = \phi_{e,ef}(x)\phi_{f,ef}(y) \quad (x \in G_e, y \in G_f). \quad (3.1.2)$$

Proposition 3.1.9 ([5], Proposition 5.2.11)

With the above notation, $(\bigcup_{e \in E} G_e, \otimes)$ is a Clifford semigroup.

Proof The multiplication \otimes is well-defined, since $ef \leq e, f$. Let $x \in G_e, y \in G_f$ and $z \in G_g$ and put $i = efg$. By (3.1.2) we have

$$\begin{aligned} (x \otimes y) \otimes z &= \phi_{ef,i}(\phi_{e,ef}(x)\phi_{f,ef}(y))\phi_{g,i}(z) = \\ \phi_{ef,i}(\phi_{e,ef}(x))\phi_{ef,i}(\phi_{f,ef}(y))\phi_{g,i}(z) &\stackrel{(C2)}{=} \phi_{e,i}(x)\phi_{f,i}(y)\phi_{g,i}(z). \end{aligned}$$

A similar argument shows that $x \otimes (y \otimes z)$ also reduces to the right-hand side of the above equation. Hence \otimes is associative and (S, \otimes) is a semigroup.

Now let x, y denote elements lying in the same group G_e . Then

$$x \otimes y = \phi_{e,ee}(x)\phi_{e,ee}(y) = \phi_{e,e}(x)\phi_{e,e}(y) = xy$$

due to (C1). Consequently the inverse x^{-1} of x in G_e is also an inverse with respect to \otimes since we are only calculating within the group G_e :

$$x = x \otimes x^{-1} \otimes x, \quad x^{-1} = x^{-1} \otimes x \otimes x^{-1}.$$

In particular we have demonstrated that S is a regular semigroup. Next we consider the idempotents of S . These are just the identity elements of the groups in which they are contained. Finally, let 1_e be any idempotent and $x \in G_f$ an arbitrary element of S . Then $1_e \otimes x = \phi_{e,ef}(1_e)\phi_{f,ef}(x)$. The right-hand side is equal to $\phi_{f,ef}(x)$ because a group morphism maps the identity element of one group to the identity element of the other. We have in effect:

$$1_e \otimes x = \phi_{f,ef}(x) = x \otimes 1_e.$$

²For a definition see 2.1.7

Hence, idempotents are central and together with the regularity we have proved that S is indeed a Clifford semigroup.

□

Definition 3.1.10 *The semigroup constructed above is called a strong semilattice of groups and is denoted by $S(E; G_e; \phi_{e,f})$.*

The notion ‘strong semilattice of groups’ indicates that $S(E; G_e; \phi_{e,f})$ is a particular semilattice of groups. This is indeed the case. The next theorem also demonstrates that the converse of 3.1.9 is true as well, i.e. each Clifford semigroup can be interpreted as a strong semilattice of groups.

Theorem 3.1.11 ³ *For an inverse semigroup S the following statements are equivalent:*

- (i) S is a Clifford semigroup.
- (ii) Every \mathcal{H} -class is a group.
- (iii) S is a semilattice of groups.
- (iv) S is a strong semilattice of groups.

Proof

(i) \Rightarrow (ii): Let H be an \mathcal{H} -class and let $a \in H$. We claim that $a\mathcal{H}a^2$. By 3.0.30,

$$aa^{-1} = aa^{-1} \underbrace{aa^{-1}}_{\in E_S} = a(aa^{-1})a^{-1} = a^2a^{-2} = a^2(a^2)^{-1}$$

shows that $a\mathcal{R}a^2$ and the verification of $a\mathcal{L}a^2$ is similar. Hence $H \cap H^2 \neq \emptyset$ and H is therefore a group due to 2.3.6.

(ii) \Rightarrow (iii): We are going to show that \mathcal{H} is a congruence. To this end let $a\mathcal{H}b$ and $c\mathcal{H}d$. Notice that $hh^{-1} = h^{-1}h$ for every $h \in S$, since we are calculating in the group \mathcal{H}_h . Also notice that $a^{-1}a = b^{-1}b$ and $e := cc^{-1} = dd^{-1}$ because a, b and c, d do lie in the same respective groups by assumption. Hence

$$\begin{aligned} (ac)(ac)^{-1} &= acc^{-1}a^{-1} = aea^{-1} = aeea^{-1} = (ae)(ae)^{-1} = (ae)^{-1}(ae) = \\ &ea^{-1}ae = eea^{-1}a = ea^{-1}a = eb^{-1}b = eeb^{-1}b = eb^{-1}be = \\ &(be)^{-1}(be) = (be)(be)^{-1} = beeb^{-1} = beb^{-1} = bdd^{-1}b^{-1} = (bd)(bd)^{-1} \end{aligned} \tag{3.1.3}$$

and so $ac\mathcal{R}bd$. The verification of $ac\mathcal{L}bd$ is similar and \mathcal{H} is thus a congruence in which the congruence classes are groups. It remains to show that the factor semigroup is a semilattice. First it is idempotent since we have $a\mathcal{H}a^2$ for every $a \in S$ because \mathcal{H}_a is a group by assumption. It is also commutative. To see this

³Compare to ([5], Theorem 5.2.12) and ([3], Theorem 4.2.1)

let $a, b \in S$. Certainly bb^{-1} and aa^{-1} are idempotents and using twice the same trick as in (3.1.3) we get

$$\begin{aligned} (ab)(ab)^{-1} &= a(bb^{-1})a^{-1} = (bb^{-1})(a^{-1}a) = (b^{-1}b)(aa^{-1}) = \\ &= (aa^{-1})(b^{-1}b) = b(aa^{-1})b^{-1} = (ba)(ba)^{-1}. \end{aligned}$$

Hence $ab \mathcal{R} ba$. In a similar way we obtain $ab \mathcal{L} ba$ and so $ab \mathcal{H} ba$, i.e. $\mathcal{H}_{ab} = \mathcal{H}_{ba}$. **(iii) \Rightarrow (iv)**: By assumption there exists a congruence ρ such that S/ρ is a semilattice and such that the congruence classes are groups. For $\rho(a) \geq \rho(b)$, i.e. $\rho(a)\rho(b) = \rho(b) = \rho(b)\rho(a)$, we define a mapping

$$\begin{aligned} \phi_{a,b}: \rho(a) &\rightarrow \rho(b) \\ x &\mapsto 1_b x, \end{aligned}$$

where 1_b denotes the identity element of $\rho(b)$. It is easy to see that this is a morphism, since

$$\phi_{a,b}(x)\phi_{a,b}(y) = \underbrace{1_b x}_{\in \rho(b)} 1_b y = 1_b xy = \phi_{a,b}(xy) \quad (x, y \in \rho(a)),$$

where $xy \in \rho(a)$ because S/ρ is idempotent. Property (C1) in 3.1.8 is immediate. Suppose now that $\rho(a) \geq \rho(b) \geq \rho(c)$. Then, for every $x \in \rho(a)$,

$$(\phi_{b,c} \circ \phi_{a,b})(x) = 1_c 1_b x = \phi_{b,c}(1_b x) = 1_c x = \phi_{a,c}(x).$$

This is (C2). We have used here that a group morphism maps the identity of one group to the identity of the other.

We are going to show that $S \cong S(S/\rho; \rho(a); \phi_{a,b})$. The only thing left to do is to verify that

$$\forall a, b \in S: ab = \phi_{a,ab}(a)\phi_{b,ab}(b).$$

The following calculation does the job.

$$\phi_{a,ab}(a)\phi_{b,ab}(b) = \underbrace{1_{ab} a}_{\in \rho(ab)} 1_{ab} b = 1_{ab} \underbrace{ab}_{\in \rho(ab)} = ab.$$

(iv) \Rightarrow (i): See 3.1.9. □

3.2 The natural partial order relation

In this section we define a far-reaching concept for inverse semigroups: the natural partial order relation. As mentioned before the idempotents of an inverse semigroup are of great importance. We have already seen that they form a semilattice in 3.0.22. From 2.1.4 we know that the ‘natural’ order relation for them is (2.1.2). To spell it out:

$$e \leq f \iff e = ef.$$

Although it is not yet clear this motivates

Definition 3.2.1 Let S be an inverse semigroup. The natural partial order \leq on S is defined as follows:

$$s \leq t \iff s = te$$

for some idempotent e .

Before showing that this does indeed define a partial order on S we prove a useful lemma. It provides alternative definitions for the natural partial order and will be used extensively in further discussions.

Lemma 3.2.2 ([5], Lemma 1.4.6) Let S be an inverse semigroup. Then the following statements are equivalent:

(i) $s \leq t$.

(ii) $s = ft$ for some idempotent f .

(iii) $s^{-1} \leq t^{-1}$.

(iv) $s = ss^{-1}t$.

(v) $s = ts^{-1}s$.

Proof

(i) \Rightarrow (ii): This follows directly from 3.0.29.

(ii) \Rightarrow (iii): By assumption $s = ft$ with $f \in E_S$. Hence $s^{-1} = (ft)^{-1} = t^{-1}f$, that is $s^{-1} \leq t^{-1}$.

(iii) \Rightarrow (iv): By assumption we have $s^{-1} = t^{-1}e$ for some idempotent e . Taking inverses yields $s = et$. Then $es = eet = et = s$ and thus

$$ss^{-1}t = (es)s^{-1}t = e(ss^{-1})t = ss^{-1}(et) = ss^{-1}s = s.$$

(iv) \Rightarrow (v): Let $s = ss^{-1}t$. By using 3.0.29 we get $s = te$ for some idempotent e . Hence

$$ts^{-1}s = t(te)^{-1}(te) = tet^{-1}te = tt^{-1}tee = te = s.$$

(v) \Rightarrow (i): This is immediate since $s^{-1}s$ is an idempotent.

□

Note that by (ii), the side on which the idempotent appears in the definition is irrelevant. Notice also that by (iii), taking inverses does not reverse the relation. We are now ready to prove

Proposition 3.2.3 ([5], Proposition 1.4.7) Let S be an inverse semigroup. The relation \leq defines a partial order on S .

Proof Since $s = s(s^{-1}s)$ for every s in S the relation is reflexive. Let $s \leq t$ and $t \leq s$. Then $s = ts^{-1}s$ and $t = st^{-1}t$. The following chain of equalities shows that \leq is antisymmetric:

$$s = ts^{-1}s = (st^{-1}t)s^{-1}s = ss^{-1}st^{-1}t = st^{-1}t = t.$$

Finally, suppose that $s \leq t$ and $t \leq u$. Then $s = te$ and $t = uf$ for some

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idempotents e and f . Hence $s = te = (uf)e = u(fe)$ and thus $s \leq u$ and \leq is transitive since fe is an idempotent. □

The next proposition is a simple consequence of the definition of the natural partial order.

Proposition 3.2.4 ([5], Proposition 1.4.7) *Let S be an inverse semigroup and let \leq denote the natural partial order.*

(i) *For idempotents $e, f \in S$ we have that $e \leq f$ if and only if $e = ef = fe$.*

(ii) *If $s \leq t$ and $u \leq v$ then $su \leq tv$.*

(iii) *If $s \leq t$ then $ss^{-1} \leq tt^{-1}$ and $s^{-1}s \leq t^{-1}t$.*

Proof

(i): Suppose that $e \leq f$. Then $e = gf = fh$ for some idempotents g, h . Then

$$ef = (gf)f = gf = e, \quad fe = f(fh) = fh = e.$$

The converse is immediate.

(ii): Suppose that $s = te$ and $u = vf$, where $e, f \in E_S$. Then

$$su = tev f = tevv^{-1}vf = tv \underbrace{v^{-1}ev}_{\in E_S} f$$

and hence $su \leq tv$, as required.

(iii): This is a simple consequence of (ii) and 3.2.2(iii). □

Observe that property (ii) tells us that \leq is compatible with multiplication. This is a strong indication that our definition of \leq is a ‘good’ one.

Remark 3.2.5 *The natural order relation is called ‘natural’ because it extends the natural order relation (2.1.2) defined only on its semilattice of idempotents. The term ‘natural’ is justified there by 2.1.4.*

Proposition 3.2.6 ([5], Proposition 1.4.10) *Let S be an inverse semigroup. Then the natural partial order is the equality relation if and only if S is a group.*

Proof Suppose that the natural order is the equality relation and let e, f be idempotents. Then $ef \leq e, f$. But the natural order is the equality relation and so $e = ef = f$. We have shown that S is an inverse semigroup having exactly one idempotent. Hence S is a group by 3.1.1. The converse is clear. □

3.3 The Wagner-Preston representation theorem

In section 3.1 we encountered the symmetric inverse monoid I_X . Now we are going to prove a classic result known as the Wagner-Preston representation theorem. It states that every inverse semigroup S can be interpreted as a subsemigroup of some symmetric inverse monoid. The Wagner-Preston representation theorem should be seen as an analogue of Cayley's theorem in group theory. A similar theorem states that every semigroup is isomorphic to a subsemigroup of a full transformation semigroup.

Theorem 3.3.1 ([5], Theorem 1.5.1) *Let S be an inverse semigroup. Then there exists an injective morphism $\theta: S \rightarrow I_S$ such that*

$$a \leq b \iff \theta(a) \text{ is a restriction of } \theta(b). \quad (3.3.1)$$

Proof For each element $a \in S$ define a mapping

$$\begin{aligned} \theta_a: a^{-1}aS &\rightarrow aa^{-1}S \\ x &\mapsto ax. \end{aligned}$$

This map is well-defined since for $x = a^{-1}ax' \in a^{-1}aS$ we have $ax = aa^{-1}(ax') \in aa^{-1}S$. Consider $\theta_{a^{-1}}: aa^{-1}S \rightarrow a^{-1}aS$. Then

$$\theta_{a^{-1}}(\theta_a(x)) = \theta_{a^{-1}}(\theta_a(a^{-1}ax')) = \theta_{a^{-1}}(aa^{-1}ax') = a^{-1}ax' = x$$

and

$$\theta_a(\theta_{a^{-1}}(y)) = \theta_a(\theta_{a^{-1}}(aa^{-1}y')) = \theta_a(a^{-1}aa^{-1}y') = aa^{-1}y' = y.$$

Thus θ_a is a bijection for every $a \in S$ with $\theta_a^{-1} = \theta_{a^{-1}}$.

Define $\theta: S \rightarrow I(S)$ by $\theta(a) = \theta_a$. We show the morphism property $\theta_a \circ \theta_b = \theta_{ab}$:

$$\text{dom}(\theta_a) \cap \text{im}(\theta_b) = a^{-1}aS \cap bb^{-1}S = a^{-1}abb^{-1}S,$$

where the last equality holds due to 3.0.28. Hence,

$$\begin{aligned} \text{dom}(\theta_a \circ \theta_b) &= \theta_{b^{-1}}(a^{-1}abb^{-1}S) = b^{-1}a^{-1}abb^{-1}S \stackrel{3.0.28(ii)}{=} \\ &= b^{-1}a^{-1}ab(b^{-1}b)S = b^{-1}a^{-1}abS = (ab)^{-1}abS = \text{dom}(\theta_{ab}). \end{aligned}$$

Thus $\text{dom}(\theta_a \circ \theta_b) = \text{dom}(\theta_{ab})$. It is obvious that θ_{ab} and $\theta_a \circ \theta_b$ have the same effect on elements and so θ is a morphism.

Suppose now that $a \leq b$. This is equivalent to $ba^{-1}a = a$ by 3.2.2. We also have $a^{-1}a \leq b^{-1}b$ by 3.2.4 and so $a^{-1}a = b^{-1}be$ for an idempotent e . It follows that $a^{-1}aS = b^{-1}beS \subseteq b^{-1}bS$. For $x \in a^{-1}aS$ we then have

$$\theta_b(x) = bx = b(a^{-1}ax') = b(a^{-1}aa^{-1}ax') = (ba^{-1}a)x = ax = \theta_a(x).$$

Hence θ_a is a restriction of θ_b .

Conversely, suppose that θ_a is a restriction of θ_b . By assumption then $a^{-1}aS \subseteq b^{-1}bS$. Clearly, $a^{-1} = a^{-1}aa^{-1} \in a^{-1}aS$ and therefore $a^{-1} \in \text{dom}(\theta_a), \text{dom}(\theta_b)$. Again by assumption we have $aa^{-1} = \theta_a(a^{-1}) = \theta_b(a^{-1}) = ba^{-1}$. Multiplying on the right with a yields $a = ba^{-1}a$, i.e. $a \leq b$ by 3.2.2.

In particular, (3.3.1) implies that θ is one-one and θ is therefore an embedding of S into I_S .

□

3.4 The compatibility relation

Definition 3.4.1 *Let S be an inverse semigroup. For all $s, t \in S$ the compatibility relation is defined by*

$$s \sim t \iff st^{-1}, s^{-1}t \in E_S.$$

It is clear that \sim is reflexive and symmetric. It is generally not transitive however. We will characterise those inverse semigroups having a transitive compatibility relation in chapter 6. The proof of the following remark is straightforward.

Remark 3.4.2 *The notion ‘compatibility relation’ stems from the fact that the relation \sim is compatible with multiplication in the sense that*

$$\forall c \in S: a \sim b \implies ac \sim bc, ca \sim cb.$$

The definition of the compatibility relation leads naturally to the following

Definition 3.4.3 *We call a non-empty subset A of an inverse semigroup S compatible if every two elements of A are compatible, that is if $\forall a, b \in A: a \sim b$.*

Definition 3.4.4 *We call a non-empty subset A of an inverse semigroup S an order ideal if it satisfies the following property:*

$$a \in A, s \leq a \implies s \in A.$$

Take for example the set of idempotents of an inverse semigroup. The idempotents do not only form a semilattice, they are an order ideal as well. This is immediate by the definition of the natural partial order and the fact that idempotents commute.

The next definition combines the previous ones.

Definition 3.4.5 *A subset A of an inverse semigroup S is said to be permissible if it is a compatible order ideal. The set of all permissible subsets of S is denoted $C(S)$.*

Proposition 3.4.6 ([5], Theorem 1.4.23) *For an inverse semigroup S , the set $C(S)$ is an inverse semigroup under multiplication of subsets of S . The mapping*

$$\begin{aligned} \iota: S &\rightarrow C(S) \\ s &\mapsto [s] \end{aligned}$$

is an injective morphism.

Proof Let $A, B \in C(S)$. To show closure we have to verify that $AB = \{ab: a \in A, b \in B\}$ is a compatible order ideal. Let $x \leq ab$ for some $a \in A, b \in B$. By 3.2.2, $x = a(bx^{-1}x)$. But B is an order ideal and so $bx^{-1}x = \hat{b} \in B$. Hence $x = a\hat{b} \in AB$ and AB is an order ideal. To show that AB is a compatible subset of S let $ab, cd \in AB$. Then $(ab)^{-1}cd = b^{-1}(a^{-1}c)d$. By assumption, $a \sim c$ and thus $a^{-1}c$ is an idempotent. This implies that $b^{-1}(a^{-1}c)d \leq b^{-1}d$. But $b^{-1}d$ is an idempotent and so $b^{-1}(a^{-1}c)d = (ab)^{-1}cd$ is itself an idempotent. We may similarly show that $ab(cd)^{-1}$ is an idempotent and so $ab \sim cd$. Hence AB is indeed a compatible subset of S and thus $AB \in C(S)$.

To show that $C(S)$ is inverse we verify that it is regular and that its idempotents commute. First note that for a permissible subset A , the set $A^{-1} = \{a^{-1}: a \in A\}$ is also permissible. This set turns out to be the inverse of A . Indeed, it is immediate that $A \subseteq AA^{-1}A$. To prove the reverse inclusion consider $ab^{-1}c$, where $a, b, c \in A$. As a first step we want to write ab^{-1} as uu^{-1} for some $u \in A$. Consider $u = ab^{-1}b$. This element is certainly smaller than a and thus in A since A is an order ideal. Notice also that ab^{-1} is an idempotent since a and b are compatible. We obtain

$$uu^{-1} = (ab^{-1}b)(ab^{-1}b)^{-1} = ab^{-1}bb^{-1}ba^{-1} = ab^{-1}ba^{-1} = ab^{-1}(ab^{-1})^{-1} = ab^{-1}$$

and may thus write $ab^{-1}c = uu^{-1}c \leq c$. But A is an order ideal and so $ab^{-1}c \in A$. Hence $A = AA^{-1}A$ and $C(S)$ is regular.

Next we show that idempotents commute. To this end we must first locate them. Suppose that $A^2 = A$ for a permissible subset A . Then for every $a \in A$ there exist elements $b, c \in A$ such that $a = bc$. Now $a \sim c$ and $b \sim a$ and by $a = aa^{-1}a = a(bc)^{-1}a = (ac^{-1})(b^{-1}a)$ we have in effect $A \subseteq E_S$. Conversely, let $A \in C(S)$ such that $A \subseteq E_S$. Then clearly $A \subseteq A^2$ but also $A^2 \subseteq A$ since A is an order ideal. We have shown that the idempotents of $C(S)$ are precisely the order ideals of E_S . It is immediate that they commute. Hence $C(S)$ is an inverse semigroup.

It remains to demonstrate that ι is an injective morphism. For all $s \in S$ the set $[s]$ is a permissible subset. Indeed, it is obviously an order ideal. To see that it is also compatible let $a, b \leq s$. Then $a = se, b = sf$ for some idempotents $e, f \in E_S$. Clearly,

$$a^{-1}b = (se)^{-1}(sf) = es^{-1}sf \in E_S$$

and

$$ab^{-1} = sefs^{-1} \in E_S.$$

It is easy to see that $[s] \cdot [t] = [st]$ holds for all $s, t \in S$, which means that ι

is a morphism. It is also one-one, since $[s] = [t]$ yields in particular $s \leq t$ and $t \leq s$ from which we deduce $s = t$ by antisymmetry.

□

3.5 Minimum group congruence and maximum idempotent-separating congruence

In this section we define two special congruences that play a prominent rôle in the theory of inverse semigroups, namely the minimum group congruence and the maximum idempotent-separating congruence. The key idea of the minimum group congruence is to relate an inverse semigroup S to a group via factor semigroups. The minimum group congruence appears in one of our characterisations of so-called E-unitary inverse semigroups in chapter 6.

Definition 3.5.1 *Let S be an inverse semigroup and let $a, b \in S$. We define a relation σ by*

$$a \sigma b \iff \exists x \in S: x \leq a, b.$$

The next theorem tells us amongst other things that σ is the smallest congruence such that S/σ is a group.

Theorem 3.5.2 (*[5], Theorem 2.4.1*) *Let S be an inverse semigroup.*

- (i) σ is the smallest congruence on S containing the compatibility relation \sim .
- (ii) S/σ is a group.
- (iii) If ρ is any congruence on S such that S/ρ is a group then $\sigma \subseteq \rho$.

Proof

(i): We start by showing that σ is an equivalence relation. Reflexivity and symmetry are immediate. To show that σ is transitive suppose that $a \sigma b$ and $b \sigma c$, where $a, b, c \in S$. Then there are elements $u, v \in S$ such that $u \leq a, b$ and $v \leq b, c$. In particular we have $u, v \leq b$ or to put it differently $u, v \in [b]$. We already know from the proof of 3.4.6 that $[b]$ is a compatible subset of S and therefore $u \sim v$. Consider the element $uu^{-1}v$. It is certainly smaller than v but also smaller than u because $u^{-1}v$ is an idempotent. This implies $uu^{-1}v \leq a, c$ and thus $a \sim c$. That σ is a congruence follows from 2.0.10 and the fact that the natural partial order is compatible with multiplication:

$$a \sigma b \implies \exists x: x \leq a, b \implies \forall c: xc \leq ac, bc \text{ and } \forall c: cx \leq ca, cb.$$

Next we verify that $\sim \subseteq \sigma$ holds. Let $a \sim b$. Hence in particular $a^{-1}b \in E_S$. Then $aa^{-1}b \leq a, b$ and thus $a \sigma b$. Now we show that σ is the smallest such congruence. Let ρ be any congruence such that $\sim \subseteq \rho$ and let $a \sigma b$. Then $u \leq a, b$ for some u . From the definition of the natural partial order it follows

3.5 Minimum group congruence and maximum idempotent-separating congruence

easily that $u \sim a, b$ which yields $a \rho u$ and $u \rho b$ by assumption. But ρ , being a congruence, is transitive and thus $a \rho b$. We have proved that $\sigma \subseteq \rho$, as required.

(ii): Consider the natural morphism $\natural: S \rightarrow S/\sigma, s \mapsto \sigma(s)$. By 3.0.27 we conclude that S/σ is an inverse semigroup. Let $\sigma(a) \in S/\sigma$ be an idempotent. Then $\sigma(a) = \sigma(a)\sigma(a)^{-1} = \sigma(a)\sigma(a^{-1}) = \sigma(aa^{-1})$. Hence every idempotent $\sigma(a)$ in S/σ can be represented as $\sigma(e)$ for an idempotent e . For every pair $e, f \in E_S$ we have $ef \leq e, f$ and so all idempotents are contained in a single σ -class (possibly together with non-idempotent elements). Thus S/σ is an inverse semigroup with a single idempotent and hence a group by 3.1.1.

(iii): Let ρ be any congruence such that S/ρ is a group and let $a \sigma b$. Then $u \leq a, b$ for some u in S and there exist idempotents e, f such that $ae = u = bf$. Hence $\rho(u) = \rho(ae) = \rho(a)\rho(e)$ and $\rho(u) = \rho(bf) = \rho(b)\rho(f)$, where $\rho(e)$ and $\rho(f)$ are idempotents in S/ρ . But S/ρ , being a group, contains only one idempotent, namely the identity element. It follows that $\rho(a) = \rho(u) = \rho(b)$ and thus $a \rho b$.

□

The following remark is a simple consequence of the preceding proof.

Remark 3.5.3 *All idempotents of S are contained in the same σ -class, namely the identity element of S/σ .*

Sometimes it is convenient to have the definition of the σ -relation in the following form:

Lemma 3.5.4 *Let S be an inverse semigroup. Then $a \sigma b$ if and only if there exists an idempotent i such that $ai = bi$.*

Proof Suppose that $a \sigma b$. Then $ae = x = bf$ for some idempotents e and f . Multiplying on the right with ef yields

$$a(ef) = ae(ef) = bf(ef) = b(ef).$$

Consequently, ef is the element we are looking for. The converse is trivial.

□

We now turn to the other important congruence mentioned at the beginning of this section, the maximum idempotent-separating congruence μ .

Definition 3.5.5 *Let S be an inverse semigroup. A congruence ρ is called idempotent-separating if distinct idempotents lie in different congruence classes, that is if*

$$\rho(e) = \rho(f) \implies e = f \quad (e, f \in E_S).$$

Apparently, the larger a congruence is, the more difficult to be idempotent-separating it will be. We are thus interested in the *maximum idempotent-separating congruence* which we shall call μ . Notice that the situation is contrary to the one for group congruences. There, we were interested in the smallest congruence such that the factor semigroup is a group.

Proposition 3.5.6 *For an inverse semigroup S the maximum idempotent-separating congruence μ is given by*

$$(a, b) \in \mu \iff \forall e \in E_S: a^{-1}ea = b^{-1}eb. \quad (3.5.1)$$

Proof We first demonstrate that μ is an idempotent-separating congruence. Routine verification shows that it is an equivalence. To show the congruence property let $a \mu b$ and $c \mu d$ and let e denote any idempotent. Then, by assumption,

$$(ac)^{-1}e(ac) = c^{-1}(a^{-1}ea)c = d^{-1}(a^{-1}ea)d = d^{-1}(b^{-1}eb)d = (bd)^{-1}e(bd).$$

The second equality holds since $a^{-1}ea$ is an idempotent. It follows that μ is a congruence. To see that it is also idempotent-separating assume that $e \mu f$ for idempotents e and f . Choosing e and f respectively in (3.5.1) we get

$$e = e^{-1}ee = f^{-1}ef = ef = e^{-1}fe = f^{-1}ff = f,$$

where we have used that idempotents commute in inverse semigroups. Hence μ is indeed an idempotent-separating congruence. The next step is to show that it is also the largest such congruence. Let ρ be an idempotent-separating congruence and let $e \in E_S$. For $(a, b) \in \rho$, 3.0.27 implies that $a^{-1}\rho b^{-1}$ too and thus $a^{-1}ea \rho b^{-1}eb$ since ρ is a congruence. But ρ is idempotent-separating and so $a^{-1}ea = b^{-1}eb$. It follows $(a, b) \in \mu$ and therefore $\rho \subseteq \mu$.

□

For later purposes we prove that μ is always contained in the Green's relation \mathcal{H} . In fact we have even more:

Proposition 3.5.7 ([3], Proposition 5.3.7) *Let S be an inverse semigroup. Then μ is the largest congruence contained in \mathcal{H} .*

Proof We start by showing that $\mu \subseteq \mathcal{H}$. Suppose that $a \mu b$, i.e. $\forall e \in E_S: a^{-1}ea = b^{-1}eb$. In particular $a^{-1}(bb^{-1})a = b^{-1}(bb^{-1})b = b^{-1}b$ and thus $ba^{-1}bb^{-1}a = bb^{-1}b = b$. Hence $b \in Sa$. Interchanging the rôles of a and b we also get $a \in Sb$ and therefore $a \mathcal{L} b$. We may similarly show that $a \mathcal{R} b$ and therefore $a \mathcal{H} b$. Consider now a congruence ρ with $\rho \subseteq \mathcal{H}$ and let $a \rho b$. Then by 3.0.27 we also have $a^{-1}\rho b^{-1}$ and hence $a^{-1}ea \rho b^{-1}eb$ for every idempotent e . Now $a^{-1}ea$ and $b^{-1}eb$ are both idempotents and they do lie in the same \mathcal{H} -class. But an \mathcal{H} -class can contain at most one idempotent by 2.3.7. It follows that $a^{-1}ea = b^{-1}eb$ which implies $a \mu b$ since e was arbitrary. We have shown that $\rho \subseteq \mu$ for every congruence ρ contained in \mathcal{H} .

□

The maximum idempotent-separating congruence μ can be used to define a property of inverse semigroups which will feature in chapter 5.

3.5 Minimum group congruence and maximum idempotent-separating congruence

Definition 3.5.8 *An inverse semigroup is said to be fundamental if μ is the equality relation.*

Every inverse semigroup can be made into a fundamental one by building the factor semigroup with respect to μ . The semilattice of idempotents is not affected by this composition.

Proposition 3.5.9 ([3], Proposition 5.3.8) *Let S be an inverse semigroup and let μ be the maximum idempotent-separating congruence on S . Then S/μ is fundamental and has semilattice of idempotents isomorphic to E_S .*

Proof Consider the natural morphism \natural which maps S onto S/μ . Due to 3.0.25 every idempotent in S/μ can be written as $\mu(e)$, where e is an idempotent in S . Conversely, it is clear that $\mu(e)$ is an idempotent in S/μ for every $e \in E_S$. Using an obvious notation, suppose that $\mu(a) \mu_{S/\mu} \mu(b)$. Then

$$\begin{aligned} \mu(a^{-1}ea) &= \mu(a^{-1}) \mu(e) \mu(a) \stackrel{3.0.27}{=} \mu(a)^{-1} \mu(e) \mu(a) \mu_{S/\mu} \mu(b)^{-1} \mu(e) \mu(b) = \\ &= \mu(b^{-1}) \mu(e) \mu(b) = \mu(b^{-1}eb). \end{aligned}$$

But $\mu_{S/\mu}$ is idempotent-separating and hence $\mu(a^{-1}ea) = \mu(b^{-1}eb)$. By the same argument applied to μ we eventually get the property

$$\forall e \in E_S: a^{-1}ea = b^{-1}eb$$

and hence $\mu(a) = \mu(b)$.

□

4 Primitive inverse semigroups and Brandt semigroups

This chapter deals with primitive inverse semigroups and Brandt semigroups. Immediately after an initial definition we see that primitive inverse semigroups resemble groups insofar as their natural partial order is equality when restricted to non-zero elements. In particular the idempotents are all incomparable except for the zero. Brandt semigroups are introduced as completely 0-simple inverse semigroups and are revealed to be the building blocks of primitive inverse semigroups. Another two characterisations are presented, in particular using 0-minimal ideals. The result motivates to focus on Brandt semigroups rather than primitive inverse semigroups. Using the Rees theorem we find a representation of Brandt semigroups as Rees matrix semigroups of a special form. At this point we turn to investigate a seemingly unrelated semigroup which obeys two axioms. This semigroup arises from Brandt's original approach. The chapter is concluded by the proof that the arising semigroups are in fact exactly the Brandt semigroups.

Definition 4.0.10 *An inverse semigroup S with zero is said to be primitive if every non-zero idempotent is primitive in the sense of 2.7.6.*

In the case of an inverse semigroup S without zero, a primitive idempotent is just a minimal idempotent and we could define S to be primitive if every idempotent is minimal. But then the proof of 3.2.6 tells us that such a semigroup is in fact a group. As our concern is with semigroups, we are merely interested in primitive inverse semigroups as defined above. The zero is always the smallest element of the semigroup with respect to the natural partial order. Thus if we required the order to be equality then the semigroup would consist of just a zero. We have, however, the following proposition which should be compared to 3.2.6.

Proposition 4.0.11 *([5], Proposition 3.3.1) Let S be an inverse semigroup with zero. Then the natural partial order is equality when restricted to $S \setminus \{0\}$ if and only if S is primitive.*

Proof If the natural partial order is equality when restricted to $S \setminus \{0\}$ then it is immediate that every non-zero idempotent is primitive. To prove the converse, suppose that every non-zero idempotent is minimal in $E_S \setminus \{0\}$. Let $s \leq t$. Then $s^{-1}s \leq t^{-1}t$. By assumption either $s^{-1}s = 0$ or $s^{-1}s = t^{-1}t$. In the former case we have that $s = s(s^{-1}s) = 0$ whereas in the latter we get $s = ts^{-1}s = tt^{-1}t = t$.

□

In the following definition we encounter the so-called Brandt semigroups which will turn out to be the building blocks of primitive inverse semigroups.

Definition 4.0.12 *A Brandt semigroup is a completely 0-simple inverse semigroup.*

Proposition 4.0.13 (*[7], Lemma II.3.2*) *Let S be a Brandt semigroup.*

(i) *If e is a primitive idempotent of S and $x, y \in S$ are such that $f = xey \in E_S \setminus \{0\}$, then $e = eyfxe$. In particular, every non-zero idempotent contained in the principal ideal generated by a primitive idempotent e is \mathcal{J} -equivalent to e .*

(ii) *All non-zero idempotents of S are primitive.*

Proof

(i): We first demonstrate that $eyfxe$ is an idempotent:

$$(eyfxe)^2 = eyf \underbrace{xey}_{=f} fxe = eyfxe.$$

This implies $eyfxe = (eyfxe)e \leq e$. By

$$x(eyfxe)y = (xey)f(xey) = fff = f \neq 0$$

we conclude that $eyfxe \neq 0$. Using this and the fact that e is primitive this entails $e = eyfxe$.

(ii): Assume that $0 \neq f \leq g$ for non-zero idempotents f, g . Fix a primitive idempotent e . Since S is 0-simple there exist $x, y \in S$: $xey = g$. By (i) we have $e = eygxe$. Now $f = fg = fxy$ and therefore $e = eyffxe = eyfxe$ again by (i). Multiplying $eygxe = e = eyfxe$ with x on the left and with y on the right and using $f \leq g$ provides $f = g$.

□

We have shown that Brandt semigroups are particular primitive inverse semigroups. The converse is not true in general as the next theorem tells us. It uses orthogonal sums¹ to make the relationship between Brandt semigroups and primitive inverse semigroups precise.

Lemma 4.0.14 (*[7], Lemma II.4.1*) *Let S be an inverse semigroup and let $a, b \in S$ such that $\mathcal{J}_a < \mathcal{J}_b$. Then, for every idempotent $f \in \mathcal{J}_b$, there exists an idempotent $e \in \mathcal{J}_a$ with $e < f$.*

Proof By assumption we have $\mathcal{J}_a < \mathcal{J}_b$, i.e. $SaS \subset SbS$. Due to $b\mathcal{J}f$ this yields $a = xfy$ for some $x, y \in S$. Putting $e = x^{-1}xfyy^{-1}$ we clearly have $e \leq f$ and $e^2 = e$. Furthermore, by the definition of e we have $e = x^{-1}ay^{-1}$ and $xey = xx^{-1}xfyy^{-1}y = xfy = a$. Hence $a \mathcal{J} e$. If now $e = f$, then $a \mathcal{J} e = f \mathcal{J} b$ —

¹For a definition see 2.0.12

a contradiction to the assumption. Thus $e < f$, as required. □

Theorem 4.0.15 ([7], Theorem II.4.3) *A semigroup S is a primitive inverse semigroup if and only if it is an orthogonal sum of Brandt semigroups.*

Proof First let S be a primitive inverse semigroup. Let $a, b \in S: \mathcal{J}_a < \mathcal{J}_b$. The \mathcal{J} -class \mathcal{J}_b contains an idempotent $f = bb^{-1}$. By virtue of 4.0.14 $e < f$ for some idempotent $e \in \mathcal{J}_a$. Since f is a primitive idempotent we conclude that $e = 0$ and therefore $\mathcal{J}_a = \{0\}$. Let now $a \neq 0$ be any element and suppose there exists some c in $\mathcal{J}(a)$ such that a and c are not \mathcal{J} -equivalent. Then $\mathcal{J}_c < \mathcal{J}_a$ and hence $\mathcal{J}_c = \{0\}$ by the same argument as before. Consequently, $\mathcal{J}(a) = \mathcal{J}_a \cup \{0\}$. Now let $\mathcal{J}_a \neq \mathcal{J}_b$ for $a, b \in S \setminus \{0\}$. Because $\mathcal{J}_{ab} \leq \mathcal{J}_a, \mathcal{J}_b$ we get, say, $\mathcal{J}_{ab} < \mathcal{J}_a$ and so $ab = 0$. This shows that S is an orthogonal sum of its non-zero principal ideals, each of which is a Brandt semigroup by 2.7.4: $S = \sum_{a \neq 0} \mathcal{J}(a)$.

The converse is a simple consequence of 4.0.13(ii) and the way in which orthogonal sums are defined. □

Now it becomes apparent that a primitive inverse semigroup S is a Brandt semigroup if and only if there is only one summand in the representation of S as an orthogonal sum of Brandt semigroups.

Definition 4.0.16 *For a semigroup with zero a 0-minimal right ideal is a right ideal which is minimal in the set of non-zero right ideals.*

The following proposition is an extension of 4.0.15. It presents another two equivalent conditions. The proof is omitted since conditions (C) and (D) play no further role in this thesis. See [10] for the equivalence of (A), (B) and (C) and [8] as well as [9] for the equivalence of (A), (B) and (D).

Proposition 4.0.17 ([2], Exercise 6.5.6) *The following conditions on a semigroup S with zero are equivalent.*

- (A) *S is a primitive inverse semigroup.*
- (B) *S is an orthogonal sum of Brandt semigroups.*
- (C) *If $a \in S \setminus \{0\}$, then there exists a unique element x of S such that $axa = a$.*
- (D) *S is an inverse semigroup which is the union of its 0-minimal right ideals.*

Theorem 4.0.15 entitles us to confine our investigations to Brandt semigroups henceforth.

4.1 A structure theorem for Brandt semigroups

Since we defined Brandt semigroups to be completely 0-simple inverse semigroups it seems reasonable to base our discussion of them on the Rees theorem and adapt it for inverse semigroups. In particular, a Brandt semigroup B is a completely 0-simple semigroup and therefore of the form $\mathcal{M}^0[G; I, \Lambda; P]$ for some group G , non-empty sets I and Λ and a regular sandwich matrix P . We know from 3.0.21 that in an inverse semigroup each \mathcal{L} -class and each \mathcal{R} -class contains exactly one idempotent. This causes P to contain in each row and each column exactly one non-zero entry:

Proof Suppose, by way of contradiction, that $p_{\lambda_i} \neq 0 \neq p_{\lambda_j}$, where $i \neq j$. Then $(i, p_{\lambda_i}^{-1}, \lambda)$ and $(j, p_{\lambda_j}^{-1}, \lambda)$ are (distinct) idempotents, lying in the same \mathcal{L} -class by 2.7.11 a contradiction to 3.0.21. Hence every row of P contains exactly one non-zero entry. An analogous argument applies to columns. □

Thus, there is a bijection mapping I onto Λ by the rule that $i \mapsto \lambda$ if and only if $p_{\lambda_i} \neq 0$. Since we interpreted I and Λ as index sets there is really no need to distinguish between the two of them and we can safely put $\Lambda = I$ which makes P a ‘square’ matrix. Using our new notion we can go even further and assume without loss of generality that $p_{ij} \neq 0 \iff i = j$. We thus obtain a ‘diagonal’ matrix. Again we do not need to stop here. A diagonal matrix is nice but even nicer would be the ‘identity matrix’ over G^0 by which we mean a diagonal matrix whose non-zero entries consist entirely of the identity element $e \in G$. To spell it out we define the $I \times I$ matrix

$$\Delta := (\delta_{ij}),$$

where δ denotes the ‘Kronecker-symbol’, i.e.

$$\delta_{ij} = \begin{cases} e & i = j \\ 0 & i \neq j \end{cases}$$

We are going to prove that this makes no essential difference, since the resulting Rees matrix semigroup is isomorphic to the original one:

Proof For all $i, j \in I$ there exist u_i, v_j , namely p_{ii} and e , such that $v_j \delta_{ij} u_i = e \delta_{ij} p_{ii} = p_{ij}$. Thus the preconditions of 2.7.17 are satisfied and therefore $S \cong \mathcal{M}^0[G; I, I; \Delta]$. □

The converse is also true. In total we have

Theorem 4.1.1 ([3], Theorem 5.1.8)

A semigroup is both completely 0-simple and an inverse semigroup, i.e. a Brandt semigroup, if and only if $S \cong \mathcal{M}^0[G; I, I; \Delta]$ for some group G and some index set I .

Proof We have already proved the difficult half of the assertion but have yet to show that $\mathcal{M}^0[G; I, I; \Delta]$ is an inverse semigroup. It is certainly regular by 2.7.15. It remains to point out that idempotents commute. Within $\mathcal{M}^0[G; I, I; \Delta]$, the idempotents are easily detected. They are the elements of the form (i, e, i) , where i is any element in I . By the definition of δ_{ij} it follows that

$$(i, e, i)(j, e, j) = (j, e, j)(i, e, i) = \begin{cases} 0 & i \neq j \\ (i, e, i) & i = j \end{cases}$$

□

By means of 4.1.1 a Brandt semigroup is, up to isomorphism, uniquely determined by a group G and an index set I which is why we occasionally denote it as $\mathcal{B}(G, I)$. Note that it is not possible for two index sets I, J with $|I| \neq |J|$ to appear in isomorphic Brandt semigroup representations $\mathcal{M}^0(G; I, I; \Delta) \cong \mathcal{M}^0(H; J, J; \Delta)$ since 2.7.11 would amount in the two Brandt semigroups having a different number of \mathcal{L} -classes.

In the special case of a Brandt semigroup, the multiplication introduced in (2.7.1) simplifies to

$$(g, i)(h, j) = \begin{cases} (gh, i) & i = j \\ 0 & i \neq j \end{cases}$$

$$0(g, i) = (g, i)0 = 0 \cdot 0 = 0, \tag{4.1.1}$$

where we have used the abbreviation $(g, i) := (i, g, i)$. For reasons that will become clear soon, we present the following lemma before we go on and head for another characterisation of Brandt semigroups. The proof is omitted since it is a simple consequence of (4.1.1).

Lemma 4.1.2 *Let $\mathcal{B}(G, I)$ be a Brandt semigroup. Then, for every non-zero element $a \in \mathcal{B}(G, I)$, there exist unique elements $e, f, a' \in \mathcal{B}(G, I)$, such that $ea = a = af$ and $a'a = f$.*

4.2 Another structure theorem

In the last section we described Brandt semigroups as Rees matrix semigroups of particularly easy form. In this section we choose an alternative approach and account for the name *Brandt* semigroup. Historically the name stems from the german mathematician H. Brandt who in 1927 introduced so-called Brandt groupoids. These are binary systems in which products are not always defined and which obey a fairly restrictive set of rules. If we adjoin an element 0 and define the undefined products to be 0, then we get a semigroup which turns out to be what we called a Brandt semigroup. We will, however, not bother about Brandt groupoids and define instead

Definition 4.2.1 *A Br-semigroup is a semigroup S with zero satisfying the following axioms:*

4 Primitive inverse semigroups and Brandt semigroups

(A1) To each element $a \neq 0$ of S there correspond a unique element e of S such that $ea = a$, a unique element f of S such that $af = a$ and a unique element a' of S such that $a'a = f$.

(A2) If e and f are non-zero idempotents of S , then $eSf \neq \{0\}$.

We will account for the suggestive notation after a short lemma.

Lemma 4.2.2 *With the preceding notation we have*

$$(i) \ e, f \in E_S$$

$$(ii) \ aa' = e$$

Proof

(i): $e^2a = e(ea) = ea = a$ together with the uniqueness of the left identity implies $e^2 = e$. Similarly we can show $f^2 = f$.

(ii): $a = af = a(a'a) = (aa')a$ and $a = ea$ together with the uniqueness of the left identity implies $aa' = e$.

□

Now we are able to prove

Proposition 4.2.3 *Br-semigroups are Brandt-semigroups.*

Proof We need to show that S is a completely 0-simple inverse semigroup. We begin with 0-simplicity. By 2.7.4 it suffices to confirm that

$$\forall a, b \in S \setminus \{0\} \exists x, y \in S: xay = b.$$

Let e denote the unique left identity of a and let f denote the unique right identity of b denoted in (A1). By (A2) there exists a non-zero element $c \in eSf$. Applying (A1) to this c we get an element c' such that $c'c$ is the unique right identity element of c . Because of the particular form of c we have $c'c = f$. Again by the particular form of c we know $ec = c$ and we arrive finally at

$$b = bf = b(c'c) = bc'(ec) \stackrel{4.2.2(ii)}{=} bc'(aa')c = \underbrace{bc'}_{:=x} a \underbrace{a'c}_{:=y}.$$

Axiom (A1) assures the existence of non-zero idempotents. To prove that S contains primitive idempotents, let e and f be idempotents such that $0 < e \leq f$. It follows that $ef = fe = e = ee$. The uniqueness of the right identity gives us $e = f$. Consequently S is a completely 0-simple semigroup. Notice that we do not yet know that S is an inverse semigroup and so \leq denotes the order relation (2.1.1), which is defined on E_S for every semigroup.

We still have to demonstrate that S is inverse. Let R be a (non-zero) \mathcal{R} -class containing idempotents e and f . Hence e is a left identity of R and in particular of f which yields $ef = f = ff$. From (A1) it follows that $e = f$ and so every \mathcal{R} -class contains merely one idempotent. An analogous statement holds for

\mathcal{L} -classes and by virtue of 3.0.21 we have proved that S is a completely 0-simple inverse semigroup.

□

Collecting all the pieces 4.2.3, 4.1.1 and 4.1.2 we have finally established the instructive

Theorem 4.2.4 ([1], Theorem 3.9) *The following three conditions on a semigroup S with zero are equivalent.*

- (i) *S is a Br-semigroup.*
- (ii) *S is a Brandt semigroup.*
- (iii) *S is isomorphic to a (regular) Rees $I \times I$ matrix semigroup $\mathcal{M}^0[G; I, I; \Delta]$ over a 0-group G^0 and with the $I \times I$ identity matrix Δ as sandwich matrix.*

5 Inverse ω -semigroups

This chapter is concerned with inverse ω -semigroups. These are inverse semigroups whose idempotents form a chain with respect to the natural partial order. Our goal is to present a universal construction method for them but we confine ourselves to the cases of bisimple and simple semigroups. We start off by introducing Munn semigroups, which are particularly significant inverse semigroups that can be constructed from arbitrary semilattices. What sets them apart is that for any inverse semigroup there exists a morphism into the related Munn semigroup, having the maximum idempotent-separating congruence μ as its kernel. For fundamental inverse semigroups this morphism even constitutes an embedding, enabling us to show that there is only one fundamental bisimple inverse ω -semigroup, namely the bicyclic semigroup. We then describe a method due to Bruck and Reilly for constructing a new and interesting semigroup from a given monoid. This so-called Bruck-Reilly extension makes it possible to determine all bisimple inverse ω -semigroups. Things get more complicated for simple inverse ω -semigroups and we have to work considerably to adapt our methods from the bisimple case. In the end we also obtain a nice characterisation again using Bruck-Reilly extensions.

5.1 The Munn semigroup

Definition 5.1.1 *Let (E, \cdot) be a semilattice. The uniformity relation \mathcal{U} on E is given by*

$$\mathcal{U} = \{(e, f) \in E \times E : [e] \cong [f]\}, \quad (5.1.1)$$

where \cong denotes an isomorphism between (sub)semilattices, that is an isomorphism between the respective commutative idempotent semigroups.

For each $(e, f) \in \mathcal{U}$ we define $T_{e,f}$ to be the set of all isomorphisms from $[e]$ onto $[f]$. Let

$$T_E = \bigcup \{T_{e,f} : (e, f) \in \mathcal{U}\}. \quad (5.1.2)$$

We call T_E the Munn semigroup of the semilattice E .

To justify our notation we have to furnish T_E with an associative operation. First note that $T_E \subseteq \mathcal{I}_E$, where \mathcal{I}_E is the symmetric inverse monoid. We shall show that T_E is in fact an inverse subsemigroup of \mathcal{I}_E . Let $\alpha: [e] \rightarrow [f]$ and $\beta: [g] \rightarrow [h]$ be elements of T_E . The product $\beta \circ \alpha$ of α and β in \mathcal{I}_E maps $\alpha^{-1}([f] \cap [g]) = \alpha^{-1}([fg])$ onto $\beta([f] \cap [g]) = \beta([fg])$. If we write $\alpha^{-1}(fg)$ as i and $\beta(fg)$ as j we see that

$$\begin{aligned} x \in \alpha^{-1}([fg]) &\iff \alpha(x) \in [fg] \iff \alpha(x) \leq fg \iff \\ &x \leq \alpha^{-1}(fg) \iff x \in [i] \end{aligned}$$

and similarly $\beta([fg]) = [j]$. Thus

$$\text{dom}(\beta \circ \alpha) = [i]$$

and

$$\text{im}(\beta \circ \alpha) = [j].$$

It is easy to see that $\beta \circ \alpha$ is an isomorphism and so $\beta \circ \alpha \in T_E$. Moreover, for every $\alpha: [e] \rightarrow [f]$ in T_E , the inverse $\alpha^{-1}: [f] \rightarrow [e]$ is also in T_E . Consider the semilattice of idempotents of T_E . A typical idempotent of T_E is the identical map $\text{id}_{[e]}$ and since $\text{id}_{[f]} \circ \text{id}_{[e]} = \text{id}_{[e] \cap [f]} = \text{id}_{[ef]}$, the map $e \mapsto \text{id}_{[e]}$ is an isomorphism. We have established

Proposition 5.1.2 ([3], Proposition 5.4.1) *For every semilattice E , the Munn semigroup T_E is an inverse semigroup whose semilattice of idempotents is isomorphic to E .*

The above result allows us to identify $\text{id}_{[e]}$ with e and to think of T_E as an inverse semigroup having E as its semilattice of idempotents.

Next we describe for T_E the Green's relations \mathcal{L} and \mathcal{R} . Let $\alpha: [e] \rightarrow [f] \in T_E$. Then $\alpha^{-1} \circ \alpha = \text{id}_{[e]}$ and $\alpha \circ \alpha^{-1} = \text{id}_{[f]}$. The fact that functional composition reads from right to left, implies

$$\alpha \mathcal{R} \beta \iff \text{dom}(\alpha) = \text{dom}(\beta)$$

and

$$\alpha \mathcal{L} \beta \iff \text{im}(\alpha) = \text{im}(\beta) \tag{5.1.3}$$

in T_E .

We shall illustrate this concept with an example.

Example 5.1.3 ([3], Example 5.4.3) Let $E = C_\omega = \{e_0, e_1, e_2, \dots\}$ be a semilattice with $e_0 > e_1 > e_2 > \dots$. Then $[e_n] = \{e_n, e_{n+1}, e_{n+2}, \dots\}$. An isomorphism is in particular order-preserving. It follows that there is only one isomorphism $\alpha_{m,n}$ which maps $[e_m]$ onto $[e_n]$. This isomorphism is given by

$$\alpha_{m,n}(e_k) = e_{k-m+n} \quad (m \leq k).$$

If $\alpha_{m,n}$ and $\alpha_{p,q}$ are elements of T_E , then their product $\alpha_{p,q} \circ \alpha_{m,n}$ maps $\alpha_{m,n}^{-1}([e_n] \cap [e_p]) = \alpha_{m,n}^{-1}([e_n \wedge e_p])$ onto $\alpha_{p,q}([e_n] \cap [e_p]) = \alpha_{p,q}([e_n \wedge e_p])$. If we write $t = \max(n, p)$ we can say that $\alpha_{p,q} \circ \alpha_{m,n}$ maps $[e_{t-n+m}]$ onto $[e_{t-p+q}]$. By uniqueness this yields

$$\alpha_{p,q} \circ \alpha_{m,n} = \alpha_{m-n+t, q-p+t}. \tag{5.1.4}$$

We can thus identify the Munn semigroup of the semilattice C_ω with the bicyclic semigroup already encountered in 3.1.4.

Remark 5.1.4 *In particular, the operation defined in (3.1.1) is indeed associative and the name bicyclic 'semigroup' is therefore justified. This closes the gap in the proof of 3.1.4.*

The previous example features an inverse semigroup having C_ω as its semilattice of idempotents. These semigroups are the topic of our current chapter and so we give them a name.

Definition 5.1.5 *An inverse semigroup with semilattice of idempotents isomorphic to C_ω is called an inverse ω -semigroup.*

In the literature such semigroups are often called ω -regular semigroups (for example in [7]). We prefer to call them inverse ω -semigroups, since a regular semigroup S , whose idempotents form a semilattice isomorphic to C_ω with respect to the partial order (2.1.1), is necessarily inverse. To see this let e, f be arbitrarily chosen idempotents in S . Without loss of generality $e \leq f$ and hence $ef = e = fe$. Thus idempotents commute and S is inverse by 3.0.21.

Example 5.1.3 motivates yet another definition.

Definition 5.1.6 *If a semilattice E has the property that $\mathcal{U} = E \times E$ we call it uniform.*

The Munn semigroup plays a prominent role in the theory of inverse semigroups and it is second only to the symmetric inverse monoid in its importance. In the case of the symmetric inverse monoid we have proved that every inverse semigroup can be embedded in such a semigroup. Here we may also formulate a significant representation theorem.

Definition 5.1.7 *Let $\phi: X \rightarrow Y$ be a map. The kernel of ϕ is defined to be the equivalence relation on X given by $(a, b) \in \ker(\phi) \iff \phi(a) = \phi(b)$.*

Theorem 5.1.8 ([3], Theorem 5.4.4) *For every inverse semigroup S with semilattice of idempotents E there is a morphism $\phi: S \rightarrow T_E$ whose kernel is μ , the maximum idempotent-separating congruence on S .*

Proof Let $a \in S$. We define a map $\alpha_a: [aa^{-1}] \rightarrow [a^{-1}a]$ by the rule that

$$\alpha_a(x) = a^{-1}xa.$$

The image is contained in $[a^{-1}a]$ since $a^{-1}ea = (a^{-1}ea)a^{-1}a$. A straightforward calculation shows that the map $\alpha_{a^{-1}}$ is a two-sided inverse of α_a and so α_a is a bijection. It is even an isomorphism. We have to verify the morphic property:

$$\begin{aligned} \forall e, f \in E: (a^{-1}(eaa^{-1})a)(a^{-1}(faa^{-1})a) &= a^{-1}aa^{-1}(eaa^{-1}faa^{-1})a = \\ &= a^{-1}(eaa^{-1}faa^{-1})a. \end{aligned}$$

We have shown that $\alpha_a \in T_E$ for all $a \in S$. Now we define a map

$$\begin{aligned} \phi: S &\rightarrow T_E \\ a &\mapsto \alpha_a. \end{aligned}$$

Let $a, b \in S$. If we write $i := \alpha_{a^{-1}}(a^{-1}abb^{-1}) = aa^{-1}abb^{-1}a^{-1} = (ab)(ab)^{-1}$ and $j := \alpha_b(a^{-1}abb^{-1}) = b^{-1}a^{-1}abb^{-1}b = (ab)^{-1}(ab)$ then, by the multiplication rule

in T_E , the product $\alpha_b \circ \alpha_a$ is an isomorphism from $[i]$ onto $[j]$. Moreover, for every $x \in [i]$,

$$\alpha_b(\alpha_a(x)) = b^{-1}a^{-1}xab = (ab)^{-1}x(ab),$$

and so $\alpha_b \circ \alpha_a = \alpha_{ab}$. This shows that ϕ is a morphism.

We have still to prove that the kernel of ϕ is μ . Let $(a, b) \in \mu$, then $(a, b) \in \mathcal{H}$ as demonstrated in 3.5.7. We deduce that

$$\text{dom}(\alpha_a) = [aa^{-1}] = [bb^{-1}] = \text{dom}(\alpha_b)$$

and

$$\text{im}(\alpha_a) = [a^{-1}a] = [b^{-1}b] = \text{im}(\alpha_b).$$

Also, for all $x = eaa^{-1} \in [aa^{-1}] = \text{dom}(\alpha_a)$ we get

$$\alpha_a(x) = a^{-1}(eaa^{-1})a = a^{-1}ea = b^{-1}eb = b^{-1}(ebb^{-1})b = b^{-1}(eaa^{-1})b = \alpha_b(x)$$

by the definition of μ and so $\phi(a) = \phi(b)$. Hence $\mu \subseteq \ker(\phi)$.

Conversely, suppose that $\phi(a) = \phi(b)$, i.e. $\alpha_a = \alpha_b$. Since the domains and images must agree we have

$$[aa^{-1}] = \text{dom}(\alpha_a) = \text{dom}(\alpha_b) = [bb^{-1}]$$

and thus $aa^{-1} = bb^{-1}$. For all $e \in E_S$ then

$$a^{-1}ea = a^{-1}(eaa^{-1})a = \alpha_a(eaa^{-1}) = \alpha_b(ebb^{-1}) = b^{-1}(ebb^{-1})b = b^{-1}eb$$

and so $(a, b) \in \mu$. □

In contrast to the Wagner-Preston embedding 3.3.1, the morphism ϕ is not necessarily one-one. In fact it is one-one if and only if S is fundamental¹.

Definition 5.1.9 *Let S be an inverse semigroup. An inverse subsemigroup of S is called full if it contains all idempotents of S .*

This notion allows us to obtain a useful alternative description of fundamental inverse semigroups.

Theorem 5.1.10 ([3], Theorem 5.4.5) *An inverse semigroup with semilattice of idempotents E is fundamental if and only if it is isomorphic to a full inverse subsemigroup of T_E .*

Proof Let S be a fundamental inverse semigroup with semilattice of idempotents E . By 5.1.8 and the remark following the proof of it, $\phi(S)$ is an inverse subsemigroup of T_E . Now consider for $e \in E$ the map

$$\phi(e) = \alpha_e: [ee^{-1}] = [e] \rightarrow [e^{-1}e] = [e],$$

¹For a definition see 3.5.8

where ϕ is as in the proof of 5.1.8. It maps $x \in [e]$ to $e^{-1}xe = x$. We see that $\phi(e) = \text{id}_{[e]}$ for every $e \in E$ and so $\phi(S)$ is a full inverse subsemigroup of T_E as required.

Conversely, suppose that S is a full inverse subsemigroup of T_E . In an obvious notation let $(\alpha, \beta) \in \mu_S$. Hence, using (5.1.3) and 3.5.7, $\text{dom}(\alpha) = \text{dom}(\beta) = [e]$ and $\text{im}(\alpha) = \text{im}(\beta) = [f]$ (say). Also, for every $x \in E$,

$$\alpha \circ \text{id}_{[x]} \circ \alpha^{-1} = \beta \circ \text{id}_{[x]} \circ \beta^{-1},$$

since S is full and μ is idempotent-separating by definition. In particular the domains agree and therefore

$$[\alpha(ex)] = [\beta(ex)]$$

for all $x \in E$. Hence $\alpha(ex) = \beta(ex)$ for all $ex \in [e]$ and so $\alpha = \beta$.

□

This entails a corollary.

Corollary 5.1.11 ([3], Corollary 5.4.6) *The Munn semigroup T_E is fundamental for every semilattice E .*

The next proposition is designed to provide an example of a concrete Munn semigroup. It demonstrates that in a sense, Munn semigroups are ‘big enough’ to comprise all inverse semigroups.

Proposition 5.1.12 ([3], Exercise 5.23) *Let X be a set and let $\mathcal{E} = 2^X$ be the semilattice (under intersection) of all subsets of X . Then $T_{\mathcal{E}}$ is isomorphic to \mathcal{I}_X .*

Proof First note that within the semilattice (\mathcal{E}, \cap) the corresponding order relation \leq coincides with \subseteq . Thus, for each $A \subseteq X$, the order ideal $[A]$ consists of all the subsets of A . Now consider an element $\phi \in T_{\mathcal{E}}$. It maps $\{U \in X : U \subseteq A\}$ onto $\{V \in X : V \subseteq B\}$ (say). We shall assign to ϕ a partial bijection of X which maps A onto B . In order to do that consider the singleton sets $\{x\}$, where $x \in A$. Such a singleton subset of A is mapped to a singleton subset of B . To see this, suppose that the image $\phi(\{x\})$ is not a singleton. Then there exists a set D such that $\emptyset \subset D \subset \phi(\{x\})$. Because ϕ is an (order-)isomorphism the inverse image $\phi^{-1}(D)$ must be a strict subset of $\{x\}$ but it cannot be \emptyset , since the empty set is mapped to the empty set by ϕ —a contradiction. Conversely, consider the inverse image $\phi^{-1}(\{b\})$, where $b \in B$. Suppose that $\phi^{-1}(\{b\})$ is not a singleton. Then there exists a set C such that $\emptyset \subset C \subset \phi^{-1}(\{b\})$. Consequently, $\emptyset = \phi(\emptyset) \subset \phi(C) \subset \phi(\phi^{-1}(\{b\})) = \{b\}$ which is a contradiction. This reasoning prompts us to define a partial mapping α_{ϕ} from A to B by

$$\alpha_{\phi}(a) = b \iff \phi(\{a\}) = (\{b\}).$$

We have already shown that it is well-defined, one-one and onto. Hence α_ϕ is a partial bijection for every $\phi \in T_{\mathcal{E}}$. We claim that

$$\phi \mapsto \alpha_\phi$$

defines an isomorphism Φ from $T_{\mathcal{E}}$ onto \mathcal{I}_X . An element ϕ in $T_{\mathcal{E}}$ is uniquely determined by the images of the singleton sets, since it is easy to see that for every $U \subseteq A$ we have

$$\phi(U) = \bigcup_{a \in U} \phi(a).$$

It follows that Φ is both one-one and onto. The morphism property is a consequence of the way in which partial mappings are composed in the symmetric inverse monoid.

□

Using 5.1.11 this gives us

Corollary 5.1.13 *The symmetric inverse monoid \mathcal{I}_X is fundamental for every set X .*

5.2 Bisimple inverse ω -semigroups

Let S be an inverse semigroup with semilattice of idempotents E . By 3.0.30(iii) two idempotents $e, f \in S$ are \mathcal{D} -equivalent if and only if there exists an element $a \in S$ such that $aa^{-1} = e$ and $a^{-1}a = f$. In such a case it is easy to verify that the map

$$\begin{aligned} \phi: [e] &\rightarrow [f] \\ x &\mapsto a^{-1}xa \end{aligned}$$

is an isomorphism with inverse map $y \mapsto aya^{-1}$. Accordingly we have

$$\mathcal{D} \cap (E \times E) \subseteq \mathcal{U}. \quad (5.2.1)$$

In the special case of a Munn semigroup T_E we can say more. By definition, $e\mathcal{U}f$ implies the existence of an isomorphism $\alpha \in T_E$, which maps $[e]$ onto $[f]$. Clearly then $\alpha^{-1} \circ \alpha = \text{id}_{[e]}$ and $\alpha \circ \alpha^{-1} = \text{id}_{[f]}$. If we decide to identify the idempotents $\text{id}_{[e]}$ with e as suggested by 5.1.2, this yields $e\mathcal{D}f$. We have therefore a stronger version of (5.2.1) for Munn semigroups:

$$\mathcal{D} \cap (E \times E) = \mathcal{U}. \quad (5.2.2)$$

In this chapter we deal with bisimple inverse semigroups, i.e. inverse semigroups in which every two elements are \mathcal{D} -equivalent². In particular every two idempotents are \mathcal{D} -equivalent and so we deduce that E is uniform for every bisimple inverse semigroup by virtue of (5.2.1). The converse is not true in general; not every

²Compare this to the the corresponding definition of simplicity in 2.7.2(i)

inverse semigroup having a uniform semilattice as its semilattice of idempotents is bisimple. Take for example a uniform semilattice E (that contains more than one element) and consider it as an inverse semigroup (having itself as its semilattice of idempotents). Then it is certainly not bisimple, since from $e \mathcal{D} f$ it follows by 3.0.30(iii) that $e = f$. In the Munn semigroup T_E however we benefit from (5.2.2), which yields that every two idempotents are \mathcal{D} -equivalent in T_E when E is uniform. Consequently, every two elements are \mathcal{D} -equivalent, since every element is

\mathcal{D} -equivalent (even \mathcal{L} -equivalent) to an idempotent and the Green's relations are transitive. Thus for a uniform semilattice E there is at least one bisimple inverse semigroup having E as its semilattice of idempotents. We may summarise these observations in

Proposition 5.2.1 ³ *A semilattice E can be the semilattice of idempotents of a bisimple inverse semigroup if and only if it is uniform.*

Taking the Munn semigroup of uniform semilattices gives us a useful recipe for constructing bisimple inverse semigroups but it is certainly no universal recipe, since 5.1.11 states that Munn semigroups are always fundamental and we can easily produce non-fundamental bisimple inverse semigroups. Simply take the direct product of some Munn semigroup (having a uniform semilattice) and a non-trivial group.

Let us for the moment focus on fundamental bisimple inverse semigroups. Recall that by 5.1.10, a fundamental inverse semigroup is isomorphic to a full inverse subsemigroup of the corresponding Munn semigroup. Using the additional property of bisimplicity we can specify this further. If the full inverse subsemigroup $S' = \phi(S)$ is to be bisimple, then for every $e, f \in E$ there is bound to be an element $\alpha \in S'$ with the property that $\text{dom}(\alpha) = [e]$ and $\text{im}(\alpha) = [f]$ (we are using (5.1.3) here). This leads to the following

Definition 5.2.2 *Let E be a semilattice. An inverse subsemigroup S' of T_E is called transitive if*

$$\forall e, f \in E: S' \cap T_{e,f} \neq \emptyset. \quad (5.2.3)$$

We have shown that a fundamental bisimple inverse semigroup regarded as a full inverse subsemigroup of the according Munn semigroup is transitive. On the other hand it is easy to see that a transitive subsemigroup of T_E is bisimple, which makes transitivity a necessary and sufficient condition for bisimplicity. Finally note that a transitive subsemigroup S' is always full as it contains an isomorphism α from $[e]$ onto $[f]$ for all $e, f \in E$ and hence

$$\forall e \in E: \text{id}_{[e]} = \alpha^{-1} \circ \alpha \in S'.$$

Combining 5.1.10 with 5.2.1 we thus obtain

Theorem 5.2.3 ([3], Theorem 5.6.2) *An inverse semigroup S with semilattice of idempotents E is fundamental and bisimple if and only if E is uniform and S is isomorphic to a transitive inverse subsemigroup of T_E .*

³Compare to ([3], Proposition 5.6.1)

Now consider the uniform semilattice C_ω . We already know that

$$\forall e, f \in C_\omega \exists! \alpha \in T_{C_\omega} : [e] \rightarrow [f] \quad (5.2.4)$$

and so the condition (5.2.3) can be satisfied only if

$$\forall e, f \in E : S' \cap T_{e,f} = T_{e,f},$$

that is if $S' = T_{C_\omega}$. In 5.1.3 we have identified T_{C_ω} with the bicyclic semigroup. We thus have

Corollary 5.2.4 ([3], Corollary 5.6.3) *Up to isomorphism, the only fundamental bisimple inverse ω -semigroup is the bicyclic semigroup.*

At this point we turn to investigate the relationship between the maximum idempotent-separating congruence μ and the Green's relation \mathcal{H} in an inverse ω -semigroup. In 3.5.7 we have seen that in an inverse semigroup, μ is the largest congruence contained in \mathcal{H} . Now we are going to show that for inverse ω -semigroups we have in fact equality.

Proposition 5.2.5 ([3], Theorem 5.6.5) *In an inverse ω -semigroup we have $\mathcal{H} = \mu$.*

Proof We have already dealt with $\mu \subseteq \mathcal{H}$ in 3.5.7. To show that $\mathcal{H} \subseteq \mu$ suppose that $(a, b) \in \mathcal{H}$. Then $aa^{-1} = bb^{-1}$, $a^{-1}a = b^{-1}b$ and so, following the notation of 5.1.8,

$$\begin{aligned} \text{dom}(\alpha_a) &= [aa^{-1}] = [bb^{-1}] = \text{dom}(\alpha_b), \\ \text{im}(\alpha_a) &= [a^{-1}a] = [b^{-1}b] = \text{im}(\alpha_b). \end{aligned}$$

By (5.2.4) it follows that $\alpha_a = \alpha_b$ and so $a \mu b$ again by 5.1.8.

□

Corollary 5.2.6 *In an inverse ω -semigroup the equivalence relation \mathcal{H} is even a congruence.*

The Bruck-Reilly extension

Construction 5.2.7 Let T be a monoid with identity 1 and let H_1 be the group of units of T . Let θ be a morphism from T into H_1 . Then we can make $\mathbb{N} \times T \times \mathbb{N}$ into a semigroup S by defining

$$(m, a, n) \cdot (p, b, q) := (m - n + t, \theta^{t-n}(a)\theta^{t-p}(b), q - p + t), \quad (5.2.5)$$

where $t := \max(n, p)$ and where θ^0 is interpreted as the identity map of T . To justify this assertion we have to check that \cdot is associative. Let therefore $(m, a, n), (p, b, q), (r, c, s) \in S$ and compute $((m, a, n) \cdot (p, b, q)) \cdot (r, c, s)$ and $(m, a, n) \cdot ((p, b, q) \cdot (r, c, s))$ respectively. Putting

$$u := \max(q - p + \max(n, p), r)$$

and

$$w := \max(n, p - q + \max(q, r)),$$

the former computes to

$$(m - n - q + p + u, \theta^{u-n-q+p}(a)\theta^{u-q}(b)\theta^{u-r}(c), s - r + u)$$

whereas the latter results in

$$(m - n + w, \theta^{w-n}(a)\theta^{w-p}(b)\theta^{w-r-p+q}(c), s - r - p + q + w).$$

Observe that the outer coordinates combine exactly as in the bicyclic semigroup which we know to be associative. Hence $m - n - q + p + u = m - n + w$, which yields $w = u + p - q$. From this we easily obtain that the middle coordinates are also equal and we have thus shown that \cdot is associative.

Definition 5.2.8 *Let T, θ be as above. We denote the semigroup (S, \cdot) by $BR(T, \theta)$ and call it the Bruck-Reilly extension of T determined by θ .*

Remark 5.2.9 *One ought to think of the Bruck-Reilly extension as a generalisation of the bicyclic semigroup B . Indeed, if we choose the trivial group $G = \{e\}$ and consider the Bruck-Reilly extension of it, then we obtain B .*

For reasons of convenience we shall omit \cdot and use juxtaposition instead at all times. The next lemma lists some basic properties of $BR(T, \theta)$ in view of later purposes.

Lemma 5.2.10 (*[3], Proposition 5.6.6*) *Let $S = BR(T, \theta)$ be the Bruck-Reilly extension for some monoid T and some morphism θ . Then we have*

- (i) *S is a simple semigroup with identity $(0, 1, 0)$.*
- (ii) *$(m, a, n) \mathcal{D}_S (p, b, q) \iff a \mathcal{D}_T b$.*
- (iii) *The element (m, a, n) is an idempotent in S if and only if $m = n$ and a is an idempotent in T .*
- (iv) *The Bruck-Reilly extension S is inverse if and only if T is inverse.*
- (v) *$(m, a, n) \geq (p, b, q)$ if and only if $m + t = p$, $n + t = q$ for some $t \in \mathbb{N}$ and $\theta^t(a)e = b$ for some $e \in E_T$.*

Proof

(i): For arbitrary elements $(m, a, n), (p, b, q) \in S$ a straightforward calculation shows that

$$(p, \theta(a)^{-1}, m + 1)(m, a, n)(n + 1, b, q) = (p, b, q).$$

Therefore S is simple due to 2.7.5.

(ii): As in the assertion we shall use subscripts to distinguish between the Green's relations in T and those in S . We start by considering the Green's

relation \mathcal{R}_S . Assume that $(m, a, n) \mathcal{R}_S (p, b, q)$. Then there exists $(r, c, s) \in S$ such that $(m, a, n)(r, c, s) = (p, b, q)$. Thus

$$(m - n + t, \theta^{t-n}(a) \theta^{t-r}(c), s - r + t) = (p, b, q), \quad (5.2.6)$$

where $t := \max(n, r)$. This implies $m \leq p$. For symmetry reasons we also have $p \leq m$ and thus $m = p$. Now, from $m - n + t = m$ we deduce that $n = t$ and by equating the middle coordinates in (5.2.6) we get $a \theta^{n-r}(c) = b$. Hence $\mathcal{R}_b \leq \mathcal{R}_a$. A dual argument shows that we also have $\mathcal{R}_a \leq \mathcal{R}_b$. Conversely, suppose that $a \mathcal{R}_T b$. Then $ax = b$ and $bx' = a$ for some $x, x' \in T$. As a result

$$(m, a, n)(n, x, q) = (m, b, q) \text{ and } (m, b, q)(q, x', n) = (m, a, n).$$

We have shown that

$$(m, a, n) \mathcal{R}_S (p, b, q) \iff m = p, a \mathcal{R}_T b. \quad (5.2.7)$$

It is evident that the dual statement also holds:

$$(m, a, n) \mathcal{L}_S (p, b, q) \iff n = q, a \mathcal{L}_T b. \quad (5.2.8)$$

Eventually we are able to deal with the \mathcal{D} -relations. Suppose that

$$(m, a, n) \mathcal{D}_S (p, b, q).$$

This implies the existence of an element (r, c, s) such that $(m, a, n) \mathcal{R}_S (r, c, s)$ and $(r, c, s) \mathcal{L}_S (p, b, q)$. A look at (5.2.7) and (5.2.8) makes it instantly clear that $a \mathcal{R}_T c \mathcal{L}_T b$, i.e. $a \mathcal{D}_T b$.

If on the other hand $a \mathcal{D}_T b$, then $a \mathcal{R}_T c \mathcal{L}_T b$ for some $c \in T$. Then

$$(m, a, n) \mathcal{R}_S (m, c, q) \mathcal{L}_S (p, b, q)$$

and hence $(m, a, n) \mathcal{D}_S (p, b, q)$, regardless of the choice of m, n, p, q . In total we have $(m, a, n) \mathcal{D}_S (p, b, q) \iff a \mathcal{D}_T b$. This is (ii).

(iii): An element of the form (m, e, m) , where e is an idempotent, is obviously idempotent. Conversely, assume that (m, a, n) is an idempotent in S . Then

$$(m, a, n) = (m, a, n)(m, a, n) = (m - n + t, \theta^{t-n}(a) \theta^{t-m}(a), n - m + t),$$

where t denotes the maximum of n and m . From comparing left and right-hand sides it follows that $m = n$ and consequently $a^2 = a$, as required.

(iv): Suppose that S is an inverse semigroup and $(m, a, n) \in S$. Let (p, b, q) denote the unique inverse element of (m, a, n) . Then

$$(m, a, n)(p, b, q) = (m - n + t, \theta^{t-n}(a) \theta^{t-p}(b), q - p + t)$$

is an idempotent \mathcal{R}_S -equivalent to (m, a, n) and \mathcal{L}_S -equivalent to (p, b, q) . From (5.2.7) and (5.2.8) it follows that $n = p$. We get $m - n + t = q - n + t$ by (iii) and thus $m = q$. Since $(m, a, n)^{-1} = (p, b, q) = (n, b, m)$ we have $(m, aba, n) =$

(m, a, n) and so $aba = a$. We have shown that T is regular. Now let e, f be arbitrary idempotents in T . We know that in the inverse semigroup S idempotents commute. Thus $(0, ef, 0) = (0, e, 0)(0, f, 0) = (0, f, 0)(0, e, 0) = (0, fe, 0)$ and so $ef = fe$. In effect T is a regular semigroup in which idempotents commute, i.e. an inverse semigroup.

Conversely, suppose that T is an inverse semigroup. Fix an element (m, a, n) in S . It is easy to see that (n, a^{-1}, m) is an inverse of (m, a, n) which makes S a regular semigroup. Let $(m, e, m), (n, f, n)$ be idempotents in S . Without loss of generality let $m \leq n$. Then

$$\begin{aligned} (m, e, m)(n, f, n) &= (n, \theta^{n-m}(e) f, n), \\ (n, f, n)(m, e, m) &= (n, f \theta^{n-m}(e), n). \end{aligned} \quad (5.2.9)$$

Using that morphisms map idempotents to idempotents we obtain $\theta^{n-m}(e)f = f\theta^{n-m}(e)$ since T is inverse. As a result of (5.2.9) S is then inverse as well.

(v): Using (iii) we have by definition,

$$(m, a, n) \geq (p, b, q) \iff \exists k \in \mathbb{N} \exists e \in E_T: (m, a, n)(k, e, k) = (p, b, q).$$

If $n > k$ then the left-hand side multiplies to (m, a, n) because idempotents are mapped to the identity element by θ . If on the other hand $n \leq k$, then, putting $t := k - n$, we have

$$(m, a, n)(k, e, k) = (m + t, \theta^t(a) e, n + t).$$

If, conversely, the conditions on the right-hand side of (v) are satisfied, it is clear that $(m, a, n) \geq (m + t, \theta^t(a) e, n + t)$, since $(m, a, n)(n + t, e, n + t) = (m + t, \theta^t(a) e, n + t)$.

□

Consider the special case of a Bruck-Reilly extension where T is a group (with identity e). By (ii) and (iv) $BR(T, \theta)$ then becomes a bisimple inverse semigroup with identity $(0, e, 0)$ and θ an endomorphism of T .

A look at (v) shows that

$$(0, e, 0) > (1, e, 1) > (2, e, 2) > \dots$$

since a group morphism maps the identity element to the identity element. Hence $BR(T, \theta)$ is a bicyclic inverse ω -semigroup. Perhaps more surprisingly the converse also holds.

Theorem 5.2.11 ([3], Theorem 5.6.7) *Let G be a group and let θ be an endomorphism of G . Let $S = BR(G, \theta)$ be the Bruck-Reilly extension of G determined by θ . Then S is a bisimple inverse ω -semigroup. Conversely, every bisimple inverse ω -semigroup is isomorphic to some $BR(G, \theta)$.*

Proof It remains to establish the converse. Let S be a bisimple inverse semigroup with semilattice of idempotents (isomorphic to) C_ω . We discussed the Munn

semigroup T_{C_ω} in example 5.1.3 and found it to be isomorphic to the bicyclic semigroup B . Recall that

$$T_{C_\omega} = \{\alpha_{m,n} : m, n \in \mathbb{N}\},$$

where $\alpha_{m,n}$ is the unique isomorphism from $[e_m]$ onto $[e_n]$ given by

$$\alpha_{m,n}(e_k) = e_{k-m+n} \quad (k \geq m).$$

By 5.1.8 there exists a morphism $\phi: S \rightarrow B$, whose kernel is μ . In fact we may apply 5.2.5, which supplies

$$\phi(s) = \phi(t) \iff s \mathcal{H} t. \quad (5.2.10)$$

Accordingly, consider an \mathcal{H} -class of S . It is of the form

$$H_{m,n} := \{a \in S : aa^{-1} = e_m, a^{-1}a = e_n\}. \quad (5.2.11)$$

Every element $a \in H_{m,n}$ maps to an element $\alpha_a \in T_{C_\omega}$ with domain $[aa^{-1}] = [e_m]$ and codomain $[a^{-1}a] = [e_n]$. There is only one such element, namely $\alpha_{m,n}$. We thus have

$$\phi(H_{m,n}) = \{\alpha_{m,n}\}$$

and apparently also

$$H_{m,n} = \phi^{-1}(\alpha_{m,n}). \quad (5.2.12)$$

Let now $a \in H_{m,n}$ and $b \in H_{p,q}$. Then

$$\phi(ab) = \phi(b) \circ \phi(a) = \alpha_{p,q} \circ \alpha_{m,n} = \alpha_{m-n+t, q-p+t} \quad (t := \max(n, p)).$$

Hence, using (5.2.12), $ab \in H_{m-n+t, q-p+t}$ and therefore

$$H_{m,n}H_{p,q} \subseteq H_{m-n+t, q-p+t}. \quad (5.2.13)$$

Let us denote $H_{0,0}$ by G . As suggested by our notation, G is a group, since it contains the idempotent e_0 (see 2.3.7). Choose and fix an element $a \in H_{0,1}$. Then $a^2 \in H_{0,2}$ by (5.2.13) and $a^n \in H_{0,n}$ for every $n \in \mathbb{N}$ by induction (we put $a^0 := e_0$). Additionally, a look at (5.2.11) tells us that $a^{-1} \in H_{1,0}$ and further $a^{-n} \in H_{n,0}$ by the same line of reasoning as before.

Notice that $a^n a^{-n}$ is an idempotent in $H_{0,0} = G$ for every $n \in \mathbb{N}$ and thus equal to e_0 . Similarly, $a^{-n} a^n = e_n$. Furthermore $e_0 a^n = a^n a^{-n} a^n = a^n$ and $a^{-m} e_0 = a^{-m} a^m a^{-m} = a^{-m}$. This yields $e_0 \mathcal{R} a^n$ and $e_0 \mathcal{L} a^{-m}$ respectively.

We may apply Green's Lemmas 2.3.3, 2.3.4 and conclude that the maps

$$G = H_{0,0} \rightarrow H_{0,n}, \quad g \mapsto ga^n$$

and

$$H_{0,n} \rightarrow H_{m,n}, \quad x \mapsto a^{-m}x$$

are bijections. Combining these we get a bijection $\phi: G \rightarrow H_{m,n}$ given by

$$g \mapsto a^{-m}ga^n$$

with inverse mapping $x \mapsto a^m x a^{-n}$. Since $n, m \in \mathbb{N}$ were arbitrarily chosen, every element in S can thus be written as $a^{-m} g a^n$ in a unique way. We have thus established a bijection

$$\begin{aligned} \Phi: S &\rightarrow \mathbb{N} \times G \times \mathbb{N} \\ a^{-m} g a^n &\mapsto (m, g, n). \end{aligned}$$

In the end we want to show that Φ is a morphism from S to $BR(G, \theta)$ for some θ . Due to formula (5.2.13), $ag \in H_{0,1}$. Hence we may express ag as $a^0 g' a^1 = e_0 g' a = g' a$ for exactly one $g' \in G$. Implicitly, this defines a map $\theta: G \rightarrow G$ via

$$ag = \theta(g)a. \quad (5.2.14)$$

We claim that θ is an endomorphism of G . To verify this, let $g_1, g_2 \in G$. Then

$$\theta(g_1 g_2) a = a(g_1 g_2) = (a g_1) g_2 = \theta(g_1)(a g_2) = \theta(g_1) (\theta(g_2) a) = (\theta(g_1) \theta(g_2)) a.$$

Multiplying both sides with a^{-1} and recalling that $aa^{-1} = e_0$ is the identity element of G we thus get

$$\theta(g_1 g_2) = \theta(g_1) \theta(g_2). \quad (5.2.15)$$

We may extend (5.2.14) as follows:

$$\forall g \in G: a^2 g = a(ag) = a(\theta(g)a) = (a\theta(g))a = (\theta^2(g)a)a = \theta^2(g)a^2.$$

Using induction we further get

$$\forall g \in G \forall n \in \mathbb{N}: a^n g = \theta^n(g) a^n. \quad (5.2.16)$$

Here, as always, θ^0 is interpreted as the identity map on G . Also from (5.2.14) we deduce

$$\forall g \in G: g^{-1} a^{-1} = (ag)^{-1} = (\theta(g)a)^{-1} = a^{-1} \theta(g)^{-1} = a^{-1} \theta(g^{-1}).$$

If we substitute g^{-1} by g and perform induction again we get

$$ga^{-n} = a^{-n} \theta^n(g). \quad (5.2.17)$$

Eventually we are able to prove that Φ is a morphism:

Let $s, t \in S$. Then $s = a^{-m} g a^n$ and $t = a^{-p} h a^q$ for certain $m, n, p, q \in \mathbb{N}$, $g, h \in G$. We have to distinguish between two cases. Suppose first that $n \geq p$. Then

$$\begin{aligned} (a^{-m} g a^n) (a^{-p} h a^q) &= a^{-m} g a^{n-p} \underbrace{a^p a^{-p}}_{=e_0} h a^q \stackrel{(5.2.16)}{=} a^{-m} g \theta^{n-p}(h) a^{n-p} a^q = \\ &= a^{-m} g \theta^{n-p}(h) a^{n-p+q}. \end{aligned}$$

If on the other hand $n \leq p$, then

$$\begin{aligned} (a^{-m} g a^n) (a^{-p} h a^q) &= a^{-m} g \underbrace{a^n a^{-n}}_{=e_0} a^{-(p-n)} h a^q \stackrel{(5.2.17)}{=} a^{-m} a^{-(p-n)} \theta^{p-n}(g) h a^q = \\ &= a^{-(m+p-n)} \theta^{p-n}(g) h a^q. \end{aligned}$$

These two formulae combine to the one formula

$$(a^{-m}ga^n)(a^{-p}ha^q) = a^{-(m-n+t)}\theta^{t-n}(g)\theta^{t-p}(h)a^{q-p+t},$$

where $t := \max(n, p)$ as usual. We have finally shown that Φ is a morphism and hence $S \cong BR(G, \theta)$. □

5.3 Simple inverse ω -semigroups

In this section we are aiming for a theorem analogous to 5.2.11, i.e. we want to establish a structure theorem for simple inverse ω -semigroups. The theory is developed in a similar way as in the previous section and it is worth comparing the corresponding interim results. We shall add however, that the situation for simple inverse ω -semigroups is slightly more complicated than in the bisimple case.

We first describe simple inverse semigroups in terms of the natural partial order relation and the Green's relation \mathcal{D} , which constitutes the property of bisimplicity when there is only one \mathcal{D} -class.

Lemma 5.3.1 ([3], Lemma 5.7.1) *Let S be an inverse semigroup with semilattice of idempotents E . Then S is simple if and only if*

$$\forall e, f \in E \exists g \in E: g \leq f \text{ and } e \mathcal{D} g. \quad (5.3.1)$$

Proof Suppose first that S is a simple inverse semigroup. Let $e, f \in E$. Then $e = xfy$ for some $x, y \in S$ by 2.7.5. Put $g := fyex$. Now

$$(fyex)^2 = fye \underbrace{xfy}_{=e} ex = fyex$$

shows that g is an idempotent. By

$$fg = \underbrace{ff}_{=f} yex = g$$

it is also clear that $g \leq f$. It remains to show that $e \mathcal{D} g$. Accordingly, we must find an element $u \in S$ such that $e \mathcal{L} u$ and $u \mathcal{R} g$. Consider $u := x^{-1}e$. Then

$$xu = xx^{-1}e = (xx^{-1}x)fy = xfy = e$$

and so $e \mathcal{L} u$. Furthermore,

$$ux = x^{-1}ex = x^{-1}e^2x = x^{-1}x(fyex) = x^{-1}xg = gx^{-1}x = fyexx^{-1}x = fyex = g$$

and

$$gx^{-1} = gx^{-1}xx^{-1} = x^{-1}xgx^{-1} = x^{-1} \underbrace{xfy}_{=e} exx^{-1} = x^{-1}exx^{-1} = x^{-1}xx^{-1}e =$$

$$x^{-1}e = u.$$

That is $u\mathcal{R}g$. Hence $e\mathcal{D}g$, as required.

Conversely, suppose that we have property (5.3.1) for an inverse semigroup S . It follows that

$$\mathcal{J}_e = \mathcal{J}_g \leq \mathcal{J}_f.$$

Interchanging the rôles of e and f we also obtain

$$\mathcal{J}_f \leq \mathcal{J}_e.$$

Hence $\mathcal{J}_e = \mathcal{J}_f$ and so all idempotents in S are \mathcal{J} -equivalent. But then all elements in S are \mathcal{J} -equivalent, since \mathcal{J} is transitive and for every element $a \in S$ we have $a\mathcal{J}aa^{-1}$ (even $a\mathcal{R}aa^{-1}$). This means that there is only one \mathcal{J} -class, i.e. S is simple. □

We defined a semilattice E to be uniform if the uniformity relation \mathcal{U} is maximal, i.e. $\mathcal{U} = E \times E$. Taking 5.3.1 as a model we now define a weaker property that is better suited for establishing a structure theorem for simple inverse ω -semigroups.

Definition 5.3.2 *A semilattice E is said to be subuniform if*

$$\forall e, f \in E \exists g \in E: g \leq f, [e] \cong [g]. \quad (5.3.2)$$

The name suggests that a uniform semilattice is also subuniform. This is indeed the case and easy to see: just put $g = f$ in (5.3.2).

In 5.2.1 we saw that a semilattice can be the semilattice of idempotents of a bisimple inverse semigroup if and only if it is uniform. The next proposition shows that subuniformity is a suitable counterpart to uniformity if we deal with simple inverse semigroups instead of bisimple ones.

Proposition 5.3.3 ([3], Proposition 5.7.2) *A semilattice E can be the semilattice of idempotents of a simple inverse semigroup if and only if it is subuniform.*

Proof Let S be a simple inverse semigroup with semilattice of idempotents E . We have already seen in (5.2.1) that for $e, g \in E$, $e\mathcal{D}g$ implies $[e] \cong [g]$. Hence, using 5.3.1, we have established half of the proposition.

To show the converse, suppose that E is a subuniform semilattice. Consider the Munn semigroup T_E . Here we have the benefits of (5.2.2) available. If $e, f \in E$ then $[e] \cong [g]$ for some $g \leq f$. In the Munn semigroup T_E this is equivalent to $e\mathcal{D}g$. Thus T_E is simple by 5.3.1. □

For a semilattice E we defined in 5.2.2 an inverse subsemigroup S of T_E to be transitive if $S \cap T_{e,f} \neq \emptyset$ for all $e, f \in E$. In other words S is said to be transitive if

$$\forall e, f \in E \exists \alpha \in S: \text{dom}(\alpha) = [e], \text{im}(\alpha) = [f].$$

As in the case of subuniformity we now formulate a weaker version of transitivity that matches our purposes for simple inverse semigroups.

Definition 5.3.4 *Let E be a semilattice. An inverse subsemigroup S of T_E is called subtransitive if*

$$\forall e, f \in E \exists \alpha \in S: \text{dom}(\alpha) = [e], \text{im}(\alpha) \subseteq [f]. \quad (5.3.3)$$

It is apparent that a transitive subsemigroup is also subtransitive. Hardly less apparent is the fact that a subtransitive subsemigroup S of T_E is necessarily full. To see this, let $e \in E$. By definition we find an element $\alpha \in S$ such that $\text{dom}(\alpha) = [e]$. Consequently $\alpha^{-1} \circ \alpha = \text{id}_{[e]} \in S$ and so S is a full inverse subsemigroup of T_E . By 5.1.10 we know now that S is a fundamental inverse semigroup with semilattice of idempotents isomorphic to E . If E is a subuniform semilattice, then S is simple:

Let $e, f \in E$. Subuniformity guarantees the existence of an element $\alpha \in S$ such that $\text{dom}(\alpha) = [e]$ and $\text{im}(\alpha) = [g] \subseteq [f]$. Clearly, $g \leq f$ and by $\alpha^{-1} \circ \alpha = \text{id}_{[e]}$ and $\alpha \circ \alpha^{-1} = \text{id}_{[g]}$ it follows that $e \mathcal{D} g$ in S . Lemma 5.3.1 now implies that S is simple.

The converse is also true. Suppose that S is a fundamental simple inverse semigroup. Using again 5.1.10 we have that $S \cong S'$ where S' is a full inverse subsemigroup of T_E . Now S' is simple because S is, and so, by 5.3.1, for $e, f \in E$ there exists an element $g \in E$ such that

$$\begin{aligned} g &\leq f, \\ \text{id}_{[e]} \mathcal{D} \text{id}_{[g]}. \end{aligned} \quad (5.3.4)$$

We are operating in a subsemigroup of the Munn semigroup T_E and may thus use (5.2.2). Hence (5.3.4) translates to

$$\text{dom}(\alpha) = [e], \text{im}(\alpha) = [g] \subseteq [f]$$

for some $\alpha \in S'$. In other words, S' is subtransitive.

We have established

Theorem 5.3.5 (*[3], Theorem 5.7.3*) *If E is a subuniform semilattice, then every subtransitive inverse subsemigroup of T_E is a fundamental simple inverse semigroup with semilattice of idempotents isomorphic to E . Conversely, if S is a fundamental simple inverse semigroup with (necessarily subuniform) semilattice of idempotents E , then S is isomorphic to a subtransitive inverse subsemigroup of T_E .*

Recall that, up to isomorphism, the only fundamental bisimple inverse ω -semigroup is the bicyclic semigroup B (see 5.2.4). We want to achieve a similar result for simple inverse ω -semigroups.

For $d = 1, 2, 3, \dots$ consider B_d , the restriction of the bicyclic semigroup B to

$$\{(m, n) \in \mathbb{N} \times \mathbb{N}: m \equiv n \pmod{d}\}. \quad (5.3.5)$$

It is straightforward to check that it is an inverse subsemigroup of B . The inverse semigroup B_d is a full subsemigroup of B , since for every idempotent $(m, m) \in B_d$ we have the trivial $m \equiv m \pmod{d}$. We claim that B_d is simple. Before we can prove that, we have to describe the Green's relations \mathcal{R} and \mathcal{L} on B_d .

Lemma 5.3.6 ([3], Lemma 5.7.4) *Let $d \in \mathbb{N} \setminus \{0\}$ and $(m, n), (p, q) \in B_d$. Then*

$$\begin{aligned} (m, n) \mathcal{R} (p, q) &\iff m = p, \\ (m, n) \mathcal{L} (p, q) &\iff n = q. \end{aligned} \tag{5.3.6}$$

Proof We only prove the first assertion. The proof of the second is similar. Let $(m, n) \mathcal{R} (p, q)$. By 3.0.30 this is equivalent to $(m, n)(m, n)^{-1} = (p, q)(p, q)^{-1}$. The left-hand side multiplies to (m, m) and the right-hand side to (p, p) . Hence $(m, n) \mathcal{R} (p, q) \iff m = p$, as required. □

Now we can go on to investigate the \mathcal{D} -classes of B_d . Let (m, m) and (n, n) be idempotents of B_d . By definition,

$$(m, m) \mathcal{D} (n, n) \iff \exists (r, s) \in B_d: (m, m) \mathcal{R} (r, s) \mathcal{L} (n, n).$$

According to (5.3.6) this element (r, s) could only be (m, n) . But (m, n) does not necessarily lie in B_d . Indeed, by definition we have

$$(m, n) \in B_d \iff m \equiv n \pmod{d}.$$

It follows that B_d consists of exactly d \mathcal{D} -classes, which we denote as

$$D_{(0,0)}, D_{(1,1)}, D_{(2,2)}, \dots, D_{(d-1,d-1)}.$$

We can now formulate the analogue of 5.2.4.

Proposition 5.3.7 ([3], Theorem 5.7.5) *Up to isomorphism, the only fundamental simple inverse ω - semigroups are the semigroups B_d ($d = 1, 2, 3, \dots$).*

Proof By virtue of 5.3.5, what we have to show is that the semigroups

$$B'_d = \{\alpha_{m,n}: m \equiv n \pmod{d}\}$$

are the only subtransitive inverse subsemigroups of the bicyclic semigroup

$$T_{C_\omega} = \{\alpha_{m,n}: m, n \in \mathbb{N}\}.$$

First we show that B'_d is subtransitive. Consider $e_m, e_n \in C_\omega$ and let

$$p \geq n: m \equiv p \pmod{d}.$$

Then certainly $\alpha_{m,p} \in B'_d$ and

$$\text{dom}(\alpha_{m,p}) = [e_m], \text{im}(\alpha_{m,p}) = [e_p] \subseteq [e_n].$$

We have shown that B'_d is a subtransitive inverse subsemigroup of T_{C_ω} for every $d \in \mathbb{N} \setminus \{0\}$. To demonstrate that these are the only ones suppose that S is a

subtransitive inverse subsemigroup of T_{C_ω} . Choosing $e_0, e_1 \in C_\omega$ and using the subtransitivity property we get

$$\exists \alpha \in S: \text{dom}(\alpha) = [e_0], \text{im}(\alpha) \subseteq [e_1],$$

where $\text{im}(\alpha) = [e_d]$ for some $d \geq 1$. Since $\alpha_{0,d}$ is the only element in T_{C_ω} having domain $[e_0]$ and image $[e_d]$, we deduce that S contains $\alpha_{0,d}$. Suppose that d is the least positive integer for which this holds. Using the multiplication rule (5.1.4) we get $\alpha_{0,d}^2 = \alpha_{0,2d} \in S$ and more generally $\alpha_{0,d}^k = \alpha_{0,kd} \in S$ for $k \geq 1$. Now, since S is a subtransitive inverse subsemigroup of T_{C_ω} it is necessarily full (as we have already seen in the proof of 5.3.5) and so $\alpha_{m,m} \in S$ for all $m \in \mathbb{N}$. Hence,

$$\alpha_{0,kd} \circ \alpha_{m,m} = \alpha_{m,m+kd} \in S. \quad (5.3.7)$$

Owing to equation (5.3.7) we can conclude that $B'_d \subseteq S$. To show that this is in fact an equality, suppose, by way of contradiction, that there exists $\alpha_{m,n} \in S$ such that $m \not\equiv n \pmod{d}$. Writing n as $m + kd + r$, with $0 < r < d$, by the division algorithm, we deduce that S contains

$$\alpha_{m+kd,m} \circ \alpha_{m,m+kd+r} = \alpha_{m,m+r}.$$

If $m = 0$, we have obtained a contradiction since $r < d$ and d was defined to be the least positive integer such that $\alpha_{0,d} \in S$. If on the other hand $m > 0$ then we observe next that

$$\alpha_{m-1+d,m-1} \circ \alpha_{m,m+r} \circ \alpha_{m-1,m-1+d} = \alpha_{m-1,m-1+r}$$

and we can descend further and further until the contradiction $\alpha_{0,r} \in S$ arises. Hence $S = B'_d$ for some d and S is indeed of the form (5.3.5). □

In 5.2.11 we observed that the Bruck-Reilly extension $BR(T, \theta)$ of a group T is a bisimple inverse ω -semigroup. In our effort to find a structure theorem for simple ω -semigroups it proves wise to examine Bruck-Reilly extensions of a particular type of Clifford semigroups instead.

Construction 5.3.8 Let $Y = \{0, 1, \dots, d-1\}$ be a chain with the reversed usual order. To simplify the notation we shall use the convention to denote by \leq the usual order of the natural numbers, whereas by \wedge we refer to the order of the chain, for example $4 \leq 5$ but $4 \wedge 5 = 5$. For every $i \in Y$ let G_i denote a group such that all the groups G_i are disjoint. Put $T := \bigcup_{i=0,1,\dots,d-1} G_i$. For every $0 \leq i \leq d-2$ choose and fix a morphism $\gamma_i: G_i \rightarrow G_{i+1}$. Moreover, we define for $0 \leq i < j \leq d-1$ a new morphism $\alpha_{i,j}: G_i \rightarrow G_j$ by the rule

$$\alpha_{i,j} = \gamma_{j-1} \circ \gamma_{j-2} \circ \dots \circ \gamma_i.$$

Putting $\alpha_{i,i} = \text{id}_{G_i}$ we thus have

$$\alpha_{j,k} \circ \alpha_{i,j} = \alpha_{i,k} \quad (i \leq j \leq k).$$

By virtue of 3.1.9 we conclude that the strong semilattice of groups (T, \otimes) is a Clifford semigroup. In fact the semilattice is even a chain isomorphic to Y . The idempotents of T are the identity elements of the groups G_i denoted by e_0, e_1, \dots, e_{d-1} . Recall that identity elements are mapped to identity elements by group morphisms and notice that e_0 is the identity element of the monoid T :

$$\forall i \forall x \in G_i: e_0 x = \alpha_{0,0 \wedge i}(e_0) \alpha_{i,0 \wedge i}(x) = \underbrace{\alpha_{0,i}(e_0)}_{=e_i} \underbrace{\alpha_{i,i}(x)}_{=x} = e_i x = x.$$

A similar argument shows that $x e_0 = x$ for all $x \in T$. Furthermore, a straightforward calculation yields $e_0 > e_1 > \dots > e_{d-1}$.

We shall refer to T as a *finite chain of groups of length d* .

Let T be a finite chain of groups of length d . Notice that the group of units of T is G_0 because a product in which an element $x \in G_i$ is involved does not necessarily lie in G_j for some $j \geq i$. Accordingly, let $S = BR(T, \theta)$, where θ is a morphism from T to G_0 . By 5.2.10, S is a simple inverse semigroup since T is inverse. Note that because of 5.2.10(ii) the \mathcal{D} -classes of S are the subsets $\mathbb{N} \times G_i \times \mathbb{N}$ ($i = 0, 1, \dots, d-1$). We shall show that S is an ω -semigroup. Let $(m, e_i, m), (n, e_j, n)$ be two idempotents. We assume without loss of generality that $m \geq n$ and distinguish between two cases:

For $m = n$ we have

$$(m, e_i, m) \leq (m, e_j, m) \iff (m, e_i, m)(m, e_j, m) = (m, e_i, m).$$

Bearing in mind that $(m, e_i, m)(m, e_j, m) = (m, e_i e_j, m)$, this is the case if and only if $e_i \leq e_j$ in T , i.e. if and only if $i \wedge j = i$.

For $m > n$ we have $\theta^{m-n}(e_j) = e_0$, the identity of T . Hence

$$(m, e_i, m)(n, e_j, n) = (m, e_i \theta^{m-n}(e_j), m) = (m, e_i, m)$$

and so $(m, e_i, m) < (n, e_j, n)$ regardless of the values of i and j . In effect, the idempotents of S form a chain

$$\begin{aligned} (0, e_0, 0) &> (0, e_1, 0) > \dots > (0, e_{d-1}, 0) > \\ (1, e_0, 1) &> (1, e_1, 1) > \dots > (1, e_{d-1}, 1) > \\ &\vdots \\ (d-1, e_0, d-1) &> (d-1, e_1, d-1) > \dots > (d-1, e_{d-1}, d-1). \end{aligned}$$

Thus $S = BR(T, \theta)$ is a simple inverse ω -semigroup. Remarkably, as with 5.2.11, the converse also holds. We sum this up in a theorem:

Theorem 5.3.9 ([3], Theorem 5.7.6) *Let T be a finite chain of groups of length d (≥ 1). If θ is a morphism from T into the group of units of T , then the Bruck-Reilly extension $BR(T, \theta)$ of T determined by θ is a simple inverse ω -semigroup with d \mathcal{D} -classes. Conversely, every simple inverse ω -semigroup is isomorphic to one of this type.*

Proof It remains to prove the converse half. Let S be a simple inverse semigroup whose semilattice of idempotents is $C_\omega = \{f_0, f_1, f_2, \dots\}$. Following the notation of the proof of 5.1.8, there is a morphism $\phi: S \rightarrow T_{C_\omega}$ with kernel μ , mapping a in S to the element

$$\begin{aligned} \alpha_a: [aa^{-1}] &\rightarrow [a^{-1}a], \\ x &\mapsto a^{-1}xa. \end{aligned}$$

Since S is an inverse ω -semigroup, by 5.2.5, the maximum idempotent-separating congruence μ coincides with the Green's relation \mathcal{H} and so the kernel of ϕ equals \mathcal{H} . The image of S in T_{C_ω} is an inverse subsemigroup according to 3.0.26. It is also subtransitive. To see this let $e, f \in C_\omega$. We may apply 5.3.1 since S is simple and are thus given an element $g \in C_\omega$, such that $e \mathcal{D} g$ and $g \leq f$. Now $e \mathcal{D} g$ amounts in the existence of an element $a \in S$ such that $aa^{-1} = e, a^{-1}a = g$. Moreover, it is easy to see that $g \leq f$ translates into $[g] \subseteq [f]$. Hence, $\phi(a) \in \phi(S)$ maps $[aa^{-1}] = [e]$ onto $[a^{-1}a] = [g] \subseteq [f]$, i.e. $\phi(S)$ is subtransitive. Because $\phi(S)$ is an ω -semigroup there are strong restrictions on the subtransitive subsemigroups of T_{C_ω} . Indeed, as shown in the proof of 5.3.7, $\phi(S)$ is isomorphic to B_d for some $d \geq 1$. We regard $\phi(S)$ and B_d as equal and write as a reminder:

$$\begin{aligned} \phi(S) &= \{\alpha_{m,n}: m, n \in \mathbb{N}, m \equiv n \pmod{d}\}, \\ \alpha_{m,n}: [f_m] &\rightarrow [f_n], f_k \mapsto f_{k-m+n} \quad (k \geq m). \end{aligned}$$

Also recall that

$$\alpha_{p,q} \circ \alpha_{m,n} = \alpha_{m-n+t, q-p+t} \quad (t := \max(n, p)).$$

As in the bisimple case we define

$$H_{m,n} := \{a \in S: aa^{-1} = f_m, a^{-1}a = f_n\},$$

where $H_{m,n}$ is an \mathcal{H} -class or the empty set. Other than in the bisimple case the latter can occur here. In fact we can say precisely when $H_{m,n}$ is an \mathcal{H} -class for if $H_{m,n} \neq \emptyset$, then $\phi(H_{m,n}) = \alpha_{m,n} \in \phi(S) = B_d$ and hence

$$H_{m,n} \neq \emptyset \iff m \equiv n \pmod{d}.$$

We would like to use 5.2.11. To this end we investigate which \mathcal{H} -classes build up to \mathcal{D} -classes. Let $a \in H_{m,n}$ and $b \in H_{p,q}$ (say). In particular, the respective sets are to be non-empty and so $m \equiv n \pmod{d}$ and $p \equiv q \pmod{d}$. Then

$$aa^{-1} = f_m, a^{-1}a = f_n \text{ and } bb^{-1} = f_p, b^{-1}b = f_q.$$

Now suppose that $a \mathcal{D} b$, i.e. $\exists c \in S: a \mathcal{R} c \mathcal{L} b$. Then $cc^{-1} = aa^{-1} = f_m$ and $c^{-1}c = b^{-1}b = f_q$. This means that c would need to be in $H_{m,q}$. Such a c can only be found if $H_{m,q} \neq \emptyset$, i.e. if $m \equiv q \pmod{d}$. Conversely, every two such elements are \mathcal{D} -equivalent. Summarising, we have

$$a \mathcal{D} b \iff n \equiv m \equiv q \equiv p \pmod{d}.$$

In effect, S consists of the d \mathcal{D} -classes

$$D^0, D^1, \dots, D^{d-1},$$

where

$$D^i = \bigcup \{H_{m,n} : m \equiv n \equiv i \pmod{d}\} = \bigcup \{H_{pd+i, qd+i} : p, q \in \mathbb{N}\}.$$

These \mathcal{D} -classes are worth a lemma:

Lemma 5.3.10 (*[3], Lemma 5.7.7*) *For $i = 0, 1, \dots, d-1$, the \mathcal{D} -class D^i is a bisimple inverse ω -semigroup with identity element f_i .*

Proof Since $\phi(H_{m,n}) = \alpha_{m,n}$ for $m \equiv n \pmod{d}$, the multiplication formula (5.1.4) yields that

$$H_{m,n}H_{p,q} \subseteq H_{m-n+t, q-p+t} \quad (t := \max(n, p)). \quad (5.3.8)$$

Let $H_{m,n}, H_{p,q} \subseteq D^i$, i.e. let $m \equiv n \equiv p \equiv q \equiv i \pmod{d}$. Then certainly

$$m - n + t \equiv q - p + t \equiv t \equiv i \pmod{d}$$

and therefore $H_{m,n}H_{p,q} \subseteq D^i$. Hence D^i is a semigroup. The inverse of an element $a \in H_{m,n}$ is of course contained in $H_{n,m}$ and since it is clear that $H_{m,n} \subseteq D^i \iff H_{n,m} \subseteq D^i$, it is also an inverse semigroup. Next we are going to show that D^i is an ω -semigroup. Accordingly we must locate the idempotents first. Let g denote an idempotent in $H_{pd+i, qd+i} \subseteq D^i$. We have

$$f_{pd+i} = gg^{-1} = g = g^{-1}g = f_{qd+i}.$$

The only idempotents in D^i are thus the elements f_{pd+i} in $H_{pd+i, pd+i}$. We know that they form a chain

$$f_i > f_{d+i} > f_{2d+i} > \dots$$

and so D^i is an inverse ω -semigroup.

Next we verify that D^i is bisimple. This might appear trivial, since D^i is a \mathcal{D} -class after all but it is a \mathcal{D} -class in S , which means that the connecting element might lie in $S \setminus D^i$. It suffices to demonstrate that all the idempotents in D^i are \mathcal{D} -equivalent. This is easily done. Just notice that for all elements $a \in H_{pd+i, qd+i} \subseteq D^i$ we have $aa^{-1} = f_{pd+i}$, $a^{-1}a = f_{qd+i}$ and thus f_{pd+i} and f_{qd+i} are \mathcal{D} -equivalent by 3.0.30(iii).

The only thing left to show is that f_i is the identity element of D^i . Accordingly, let $a \in H_{pd+i, qd+i} \subseteq D^i$. We wish to establish the identity $f_i a = a$. By uniqueness of inverses it suffices to show that a^{-1} is the inverse of $f_i a$:

$$(f_i a) a^{-1} (f_i a) = f_i f_{pd+i} f_i a = f_i f_{pd+i} a = f_i a a^{-1} a = f_i a,$$

$$a^{-1} (f_i a) a^{-1} = a^{-1} f_i f_{pd+i} = a^{-1} f_{pd+i} = a^{-1} a a^{-1} = a^{-1}.$$

Hence f_i is the identity element of D^i .

All in all, D^i is a bisimple inverse ω -semigroup with identity element f_i .

□

We return now to the proof of 5.3.9. As a straightforward consequence of (5.3.8) we observe that the group of units of D^i is $H_{i,i}$, which we shall denote by G_i . Let

$$T := \bigcup_{i=0}^{d-1} G_i.$$

We are going to demonstrate that T is a finite chain of groups (of length d). For this purpose we show that T agrees with the construction given in 5.3.8. For $0 \leq i \leq j \leq d-1$ let $\phi_{i,j}: G_i \rightarrow G_j$ be defined by the rule

$$x \mapsto f_j x.$$

A look at (5.3.8) reveals that this is well-defined. In particular, for $i = j$ we have $\phi_{i,i}(x) = f_i x = x$, since f_i is the identity of G_i and so $\phi_{i,i}$ is the identity map on G_i . The map $\phi_{i,j}$ is a morphism:

$$\phi_{i,j}(x)\phi_{i,j}(y) = (f_j x)(f_j y) = \underbrace{f_j x}_{\in G_j} f_j y = f_j(xy) = \phi_{i,j}(xy).$$

Next, suppose that $0 \leq i \leq j \leq k \leq d-1$. For all $x \in G_i$ we have

$$\phi_{j,k}(\phi_{i,j}(x)) = f_k(f_j x) = (f_k f_j)x = f_k x = \phi_{i,k}(x)$$

and so $\phi_{j,k} \circ \phi_{i,j} = \phi_{i,k}$. Finally we verify that the operation \otimes defined in (3.1.2) agrees with the multiplication on T inherited from S . Let $x \in G_i, y \in G_j$ and assume first that $i \leq j$. Then

$$x \otimes y = \phi_{i,i \wedge j}(x)\phi_{j,i \wedge j}(y) = \phi_{i,j}(x)\phi_{j,j}(y) = (f_j x)y = f_j \underbrace{xy}_{\in G_j} = xy.$$

If on the other hand $i \geq j$, then similar

$$x \otimes y = \phi_{i,i \wedge j}(x)\phi_{j,i \wedge j}(y) = \phi_{i,i}(x)\phi_{j,i}(y) = x(f_i y) = \underbrace{x}_{\in G_i} f_i y = xy.$$

Hence T is a strong semilattice of groups and consequently a Clifford semigroup by 3.1.9. It is even a finite chain (of length d) as can be seen easily by considering formula (5.3.8). At this point it is worth observing that f_0 is the identity element of T . Let $x \in G_i \subseteq T$ (say), then

$$x f_0 = x \otimes f_0 = \phi_{i,i \wedge 0}(x)\phi_{0,i \wedge 0}(f_0) = \phi_{i,i}(x)\phi_{0,i}(f_0) = x(f_i f_0) = x f_i = x$$

and

$$f_0 x = \phi_{0,0 \wedge i}(f_0)\phi_{i,0 \wedge i}(x) = \phi_{0,i}(f_0)\phi_{i,i}(x) = (f_i f_0)x = f_i x = x.$$

We should mention as well that the group of units of T is G_0 . This follows from (5.3.8) and the fact that G_0 is a group containing the identity f_0 .

Lemma 5.3.10 suggests that we should apply 5.2.11 to each \mathcal{D} -class D^i . This is almost what we do. In the proof of 5.2.11 we chose an $a \in H_{0,1}$ and expressed each element of $H_{m,n}$ uniquely as $a^{-m}ga^n$ ($g \in G$). Analogously, we would choose for each i an element a_i in $H_{i,d+i}$ and then express each element of D^i as $a_i^{-m}g_ia_i^n \in H_{md+i,nd+i}$, with $m, n \in \mathbb{N}, g_i \in G_i$. It makes things easier however to make the elements a_i depend on an arbitrarily chosen element $a_0 = a$ in $H_{0,d}$. We immediately have $a^{-1} \in H_{d,0}$ and by definition of $H_{0,d}$,

$$aa^{-1} = f_0, \quad a^{-1}a = f_d.$$

Moreover, for $i \in \{0, 1, \dots, d-1\}$, we have

$$f_ia \in H_{i,i}H_{0,d} \subseteq H_{i,d+i}.$$

Since f_i is the identity of D^i it follows that $(f_ia)f_i = f_ia$ and thus $(f_ia)^2 = f_ia^2$. This generalises to

$$(f_ia)^n = f_ia^n \quad (n \in \mathbb{N} \setminus \{0\}).$$

Taking inverses yields

$$(f_ia)^{-n} = a^{-n}f_i \quad (n \in \mathbb{N} \setminus \{0\}).$$

Using f_ia as a_i in the way indicated above, we express each element of D^i uniquely as

$$(f_ia)^{-m}g_i(f_ia)^n = a^{-m}f_ig_if_ia^n = a^{-m}g_ia^n \quad (m, n \in \mathbb{N} \setminus \{0\}, g_i \in G_i). \quad (5.3.9)$$

The last equality stems from the fact that f_i is the identity of G_i . The element $a^{-m}g_ia^n$ belongs to the \mathcal{H} -class $H_{md+i,nd+i}$. If we agree to formally interpret a^0 as 1 we may extend the validity of (5.3.9) by allowing $m, n \in \mathbb{N}$. We have thus established a bijection $\Psi: S \rightarrow \mathbb{N} \times T \times \mathbb{N}$ defined by

$$\Psi(a^{-m}g_ia^n) = (m, g_i, n).$$

In order to make Ψ an isomorphism we ought to find a suitable morphism θ which maps T into its group of units G_0 . Again we are trying to imitate the proof of 5.2.11. Note that for all $g_i \in G_i$ ($i = 0, 1, \dots, d-1$) we have $ag_i \in H_{0,d}H_{i,i} \subseteq H_{0,d}$. Further, we know from the proof of 5.2.11 that ag_i is uniquely expressible as g_0a with $g_0 \in G_0$. This implicitly defines a map ϕ_i from G_i to G_0 by

$$ag_i = \phi_i(g_i)a.$$

Since the groups G_0, G_1, \dots, G_{d-1} are disjoint we can piece together a map $\theta: T \rightarrow G_0$:

$$\theta(x) = \phi_i(x) \quad (x \in G_i). \quad (5.3.10)$$

This map θ will turn out to be the morphism we are looking for. To show the morphic property, consider, without loss of generality $x \in G_i, y \in G_j$ with $i \leq j$. Then $xy \in G_j$ and so by (5.3.10),

$$a(xy) = \phi_j(xy)a = \theta(xy)a.$$

On the other hand

$$a(xy) = (ax)y = \phi_i(x)(ay) = \phi_i(x)\phi_j(y)a = (\theta(x)\theta(y))a.$$

Hence $\theta(xy)a = (\theta(x)\theta(y))a$. Multiplying both sides with a^{-1} on the right we get

$$\theta(xy)\underbrace{aa^{-1}}_{=f_0} = (\theta(x)\theta(y))\underbrace{aa^{-1}}_{=f_0}$$

and thus the desired $\theta(xy) = (\theta(x)\theta(y))$, since f_0 is the identity element of T . Exactly as in the proof of 5.2.11 we deduce for every $x \in T$ and every $k \in \mathbb{N}$ that

$$a^k x = \theta^k(x)a^k \text{ and } xa^{-k} = a^{-k}\theta^k(x),$$

where a^0 is interpreted as f_0 and $\theta^0(x)$ as x .

Now we have prepared all ingredients to finish off the proof:

Let $x, y \in T$ and let $m, n, p, q \in \mathbb{N}$. We have to distinguish between the two cases $n \geq p$ and $n \leq p$.

Remembering that f_0 is the identity element of T , suppose first that $n \geq p$. Then

$$\begin{aligned} (a^{-m}xa^n)(a^{-p}ya^q) &= a^{-m}xa^{n-p}(aa^{-1})ya^q = a^{-m}xa^{n-p}(f_0y)a^q = a^{-m}xa^{n-p}ya^q = \\ &= a^{-m}x\theta^{n-p}(y)a^{n-p}a^q = a^{-m}x\theta^{n-p}(y)a^{q-p+n}. \end{aligned}$$

If on the other hand $n \leq p$, then

$$\begin{aligned} (a^{-m}xa^n)(a^{-p}ya^q) &= a^{-m}(xf_0)a^{-(p-n)}ya^q = a^{-m}xa^{-(p-n)}ya^q = \\ &= a^{-m}a^{-(p-n)}\theta^{p-n}(x)ya^q = a^{-(m-n+p)}\theta^{p-n}(x)ya^q. \end{aligned}$$

Writing $t := \max(n, p)$ as usual, we can combine these two statements into a single one:

$$(a^{-m}xa^n)(a^{-p}ya^q) = a^{-(m-n+t)}\theta^{t-n}(x)\theta^{t-p}(y)a^{q-p+t}.$$

It is now clear that Ψ is an isomorphism from S onto $BR(T, \theta)$.

□

We have successfully characterised both bisimple and simple inverse ω -semigroups. For a more general approach on inverse ω -semigroups see for example [5].

6 E-unitary inverse semigroups

This chapter revolves around E-unitary inverse semigroups, which indisputably form one of the most important classes in inverse semigroup theory. One of the reasons for this is McAlister's covering theorem which is presented briefly. In addition to that, E-unitary inverse semigroups tend to pop up in many 'practical' contexts, but we do not cover these applications in this thesis. E-unitary inverse semigroups are distinguished among inverse semigroups by having the property that every element greater than an idempotent (with respect to the natural partial order) is itself idempotent. In general inverse semigroups we can merely presuppose that elements smaller than idempotent are idempotents. In the course of the chapter we gather several further (equivalent) conditions, involving in particular the compatibility relation \sim and the minimum group congruence σ , which might as well serve as a defining property for an inverse semigroup to be E-unitary. Some examples of E-unitary inverse semigroups are assembled in an own section, many of which should be familiar from earlier parts of the thesis. In particular we investigate Clifford semigroups and Bruck-Reilly extensions and determine exactly when they are E-unitary. This allows us in turn to give nice characterisations of E-unitary bisimple inverse ω -semigroups and E-unitary simple inverse ω -semigroups respectively. The final section is dedicated to P-semigroups, a special type of E-unitary inverse semigroups arising from a slightly sophisticated construction due to McAlister. The main result states that every E-unitary inverse semigroup is isomorphic to one of this kind. Eventually, some easy verified properties of P-semigroups are proven.

Definition 6.0.11 *A subset A of an inverse semigroup S is said to be left [right] unitary if $a \in A, s \in S$ and $as \in A$ [$sa \in A$] imply $s \in A$.*

Proposition 6.0.12 ¹ *Let S be an inverse semigroup and let σ be the minimum group congruence on S . The following statements are equivalent:*

- (i) E_S is left unitary.
- (ii) E_S is right unitary.
- (iii) If e is an idempotent and $e \leq s$, then s is an idempotent.
- (iv) $\mathcal{L} \cap \sigma$ is the identity relation.

Proof

(i) \Rightarrow (ii): Suppose first that E_S is left unitary and that e and se are both in E_S . By 3.0.29, $se = fs$ for some $f \in E_S$. Hence $s \in E_S$ by assumption.

¹Compare to ([3], Theorem 5.9.1) and ([5], Theorem 2.4.3)

(ii) \Rightarrow (iii): Suppose that E_S is right unitary and that $e \leq s$ for an idempotent e . Then $se = e \in E_S$ and thus $s \in E_S$ as required.

(iii) \Rightarrow (iv): Suppose that every element greater than an idempotent is an idempotent and let a, b in S such that $a \mathcal{L} b$, $a \sigma b$. Then $a^{-1}a = b^{-1}b$ because a and b are \mathcal{L} -equivalent. Since $a \sigma b$, there exists an element $c \in S$ so that $c \leq a, b$. Consequently $ea = c = fb$ for certain idempotents e, f . Now we have

$$E_S \ni cc^{-1} = ea(fb)^{-1} = e(ab^{-1})f \leq ab^{-1}.$$

Hence $ab^{-1} \in E_S$. It follows that

$$a = aa^{-1}a = ab^{-1}b = (ab^{-1})^{-1}(ab^{-1})b = b(a^{-1}a)b^{-1}b = b(b^{-1}b)b^{-1}b = b.$$

(iv) \Rightarrow (i): Suppose that $e, es \in E_S$. Note that $s \mathcal{L} s^{-1}s$. Let i be any idempotent. Clearly, $(es)i \leq i$ and $(es)i \leq si \leq s$. Hence $\forall i \in E_S: s \sigma i$. In particular we have $s \sigma s^{-1}s$. Combining both considerations gives $(s, s^{-1}s) \in \mathcal{L} \cap \sigma$. By assumption this yields $s = s^{-1}s$ and thus $s \in E_S$.

□

Definition 6.0.13 *We call an inverse semigroup E-unitary if it satisfies one (and thus all) of the properties in 6.0.12.*

An interesting feature of E-unitary inverse semigroups is that their compatibility relations are congruences. More specifically we have

Theorem 6.0.14 ([5], Theorem 2.4.4) *Let S be an inverse semigroup. Then the compatibility relation is transitive if and only if S is E-unitary.*

Proof Suppose that \sim is transitive and that $e \leq s$ for an idempotent e and some $s \in S$. Clearly, $se^{-1} = se = e \in E_S$. Moreover we have $s^{-1}e \leq s^{-1}s$ and so also $s^{-1}e \in E_S$. Hence $s \sim e$. On the other hand we have $e \sim s^{-1}s$ since both e and $s^{-1}s$ are idempotents. Thus $s \sim s^{-1}s$ by our assumption that \sim is transitive. It follows that S is E-unitary, since $s = ss^{-1}s = s(s^{-1}s)^{-1} \in E_S$.

Suppose conversely that S is E-unitary and that $s \sim t$ and $t \sim u$. By assumption $s^{-1}t$ and $t^{-1}u$ are idempotents and therefore $(s^{-1}t)(t^{-1}u)$ is one as well. Now,

$$(s^{-1}t)(t^{-1}u) = s^{-1}(tt^{-1})u \leq s^{-1}u$$

and similarly

$$E_S \ni (st^{-1})(tu^{-1}) \leq su^{-1}.$$

But we assumed S to be E-unitary and so both $s^{-1}u$ and su^{-1} are idempotents. Hence $s \sim u$ and \sim is transitive.

□

Corollary 6.0.15 *Let S be an E-unitary inverse semigroup. Then the compatibility relation \sim is a congruence.*

Proof The compatibility relation is apparently reflexive and symmetric. By virtue of 6.0.14 it is also transitive. In addition to that, it is compatible with multiplication by 3.4.2 and thus a congruence.

□

Definition 6.0.16 *Let S and T denote inverse semigroups. We say that S is a cover of T if there exists a surjective morphism from S onto T .*

Definition 6.0.17 *Let S and T be inverse semigroups. A morphism $\theta: S \rightarrow T$ is said to be idempotent-separating if $\ker(\theta)$ is idempotent-separating (in the sense of 3.5.5).*

The theorem below is known as the McAlister covering theorem. It states that every inverse semigroup has an E-unitary cover. More precisely we have

Theorem 6.0.18 ([5], Theorem 2.2.4) *For every (finite) inverse semigroup S there is a (finite) E-unitary inverse semigroup P and a surjective, idempotent-separating morphism θ from P onto S .*

The McAlister covering theorem gives a first insight into the importance of E-unitary semigroups. Their class is large enough to comprise all inverse semigroups as idempotent-separating morphic images. A related result is mentioned in chapter 7. The proof is omitted.

Definition 6.0.19 *A congruence ρ on an inverse semigroup S is said to be idempotent-pure if $e \in E_S, (e, s) \in \rho \implies s \in E_S$.*

Lemma 6.0.20 ([5], Proposition 2.4.5) *Let S be an inverse semigroup. Then a congruence ρ is idempotent-pure if and only if $\rho \subseteq \sim$.*

Proof Suppose that ρ is an idempotent-pure congruence and consider $a, b \in S$ so that $a \rho b$. Then also $ab^{-1} \rho bb^{-1}$ and ab^{-1} is an idempotent, since bb^{-1} is one. We may similarly show that $a^{-1}b$ is an idempotent and so $a \sim b$. Hence $\rho \subseteq \sim$. Conversely, suppose that $\rho \subseteq \sim$ for a congruence ρ . Suppose further that $e \rho s$, where e is an idempotent. Then by 3.0.27, $e^{-1} \rho s^{-1}$ and thus $e = e^{-1}e \rho s^{-1}s$. Now ρ is in particular transitive and we have therefore $s \rho s^{-1}s$. By assumption $\rho \subseteq \sim$ and thus $s \sim s^{-1}s$. But then $s = ss^{-1}s = s(s^{-1}s)^{-1} \in E_S$. As a result ρ is idempotent-pure.

□

The next theorem provides quite a few alternative characterisations of E-unitary inverse semigroups. Properties (v) and (vi) are needed in chapter 7.

Theorem 6.0.21 ² *Let S be an inverse semigroup. Then the following conditions are equivalent:*

²Compare to ([5], Theorem 2.4.6)

(i) S is *E*-unitary.

(ii) $\sim = \sigma$.

(iii) σ is idempotent-pure.

(iv) $\forall e \in E_S: \sigma(e) = E_S$.

(v) HH^{-1} is an ideal of E_S for every σ -class H .

(vi) HeH^{-1} is an ideal of E_S for every $e \in E_S$ and every σ -class H .

Proof

(i) \Rightarrow (ii): By 3.5.2(i) we already know that $\sim \subseteq \sigma$. For the other inclusion let $a \sigma b$, i.e. $\exists x \in S: x \leq a, b$. Hence $ea = x = fb$ for some idempotents e, f and

$$E_S \ni xx^{-1} = ea(fb)^{-1} = e(ab^{-1})f \leq ab^{-1}.$$

But S is *E*-unitary and so $ab^{-1} \in E_S$. We may similarly show that $a^{-1}b \in E_S$ and thus $a \sim b$.

(ii) \Rightarrow (iii): This follows immediately from 6.0.20.

(iii) \Rightarrow (iv): By the definition of an idempotent-pure congruence the σ -class $\sigma(e)$ consists of idempotents only. On the other hand it contains all of them since idempotents are always σ -equivalent. We conclude that $\sigma(e) = E_S$ for every $e \in E_S$.

(iv) \Rightarrow (v): Let $a \sigma b$. Then $ab^{-1} \sigma bb^{-1}$. It follows that ab^{-1} is an idempotent by assumption. Thus $HH^{-1} \subseteq E_S$. Let $e \in E_S$. Then

$$eab^{-1} = ab^{-1}e = a(eb)^{-1} \in HH^{-1},$$

since $b \sigma eb$. Hence HH^{-1} is indeed an ideal of E_S .

(v) \Rightarrow (vi): Let $s, t \in H$ and $e \in E_S$. Then by the hypothesis $st^{-1} \in E_S$ and thus, using 3.0.29,

$$set^{-1} = st^{-1}e' \in E_S.$$

It follows that $HeH^{-1} \subseteq E_S$. For all idempotents f we have $f(set^{-1}) = (fs)et^{-1} \in HeH^{-1}$ since $fs \sigma s$. Hence HeH^{-1} is an ideal of E_S for every $e \in E_S$.

(vi) \Rightarrow (i): Let $ae, e \in E_S$. Clearly we have $a \sigma ae$. Also by $ae = (ae)e$ we have $ae \leq e$ and thus $ae \sigma e$. Hence $a \sigma e$ and so $a \sigma e = e^{-1}e \sigma a^{-1}a$. Putting $H = \sigma(a)$ we get

$$a = a(a^{-1}a)(a^{-1}a)^{-1} \in \sigma(a)(a^{-1}a)(\sigma(a))^{-1} = H(a^{-1}a)H^{-1} \subseteq E_S$$

by assumption. It follows that S is *E*-unitary.

□

6.1 Examples of E-unitary inverse semigroups

Example 6.1.1 Consider the bicyclic semigroup B already encountered in 3.1.4. We are going to show that it is an E-unitary semigroup. Before we do that we have to locate the idempotents and describe the natural partial order on B . Accordingly, suppose that (m, n) is an idempotent in B . Then

$$(m, n) = (m, n)(m, n) = (m - n + \max(m, n), n - m + \max(m, n)).$$

Comparing left and right coordinates of the respective left and right-hand sides yields $m = \max(m, n) = n$. Conversely it is immediate that every element $(m, m) \in B$ is an idempotent of B . Hence $E_B = \{(m, m) : m \in \mathbb{N}\}$.

Next we describe the natural partial order. Suppose that $(m, n) \leq (p, q)$ with $(m, n), (p, q) \in B$. By definition $(m, n) = (p, q)(s, s)$ for some $s \in \mathbb{N}$. The right-hand side multiplies to $(p - q + \max(q, s), \max(q, s))$. Thus $m = p - q + \max(q, s)$ and $n = \max(q, s)$. The latter implies in particular that $n \geq q$. Substituting n for $\max(q, s)$ in the former equation gives $m = p - q + n$ or to express it differently, $m - n = p - q$. We conclude that $m - n = p - q$ and $n \geq q$ are necessary conditions for $(m, n) \leq (p, q)$. This conditions are also sufficient. Suppose that $m - n = p - q$ and $n \geq q$. Then

$$(p, q)(n, n) = (p - q + \max(q, n), \max(q, n)) = (p - q + n, n) = (m, n)$$

and therefore $(m, n) \leq (p, q)$.

Now we can show that B is E-unitary. Let $(s, s) \leq (m, n)$. Then by the above we have in particular $s - s = m - n$ and hence $m = n$. Consequently, $(m, n) = (m, m)$ is an idempotent and B is E-unitary.

Corollary 6.1.2 Recall B_d , the inverse semigroup we already encountered in (5.3.5). We have seen there that it is an inverse subsemigroup of the E-unitary inverse semigroup B . As such it is E-unitary too.

Our next example features semidirect products of semilattices by groups³.

Proposition 6.1.3 ⁴ Let the group G act on the semilattice E by order automorphisms. Then the semidirect product of E by G is an E-unitary inverse semigroup.

Proof We start by showing that the semidirect product is inverse. Note that

$$\forall e \in E \forall g \in G: (e, g)(g^{-1}e, g^{-1})(e, g) = (e, g).$$

Hence $E \rtimes G$ is regular. It is straightforward to check that the idempotents are the elements of the form $(e, 1)$, where 1 is the identity element of G and e is any element of E . By

$$(e, 1)(f, 1) = (e \wedge f, 1) = (f \wedge e, 1) = (f, 1)(e, 1),$$

³For a definition of these see 2.5.2

⁴Compare to ([5], Theorem 7.1.1)

idempotents do commute.

Suppose now that $(e, 1) \leq (f, g)$. Then by 3.2.2(iv),

$$(e, 1) = (e, 1)(f, g).$$

The right-hand side multiplies to $(e \wedge f, g)$ which yields $g = 1$. Hence $(f, g) = (f, 1)$ is an idempotent and $E \times G$ is indeed E-unitary.

□

Corollary 6.1.4 *A direct product of a semilattice and a group is E-unitary.*

Proof Just notice that a direct product of a group and a semilattice can be interpreted as a special case of a semidirect product of a semilattice by a group by assuming that each group element acts as the identity mapping.

□

Remark 6.1.5 *Inverse subsemigroups of semidirect products of semilattices by groups are also E-unitary. We omit the proof of this and refer the reader to [5].*

In fact one can say exactly which E-unitary inverse semigroups are semidirect products of semilattices by groups:

Proposition 6.1.6 ([5], Theorem 7.1.2) *Let S be an inverse semigroup. Then S is isomorphic to a semidirect product of a semilattice by a group if and only if S is E-unitary and for each $a \in S, e \in E_S$ there exists $b \in S$ such that $a \sim b$ and $b^{-1}b = e$.*

Definition 6.1.7 *Let X be a non-empty set. An inverse semigroup $FIS(X)$, equipped with a function $\iota: X \rightarrow FIS(X)$, is said to be a free inverse semigroup on X if for every inverse semigroup S and function $\kappa: X \rightarrow S$ there exists a unique morphism $\theta: FIS(X) \rightarrow S$ such that $\theta \circ \iota = \kappa$. We may likewise define free inverse monoids.*

Free inverse semigroups are prominent examples of E-unitary semigroups. A proof of a stronger version of the following proposition can be found in ([5], Theorem 6.3.3).

Proposition 6.1.8 *Free inverse semigroups are E-unitary. Furthermore, each non-idempotent σ -class contains a maximum element.*

We discussed primitive inverse semigroups in chapter 4. It is an easy task to determine which of these are E-unitary. An E-unitary primitive inverse semigroup S has a zero element by definition and we know that a zero element is always the smallest element of an inverse semigroup with respect to the natural partial order. This implies that S consists only of idempotents because S is E-unitary.

It is thus a semilattice such that $e \wedge f = 0$ whenever $e \neq f$. Conversely, it is obvious that all such semilattices are E-unitary primitive inverse semigroups. Recall the Bruck-Reilly extension⁵ $BR(T, \theta)$, where T is a monoid and θ is a morphism from T into its group of units. The next proposition establishes a connection between E-unitariness of T and $BR(T, \theta)$.

Proposition 6.1.9 ([3], Exercise 5.20) *Let T be an inverse monoid and let θ be an endomorphism into the group of units of T . Then the Bruck-Reilly extension $S = BR(T, \theta)$ is E-unitary if and only if T is E-unitary and $\sigma = \ker(\theta)$.*

Proof Suppose first that $BR(T, \theta)$ is E-unitary. From 5.2.10(iii) we know that the idempotents of S are the elements of the form (m, e, m) , where e is an idempotent in T .

Let $e, ae \in E_T$. Then $(0, e, 0)$ and $(0, ae, 0)$ are idempotents in S . But $(0, ae, 0) = (0, a, 0)(0, e, 0) \leq (0, a, 0)$ and so $(0, a, 0) \in E_S$ by assumption. This implies in particular that $a \in E_T$. Hence T is E-unitary.

Let $a, b \in T$ such that $a \sigma b$. Then $ae = bf$ for certain $e, f \in E_T$. Since idempotents are mapped to idempotents and since the only idempotent in the group of units is the identity element, we get

$$\theta(a) = \theta(a)\theta(e) = \theta(ae) = \theta(bf) = \theta(b)\theta(f) = \theta(b).$$

Hence $\sigma \subseteq \ker(\theta)$.

To show the reverse inclusion let $a, b \in \ker(\theta)$, i.e. $\theta(a) = \theta(b)$. Then

$$\begin{aligned} (1, 1, 1)(0, a, 0)(0, b^{-1}, 0)(1, 1, 1) &= (1, \theta(a), 1)(1, \theta(b^{-1}), 1) = \\ &= (1, \theta(a)\theta(b)^{-1}, 1) = (1, 1, 1). \end{aligned}$$

Hence, since $BR(T, \theta)$ is E-unitary, $(0, ab^{-1}, 0) = (0, a, 0)(0, b^{-1}, 0)$ is an idempotent. This implies that $ab^{-1} \in E_T$. We may similarly show that $a^{-1}b$ is also an idempotent and we thus have $a \sim b$. But T is E-unitary and thus, by 6.0.21(ii), $a \sigma b$. As a result, $\ker(\theta) = \sigma$.

Conversely, suppose that T is an E-unitary inverse semigroup and $\ker(\theta) = \sigma$. Let $(p, a, q) \geq (m, e, m)$. We have to show that (p, a, q) is idempotent. By assumption we have

$$(m, e, m) = (p, a, q)(m, e, m) = (p - q + t, \theta^{t-q}(a)\theta^{t-m}(e), t),$$

where $t := \max(q, m)$. By comparing left and right-hand sides we obtain $p = q$ and $e = \theta^{t-q}(a)\theta^{t-m}(e)$. Now $\theta^{t-m}(e)$ is an idempotent since idempotents are mapped to idempotents by morphisms. Hence $\theta^{t-q}(a)$ is an idempotent because T was assumed to be E-unitary. If $m \leq q$ then $\theta^{t-q}(a) = a$ and we are finished. If on the other hand $m > q$, then we have

$$\theta^{t-q}(a) = \theta^{m-q}(a) = \theta^{m-q}(a)^2 = \theta^{m-q}(a^2),$$

where $m - q$ is a positive integer. By assumption we conclude that

$$\theta^{m-q-1}(a) \sigma \theta^{m-q-1}(a^2) = \theta^{m-q-1}(a)^2.$$

⁵For a definition see 5.2.8

This implies that

$$\theta^{m-q-1}(a) = \theta^{m-q-1}(a)^2,$$

since in an E-unitary inverse semigroup we have

Lemma 6.1.10 $x \sigma x^2 \implies x = x^2$.

Proof The assumption is equivalent to $x \sim x^2$. In particular, $x^{-1}x^2 = (x^{-1}x)x \leq x$ is an idempotent and thus x is one too. Hence $x = x^2$. □

We may repeat the argument and get $\theta^{m-q-2}(a) \sigma \theta^{m-q-2}(a^2)$ from which we deduce $\theta^{m-q-2}(a) = \theta^{m-q-2}(a)^2$ using the lemma. Descending further and further we finally arrive at $a = a^2$. Combined with $p = q$ from above this implies that $BR(T, \theta)$ is indeed E-unitary. □

As an application of the previous proposition we can easily characterise E-unitary bisimple inverse ω -semigroups. Remember that we have proved in 5.2.11 that every bisimple inverse ω -semigroup is isomorphic to some $BR(T, \theta)$ for a group T .

Theorem 6.1.11 *A bisimple inverse ω -semigroup is E-unitary if and only if θ is one-one.*

Proof We merely have to check the conditions in 6.1.9. This is quickly done: Every group is E-unitary with the σ -relation being the equality relation due to 3.2.6. □

Our next goal is to characterise the E-unitary simple inverse ω -semigroups. In 5.3.9 we proved that each simple inverse ω -semigroup is isomorphic to some Bruck-Reilly extension $BR(T, \theta)$, where T is a finite chain of groups of length d and θ is a morphism from T into its group of units. Within T the multiplication is defined via morphism $\gamma_i: G_i \rightarrow G_{i+1}$ ($i = 0, \dots, d-2$). With the help of 6.1.9 we can say exactly when $BR(T, \theta)$ is E-unitary, namely when T is E-unitary and $\sigma_T = \ker(\theta)$. In order to obtain a more elegant criterion we formulate a proposition that tells us when a Clifford semigroup is E-unitary—which is also of independent interest—and investigate the σ -relation on finite chains of groups. Recall that each Clifford semigroup is (up to isomorphism) a strong semilattice of groups (see 3.1.11).

Theorem 6.1.12 ([3], Exercise 5.20) *Let $S = S(E; G_e; \phi_{e,f})$ be a strong semilattice of groups, i.e. a Clifford semigroup.*

- (i) *S is E-unitary if and only if the connecting morphisms $\phi_{e,f}$ ($e, f \in E$, $e \geq f$) are all one-one.*

$$(ii) \ a \sigma b \iff \exists l \in E: \phi_{e,l}(a) = \phi_{f,l}(b) \quad (a \in G_e, b \in G_f).$$

Proof

(i): First notice that the idempotents of S are the group identities, since for a, b in the same group, $a \otimes b = ab$, where juxtaposition on the right denotes multiplication in the group. Let $a \geq b$ for $a \in G_e, b \in G_f$. Then, by definition,

$$b = a1_l = \phi_{e,el}(a)\phi_{l,el}(1_l) \quad (6.1.1)$$

for some $l \in E$ with 1_l the identity element of G_l . The right-hand side simplifies to $\phi_{e,el}(a)$, since identities are mapped to identities by group morphisms and $\phi_{e,el}(a) \in G_{el}$. The codomain of $\phi_{e,el}$ is G_{el} whereas $b \in G_f$. Thus $f = el$ and consequently $e \geq f$. As a result, (6.1.1) becomes

$$b = \phi_{e,f}(a). \quad (6.1.2)$$

On the other hand it is easy to see that (6.1.2) implies $a \geq b$. We summarise

$$a \geq b \iff e \geq f, \phi_{e,f}(a) = b \quad (a \in G_e, b \in G_f). \quad (6.1.3)$$

Suppose now that S is E-unitary and fix elements $e, f \in E$ such that $e \geq f$. By (6.1.3) we get in this case

$$a \geq 1_f \iff \phi_{e,f}(a) = 1_f. \quad (6.1.4)$$

By assumption $a \in G_e$ is idempotent and therefore $a = 1_e$. We have shown that the only element in G_e which is mapped to 1_f by $\phi_{e,f}$ is 1_e , i.e. $\ker(\phi_{e,f}) = \{1_e\}$ (here of course \ker means the notion for groups). Hence $\phi_{e,f}$ is one-one.

Conversely, suppose that $\phi_{e,f}$ is one-one for every pair $e, f \in E$ such that $e \geq f$. Let $a \geq 1_f$ for an idempotent 1_f with $a \in G_e$ (say). By (6.1.4) this is equivalent to $\phi_{e,f}(a) = 1_f$. But $\phi_{e,f}(1_e) = 1_f$ and $\phi_{e,f}$ is one-one by assumption. We conclude that $a = 1_e$. Hence a is an idempotent and S is E-unitary, as required.

(ii): Suppose that $a \sigma b$ for $a \in G_e, b \in G_f$. Then there exists $x \in S$ such that $a \geq x$ and $b \geq x$ for some $x \in G_l$. By (6.1.3) this yields $e, f \geq l$ and $\phi_{e,l}(a) = \phi_{f,l}(b)$.

If on the other hand there exists an element $l \in E$ such that $\phi_{e,l}(a) = \phi_{f,l}(b)$, then clearly $a1_l = b1_l$ and hence $a \sigma b$.

□

Theorem 6.1.13 *With the notation used in 5.3.8, a simple inverse ω -semigroup $BR(T, \theta)$ is E-unitary if and only if γ_i is one-one for all $i \in \{0, \dots, d-2\}$ and $\theta(a) = \theta(b) \iff \alpha_{j,k}(a) = b \quad (a \in G_j, b \in G_k, j \leq k)$.*

Proof By 6.1.9 $S = BR(T, \theta)$ is E-unitary if and only if T is E-unitary and $\ker(\theta) = \sigma_T$. From 6.1.12 it follows that this is the case exactly when all connecting morphisms are one-one and $\theta(a) = \theta(b) \iff \exists l \geq j, k: \alpha_{j,l}(a) = \alpha_{k,l}(b) \quad (a \in G_j, b \in G_k)$. But $S = BR(T, \theta)$ is not just any Clifford semigroup.

It is a finite chain of groups. By our definition of the connecting morphisms α , the first condition is satisfied if γ_i is one-one for $i \in \{0, \dots, d-2\}$. The second condition simplifies as well. Suppose without loss of generality that $a \in G_j, b \in G_k$, with $j \leq k$. Then $\alpha_{j,l}(a) = \alpha_{k,l}(b)$. Since the morphisms are one-one this implies $\alpha_{j,l-1}(a) = \alpha_{k,l-1}(b)$ from which it follows that $\alpha_{j,l-2}(a) = \alpha_{k,l-2}(b)$ and so on until we get $\alpha_{j,k}(a) = \alpha_{k,k}(b) = b$. This completes the proof.

□

The next proposition gives a further example for an E-unitary inverse semigroup.

Proposition 6.1.14 ([5], Proposition 7.1.3) *Let S be an inverse semigroup in which each σ -class contains a maximum element. Then S is an E-unitary inverse monoid.*

Proof Let e be any idempotent of S . Then

$$\sigma(e) = \sigma(ee) = \sigma(e)\sigma(e)$$

and so $\sigma(e)$ is the identity element of the group S/σ . Hence $E_S \subseteq \sigma(e)$. By assumption $\sigma(e)$ contains a maximum element denoted by i . We have seen that i is larger than every idempotent and we shall show that it is an idempotent itself. Clearly $i^{-1}i\sigma e^{-1}e = e$ and therefore $i^{-1}i \leq i$ since i is the maximum element. Multiplying on the left with i yields $i \leq i^2$. On the other hand $i^2\sigma ee = e$ and so $i^2 \leq i$. It follows that $i = i^2$. As a result all elements of $\sigma(e)$ are idempotents since i is the maximum element. Combining this with the observation above we obtain $\sigma(e) = E_S$. Thus S is E-unitary by 6.0.21.

At last we show that i is the identity element of S . For every $s \in S$ we have

$$is = i(ss^{-1})s = (ss^{-1})s = s$$

and

$$si = s(s^{-1}s)i = s(s^{-1}s) = s.$$

To sum it up, S is an E-unitary inverse monoid, as required.

□

This motivates the following

Definition 6.1.15 *An F-inverse semigroup is an inverse semigroup in which every σ -class has a greatest element.*

Keep in mind that an F-inverse semigroup is always a monoid. We will embark on F-inverse semigroups in detail in chapter 7.

6.2 The P-theorem

We now present a recipe due to McAllister for constructing E-unitary inverse semigroups. The strong point of the seemingly arbitrary construction becomes apparent when we show that the E-unitary inverse semigroups constructed in such a way constitute in fact all of them.

Construction 6.2.1 Let \mathcal{X} be a poset and let \mathcal{Y} be a subset of \mathcal{X} . Let G be a group which acts on \mathcal{X} by order automorphisms⁶. Suppose that the triple $(G, \mathcal{X}, \mathcal{Y})$ satisfies the following properties:

(P1) \mathcal{Y} is a semilattice with respect to \leq .

(P2) \mathcal{Y} is an order ideal, i.e.

$$A \in \mathcal{Y}, X \leq A \implies X \in \mathcal{Y}.$$

(P3) $G\mathcal{Y} = \mathcal{X}$, i.e. $\forall X \in \mathcal{X} \exists g \in G, A \in \mathcal{Y}: gA = X$.

(P4) $g \in G \implies g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$, where $g\mathcal{Y}$ denotes the set $\{gY : Y \in \mathcal{Y}\}$.

Definition 6.2.2 A triple $(G, \mathcal{X}, \mathcal{Y})$ having the properties (P1)–(P4) is called a McAlister triple.

We can use McAlister triples to define a particularly interesting semigroup:

Proposition 6.2.3 Let $(G, \mathcal{X}, \mathcal{Y})$ be a McAlister triple. Then the set

$$S = \mathcal{P}(G, \mathcal{X}, \mathcal{Y}) := \{(A, g) \in \mathcal{Y} \times G : g^{-1}A \in \mathcal{Y}\} \quad (6.2.1)$$

becomes an E-unitary inverse semigroup under the multiplication rule

$$(A, g)(B, h) = (A \wedge gB, gh). \quad (6.2.2)$$

Proof We start by showing that the multiplication is well-defined. Note first that $g^{-1}A \wedge B$ exists since both $g^{-1}A$ and B are elements of the semilattice \mathcal{Y} . Applying 2.4.3 yields that $A \wedge gB = g(g^{-1}A \wedge B)$ also exists. It is in \mathcal{Y} since $A \wedge gB \leq A \in \mathcal{Y}$ and \mathcal{Y} is an order ideal. Also using 2.4.3 we get

$$(gh)^{-1}(A \wedge gB) = h^{-1}g^{-1}(A \wedge gB) = h^{-1}g^{-1}A \wedge h^{-1}B \leq h^{-1}B \in \mathcal{Y}$$

and thus $(gh)^{-1}(A \wedge gB) \in \mathcal{Y}$, since \mathcal{Y} is an order ideal. We have shown that $(A, g)(B, h)$ is indeed in $S = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$.

Next we show that the multiplication is associative. Accordingly, let $(A, g), (B, h), (C, i) \in S$. It suffices to verify that the first coordinates of the products $((A, g)(B, h))(C, i)$ and $(A, g)((B, h)(C, i))$ agree. The left-hand side of the first term computes to $(A \wedge gB) \wedge ghC$, whereas the left-hand side of the second comes down to $A \wedge g(B \wedge hC)$. This is of course equal and S is therefore a semigroup.

⁶For a definition see 2.4.2

It is also inverse. To this end we check that it is regular and that idempotents commute. By definition, $(A, g) \in S \iff (g^{-1}A, g^{-1}) \in S$. Now by

$$(A, g) (g^{-1}A, g^{-1}) (A, g) = (A \wedge \underbrace{gg^{-1}}_{=1} A, 1)(A, g) = (A \wedge 1A, 1g) = (A, g)$$

and

$$(g^{-1}A, g^{-1}) (A, g) (g^{-1}A, g^{-1}) = (g^{-1}A, 1) (g^{-1}A, g^{-1}) = (g^{-1}A, g^{-1})$$

we see that S is regular. To show that idempotents commute we must first identify them:

$$(A, g) = (A, g)^2 = (A \wedge gA, g^2) \implies g = 1.$$

Conversely, every element of the form $(A, 1)$ is idempotent. It follows that

$$(A, 1)(B, 1) = (A \wedge 1B, 1) = (A \wedge B, 1) = (B \wedge A, 1) = (B, 1)(A, 1)$$

and thus idempotents do indeed commute. We have shown that S is an inverse semigroup.

It remains to show that S is E-unitary. In view of that suppose that $(A, 1) \leq (B, g)$ which means that there exists an idempotent $(C, 1)$ such that $(A, 1) = (B, g)(C, 1)$. But $(B, g)(C, 1) = (B \wedge gC, g)$ and we have in particular $g = 1$. In effect $(B, g) = (B, 1)$ is an idempotent and S is indeed E-unitary.

□

Definition 6.2.4 *Semigroups of the form $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$, where $(G, \mathcal{X}, \mathcal{Y})$ is a McAlister triple, are called P-semigroups.*

We have proved that the semigroup S as defined above is an E-unitary inverse semigroup. This provides some justification for the expensive and seemingly arbitrary construction of S . The next theorem however does even more so:

Theorem 6.2.5 (*[3], Theorem 5.9.2*) *Let $(G, \mathcal{X}, \mathcal{Y})$ be a McAlister triple. Then $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ is an E-unitary inverse semigroup. Conversely, every E-unitary inverse semigroup is isomorphic to a P-semigroup.*

Proof We have already proved the easy half. The converse requires considerably more effort. Let S be an E-unitary inverse semigroup. We have to find a McAlister triple $(G, \mathcal{X}, \mathcal{Y})$ such that $S \cong \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$. Put $G := S/\sigma$ and for each $s \in S$ define s^0 in $E_S \times G$ by

$$s^0 := (s^{-1}s, \sigma(s)).$$

If T is a subset of S , then by T^0 we mean $\{t^0 : t \in T\}$. Notice that by virtue of 6.0.12(iv),

$$s^0 = t^0 \implies s = t. \tag{6.2.3}$$

Recall that a principal right ideal of S is a set of the form aS for some $a \in S$. By the very definition of the Green's relation \mathcal{R} we have $aS = bS \iff a \mathcal{R} b$. By 3.0.21 (iv) every principal right ideal can thus be uniquely written as eS with $e \in E_S$. As a result the set \mathfrak{R} of all principal right ideals of S is in one-one correspondence with E_S and so is

$$\mathcal{Y} := \{A^0 : A \in \mathfrak{R}\}$$

by (6.2.3). The correspondence is even a semilattice isomorphism between E_S and (\mathcal{Y}, \cap) , since for $e, f \in E_S$ we have $(efS)^0 = (eS)^0 \cap (fS)^0 \iff efS = eS \cap fS$ by (6.2.3), where the right-hand side holds because of 3.0.28.

We define an action of G on $E_S \times G$ by

$$g(e, h) := (e, gh). \quad (6.2.4)$$

More generally by gA^0 ($g \in G, A^0 \in \mathcal{Y}$) we mean $\{g(e, h) : (e, h) \in A^0\}$. This enables us to define

$$\mathcal{X} := G\mathcal{Y} = \{gA^0 : g \in G, A^0 \in \mathcal{Y}\}.$$

Both \mathcal{Y} and \mathcal{X} are sets of subsets of $E \times G$ and partially ordered by set inclusion. Moreover we have $\mathcal{Y} \subseteq \mathcal{X}$.

Next, using (6.2.4), we define a group action of G on \mathcal{X} by

$$g(hA^0) := (gh)A^0 \quad (g \in G, hA^0 \in \mathcal{X}). \quad (6.2.5)$$

Of course we have to check that this definition is well-defined and does not depend on the particular representation of the element in \mathcal{X} . In view of that let $hA^0 = kB^0$ with $h, k \in G, A^0, B^0 \in \mathcal{Y}$. Then within \mathcal{Y} we have

$$A^0 = (h^{-1}k)B^0 = ((gh)^{-1}(gk))B^0$$

and therefore $(gh)A^0 = (gk)B^0$.

Now it is easy to verify the conditions in (2.4.1) and it follows that (6.2.5) defines a group action on \mathcal{X} . We have already shown that $(G, \mathcal{X}, \mathcal{Y})$ satisfies (P1) and (P3). Next we take care of (P2). We have to prove that \mathcal{Y} is an order ideal of \mathcal{X} . Accordingly, let $g(eS)^0$ be a typical element of \mathcal{X} and suppose that $g(eS)^0 \subseteq (fS)^0$ for some f in E . Among the elements of $(eS)^0$ is the element $(ee)^0 = e^0 = (e, \sigma(e)) = (e, 1)$, where 1 denotes the identity element of G . The last equality holds due to 3.5.3. Hence

$$(e, g) = g(e, 1) = u^0 = (u^{-1}u, \sigma(u))$$

for some u in fS . Comparinig left- and right-hand sides we get $g = \sigma(u)$ and $e = u^{-1}u$. This yields

$$\begin{aligned} g(eS)^0 &= \sigma(u) (u^{-1}uS)^0 = \sigma(u) \{(u^{-1}us)^0 : s \in S\} = \\ &= \sigma(u) \{(s^{-1}u^{-1}us, \sigma(u^{-1}us)) : s \in S\} = \{((us)^{-1}us, \underbrace{\sigma(uu^{-1}us)}_{=u}) : s \in S\} = (uS)^0. \end{aligned}$$

Hence $g(eS)^0 \in \mathcal{Y}$ and \mathcal{Y} is an order ideal of \mathcal{X} . This is (P2).

Eventually we deal with (P4). Let $g \in G$ and choose $s \in S$ such that $\sigma(s)^{-1} = g$. Then

$$\begin{aligned} g(sS)^0 &= \sigma(s)^{-1} \{((st)^{-1}(st), \sigma(st)) : t \in S\} = \{((st)^{-1}(st), \sigma(s^{-1}st)) : t \in S\} = \\ &= \left\{ \left((s^{-1}st)^{-1} (s^{-1}st), \sigma(s^{-1}st) \right) : t \in S \right\} = (s^{-1}sS)^0 \in \mathcal{Y}. \end{aligned}$$

Hence $g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$, i.e. (P4). Summarising, $(G, \mathcal{X}, \mathcal{Y})$ is a McAlister triple. It remains to show that $S \cong \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$. Accordingly, we define a map by

$$\begin{aligned} \phi: S &\rightarrow \mathcal{P}(G, \mathcal{X}, \mathcal{Y}), \\ s &\mapsto ((sS)^0, \sigma(s)). \end{aligned}$$

It is well-defined as can be seen from the verification of (P4). We claim that ϕ is the isomorphism we are looking for. First of all it is one-one, since for all $s, t \in S$,

$$\phi(s) = \phi(t) \implies (sS)^0 = (tS)^0, s\sigma t \implies sS = tS, s\sigma t \implies (s, t) \in \sigma \cap \mathcal{R} \implies s = t.$$

The last implication follows from 6.0.12(iv) (the appearance of \mathcal{R} instead of \mathcal{L} makes no difference). It is also onto:

Suppose that $((sS)^0, g) \in \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$. By definition, this is equivalent to $g^{-1}(sS)^0 \in \mathcal{Y}$. Hence there exists $t \in E_S$ such that $g^{-1}(sS)^0 = (tS)^0$. In particular,

$$(ss^{-1}, g^{-1}) \stackrel{3.5.3}{=} (ss^{-1}, g^{-1}\sigma(ss^{-1})) = g^{-1}(ss^{-1})^0 = u^0 = (u^{-1}u, \sigma(u))$$

for some $u \in tS$. Comparing left- and right-hand sides we see that $s\mathcal{R}u^{-1}$ and $g^{-1} = \sigma(u)$. Using 3.0.27 and our first definition of the Green's relation \mathcal{R} this implies

$$((sS)^0, g) = \left((u^{-1}S)^0, \sigma(u^{-1}) \right) = \phi(u^{-1}).$$

Hence ϕ is onto. The only thing left to show is the morphism property of ϕ . In $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ we have

$$((sS)^0, \sigma(s)) ((tS)^0, \sigma(t)) = ((sS)^0 \cap (\sigma(s))(tS)^0, \sigma(st))$$

and it suffices therefore to show that

$$(sS)^0 \cap (\sigma(s))(tS)^0 = (stS)^0.$$

Suppose first that $(e, g) \in (sS)^0 \cap (\sigma(s))(tS)^0$. Then

$$(e, g) = (su)^0 = \sigma(s)(tv)^0$$

for some $u, v \in S$. Applying $\sigma(s)^{-1}$, this yields $\sigma(s)^{-1}(su)^0 = (tv)^0$ and thus

$$\begin{aligned} (tv)^0 &= \sigma(s)^{-1} \left((su)^{-1}su, \sigma(su) \right) = \left((su)^{-1}su, \sigma(s^{-1})\sigma(su) \right) = \\ &= \left((s^{-1}su)^{-1} s^{-1}su, \sigma(s^{-1}su) \right) = (s^{-1}su)^0. \end{aligned}$$

By (6.2.3) we thus get $tv = s^{-1}su$ and consequently $stv = ss^{-1}su = su$. It follows that $(e, g) = (stv)^0 \in (stS)^0$ as required.

Conversely, suppose that $(e, g) \in (stS)^0$. Obviously $(e, g) \in (sS)^0$ and $(e, g) = (stu)^0$ for some u in S . Put $v := (st)^{-1}(st)u$. A straightforward calculation shows that $(tv)^{-1}(tv) = (stu)^{-1}(stu)$. Since $(st)^{-1}(st)u \leq u$ we also have $u\sigma v$ and therefore $stu\sigma stv$. Hence

$$(e, g) = (stu)^0 = ((stu)^{-1}(stu), \sigma(stu)) = ((tv)^{-1}(tv), \sigma(s)\sigma(tv)) = \sigma(s) ((tv)^{-1}(tv), \sigma(tv)) \in \sigma(s)(tS)^0.$$

Thus (e, g) is indeed in $(sS)^0 \cap (\sigma(s))(tS)^0$ and ϕ is the desired isomorphism: $S \cong \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$.

□

In 6.1.12(i) we characterised those Clifford semigroups which are also E-unitary taking a general strong semilattice of groups as our starting point. One can also do it the other way round and begin with a P-semigroup:

Proposition 6.2.6 ([7], Proposition VII.5.18) *An E-unitary inverse semigroup $S = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ is a Clifford semigroup if and only if the action of G on \mathcal{Y} is trivial, in the sense that $gA = A$ for every g in G and A in \mathcal{Y} .*

Remark 6.2.7 *In fact, the E-unitary P-semigroup appearing in the proposition above is the direct product of G and \mathcal{Y} . This can be seen from the definition of the set $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ in (6.2.1) and the fact that*

$$A, g^{-1}A \in \mathcal{Y} \iff A \in \mathcal{Y} \cap g\mathcal{Y} = \mathcal{Y} \cap \mathcal{Y} = \mathcal{Y}$$

in our case.

We conclude this chapter with a lemma that provides basic properties of the P-semigroup $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$.

Proposition 6.2.8 ⁷ *Let $S = \mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ be a P-semigroup and let $(A, g), (B, h) \in S$. Then we have the following basic properties:*

$$(i) \ E_S \cong \mathcal{Y}$$

$$(ii) \ (A, g) \leq (B, h) \iff A \leq B, g = h$$

$$(iii) \ (A, g)\sigma(B, h) \iff g = h$$

$$(iv) \ S/\sigma \cong G$$

$$(v) \ (A, g)^{-1} = (g^{-1}A, g^{-1})$$

⁷Compare to ([3], Proposition 5.9.4) and ([5], Theorem 7.1.1)

Proof

(i): Let (A, g) be an idempotent in S . Then $(A, g) = (A, g)^2 = (A \wedge gA, g^2)$. Hence it is necessary for an idempotent to have the identity 1 as its group coordinate. Conversely, it is clear that each such element in $\mathcal{P}(G, \mathcal{X}, \mathcal{Y})$ is an idempotent. Since the multiplication in E_S concerns therefore only the first coordinate we thus have an isomorphism from E_S onto \mathcal{Y} via $(A, 1) \mapsto A$.

(ii): Suppose that $(A, g) \leq (B, h)$. Then $(A, g) = (C, 1)(B, h) = (C \wedge B, h)$ for some C in \mathcal{Y} . Hence $A \leq B$ and $g = h$. If conversely $A \leq B$ and $g = h$, then $A = A \wedge B$ and thus $(A, g) = (A, 1)(B, g)$. That is $(A, g) \leq (B, g) = (B, h)$.

(iii): Suppose that $(A, g) \sigma (B, h)$. Then $(C, k) \leq (A, g), (B, h)$ for some (C, k) in S . By **(ii)** this implies $g = k = h$.

To establish the converse note that $(A \wedge B, g) \leq (A, g), (B, g)$, also by **(ii)**. Hence $(A, g) \sigma (B, g)$ for every pair $(A, g), (B, g) \in S$.

(iv): We have shown in **(iii)** that there is a one-one correspondence between σ -classes in S and elements in the group G . Since the multiplication in the second coordinate coincides with the one in G , the mapping $\{(A, g) \in \mathcal{P}(G, \mathcal{X}, \mathcal{Y})\} \mapsto g$ is the isomorphism we are looking for.

(v): Straightforward calculation shows that both $(A, g)(g^{-1}A, g^{-1})(A, g) = (A, g)$ and $(g^{-1}A, g^{-1})(A, g)(g^{-1}A, g^{-1}) = (g^{-1}A, g^{-1})$ hold. We know further that S is an inverse semigroup. Hence $(g^{-1}A, g^{-1})$ is the unique inverse of (A, g) .

□

7 F-inverse semigroups

This chapter is about F-inverse semigroups. We have already defined them in 6.1.15 and know that they are particular E-unitary inverse semigroups. Ultimately, we wish to establish a result analogous to 6.2.5 for F-inverse semigroups making use of the P-theorem and the fact that the semigroup is actually F-inverse. Before we embark on this task, we give criteria of F-inversibility for many of the semigroups featuring earlier in this thesis, including bisimple and simple inverse ω -semigroups and we list some F-inverse semigroups such as the bicyclic semigroup.

Theorem 7.0.9 ([5], Proposition 7.1.4) *Every E-unitary inverse semigroup S can be embedded in an F-inverse semigroup T such that T/σ_T is isomorphic to S/σ_S .*

Proof Recall that for an inverse semigroup S we defined $C(S)$ to be the inverse semigroup consisting of all permissible subsets¹ of S . Recall also that the idempotents of $C(S)$ are the order ideals of E_S . By virtue of 3.4.6 we have an embedding ι of S in $C(S)$ with $\iota(s) = [s]$. We are going to show that $C(S)$ is in fact the F-inverse monoid we are looking for.

We first characterise the natural partial order in $C(S)$. Let $A, B \in C(S)$. By 3.2.2, $A \leq B \iff A = AA^{-1}B$. From the proof of 3.4.6 we know that $AA^{-1} = \{aa^{-1} : a \in A\}$ consists entirely of idempotents. Hence every element of A is less or equal than an element in B . It follows that $A \subseteq B$ since B is an order ideal. Conversely, suppose that $A \subseteq B$. Then clearly $A = AA^{-1}A \subseteq AA^{-1}B$. To prove that the reverse inclusion also holds let $aa^{-1}b \in AA^{-1}B$. By assumption a and b are both elements of the compatible subset B and thus $a^{-1}b \in E_S$. Hence $a(a^{-1}b) \leq a$ and so $aa^{-1}b \in A$, since A is an order ideal. We have shown that in $C(S)$ the natural partial order coincides with subset inclusion.

Now we are able to show that $C(S)$ is F-inverse. We will use subscripts to avoid any ambiguity regarding the respective σ -relations. Let $s \in S$. Then the σ -class $\sigma(s)$, regarded as a set, is compatible, since for the E-unitary inverse semigroup S we have that $\sigma = \sim$. It is also an order ideal. To see this let $t \in \sigma(s)$ and $u \leq t$. Clearly $u \sigma t$, which together with $t \sigma s$ and the transitivity of σ yields $u \sigma s$. Hence $u \in \sigma(s)$ and $\sigma(s)$ is an order ideal. We prove that $\sigma(s)$ is the maximum element of its $\sigma_{C(S)}$ -class. Let $A \in C(S)$ such that $A \sigma_{C(S)} \sigma(s)$. Then $B \subseteq A, \sigma(s)$ for some permissible subset B . An element $b \in B$ is compatible both with every element in A and in $\sigma(s)$. But S is E-unitary and thus the compatibility relation is transitive. It follows that every element in A is compatible with s . In an E-unitary inverse semigroup we have $\sim = \sigma$ which yields $A \subseteq \sigma(s)$. Hence

¹For a definition see 3.4.5

$\sigma(s)$ is indeed the maximum element of its $\sigma_{C(S)}$ -class. On the other hand it is easy to see that every $\sigma_{C(S)}$ -class contains an element of the form $\sigma(s)$ for some $s \in S$. Thus $C(S)$ is *F*-inverse.

It remains to verify that S/σ is isomorphic to $C(S)/\sigma_{C(S)}$. We define a map θ by

$$\begin{aligned} \theta: S/\sigma &\rightarrow C(S)/\sigma_{C(S)} \\ \sigma(s) &\mapsto \sigma_{C(S)}(\sigma(s)). \end{aligned}$$

By the considerations above θ is both one-one and onto. To show the morphism property let $\sigma(s), \sigma(t) \in S/\sigma$. Then

$$\begin{aligned} \theta(\sigma(s))\theta(\sigma(t)) &= \sigma_{C(S)}(\sigma(s))\sigma_{C(S)}(\sigma(t)) = \sigma_{C(S)}(\sigma(s)\sigma(t)) = \sigma_{C(S)}(\sigma(st)) = \\ &= \theta(\sigma(st)). \end{aligned}$$

Thus θ is indeed an isomorphism and the proof is completed. □

The theorem states in a sense that the class of *F*-inverse semigroups, although strictly smaller than the class of *E*-unitary ones, is essentially as ‘big’.

7.1 Examples of *F*-inverse semigroups

Groups are of course *F*-inverse, since for them the σ -relation is the identity relation due to 3.2.6. Primitive inverse semigroups on the other hand are almost never *F*-inverse; the only primitive *F*-inverse semigroup is the two-element semigroup $(\{0, 1\}, \cdot)$. To see this assume that there exist distinct non-zero elements a and b . Obviously, a and b do lie in the same σ -class, since $0 \leq a, b$. The natural partial order is equality when restricted to non-zero elements due to 4.0.11 and so there is no maximum element in this σ -class (not even an upper bound for a and b). This means that if a primitive inverse semigroup S is to be *F*-inverse there can only be one non-zero element. This element must be the primitive idempotent and thus $S \cong (\{0, 1\}, \cdot)$.

Clifford semigroups are not *F*-inverse in general, even if they are assumed to be *E*-unitary and therefore have injective morphisms when regarded as a strong semilattice of groups $S(E; G_e; \phi_{e,f})$ (see 6.1.12(i)). Just consider the semilattice (\mathbb{N}, \leq) with copies of the trivial group $\{e\}$ attached to each element. Then, all elements are σ -equivalent, but there is no maximum.

In 6.1.12(ii) we saw that in a Clifford semigroup $S = S(E; G_e; \phi_{e,f})$,

$$a \sigma b \iff \exists l \in E: \phi_{e,l}(a) = \phi_{f,l}(b) \quad (a \in G_e, b \in G_f).$$

For an *E*-unitary Clifford semigroup, by 6.1.12(i), this translates to

$$a \sigma b \iff \phi_{e,e \wedge f}(a) = \phi_{f,e \wedge f}(b).$$

Hence S is *F*-inverse if, for every $a \in S$, there exists an element $i \in G_i$ with $i \geq e$, such that

$$(i) \quad \phi_{i,e}(i) = a \quad (a \in G_e),$$

$$(ii) \quad \phi_{x,i \wedge x}(x) = \phi_{i,i \wedge x}(i) \implies \phi_{i,x}(i) = x \quad (x \in G_x).$$

In the special case of a finite chain of groups $T = \bigcup_{i=0,1,\dots,d-1} G_i$ we only need to demand that the connecting morphisms are one-one. The reason is that we can only descend finitely many times in the natural partial order and the injective nature of the morphisms guarantees that our ‘way up’ (with respect to the order in T) is uniquely determined. The σ -classes are in fact chains. For $g \in G_0$ the corresponding σ -class is

$$g > g1_1 > g1_2 > \dots > g1_{d-1}.$$

By 6.1.12(i) and 6.1.14 this yields

Corollary 7.1.1 *A finite chain of groups is F -inverse if and only if it is E -unitary.*

Next we want to find out which bisimple inverse ω -semigroups are F -inverse.

Theorem 7.1.2 *A bisimple inverse ω -semigroup S is F -inverse if and only if it is E -unitary.*

Proof Thanks to 6.1.14 we only need to establish the converse half. Suppose that $S \cong BR(G, \theta)$ is E -unitary, where G is a group and θ is an endomorphism of G . By 6.1.9 this is equivalent to θ being one-one, since the σ -relation coincides with the identity relation for a group. Let $(m, a, n) \sigma (p, b, q)$. Without loss of generality we deduce from 5.2.10(v) that $m+t = p$, $n+t = q$ and $\theta^{k+t}(a) = \theta^k(b)$ for certain $k, t \in \mathbb{N}$. But θ is one-one and so $\theta^t(a) = b$. Hence

$$(m, a, n)(n+t, 1, n+t) = (p, b, q),$$

i.e. $(m, a, n) \geq (p, b, q)$. In effect the σ -classes are actually chains and these chains must terminate because an infinite regression is not possible in \mathbb{N} . Thus $BR(G, \theta)$ is F -inverse. □

The following corollary deals with the already familiar bicyclic semigroup.

Corollary 7.1.3 *The bicyclic semigroup B is F -inverse.*

Proof Consider the trivial group $G = \{e\}$. Then $BR(\{e\}, \theta) \cong B$, where—most trivially— θ is not only uniquely determined but also one-one. As shown in the preceding proof, $BR(\{e\}, \theta)$ is then E -unitary and thus F -inverse. □

Encouraged by the easy characterisation of F -inverse bisimple inverse ω -semigroups we investigate simple inverse ω -semigroups next. Although slightly more complicated to prove, the result is essentially the same for them:

Theorem 7.1.4 *A simple inverse ω -semigroup S is F -inverse if and only if it is E -unitary.*

Proof As for the bisimple case we need of course only to prove the converse half. Accordingly, suppose that $S \cong BR(T, \theta)$ is E -unitary, where $T = \bigcup_{i=0,1,\dots,d-1} G_i$ is a finite chain of groups and θ is a morphism from T into its group of units G_0 . As in the previous theorem we may use 6.1.9 to obtain more information. From this it follows that T is E -unitary and so the morphisms $\alpha_{i,j}$ are all one-one. Moreover, we know the kernel of θ to be σ_T .

We are now ready to investigate the σ -relation on S (we identify S with $BR(T, \theta)$). Suppose that $(m, a, n) \sigma_S (p, b, q)$, where $m \leq p$, $a \in G_i, b \in G_j$. Then 5.2.10(v) yields that $m - n = p - q$ and $\theta^{k+t}(a)e = \theta^k(b)f$, where $t := p - m$, $k \in \mathbb{N}$ and $e, f \in E_T$. Multiplying on the right with ef and using that idempotents commute we get

$$\theta^{k+t}(a)ef = \theta^k(b)ef.$$

The idempotent ef is a group identity for some group G_l , hence $ef = 1_l$. We assume without loss of generality that $k > 0$. Thus $\theta^{k+t}(a), \theta^k(b) \in G_0$. In effect, the above equation translates to

$$\alpha_{0,l}(\theta^{k+t}(a)) = \alpha_{0,l}(\theta^k(b)).$$

But $\alpha_{0,l}$ is one-one and so $\theta^{k+t}(a) = \theta^k(b)$. Now we make use of the fact that $\ker(\theta) = \sigma_T$ and conclude $\theta^{k+t-1}(a) \sigma \theta^{k-1}(b)$. We may repeat the above argument until we get

$$\theta^t(a) \sigma b. \tag{7.1.1}$$

If $t > 0$, then $\theta^t(a) \in G_0$ (and $b \in G_j$). From (7.1.1) we deduce

$$\alpha_{0,j}(\theta^t(a)) = \theta^t(a)1_j = b,$$

using again that the connecting morphisms are one-one. Hence $(m, a, n) \geq (p, b, q)$ by 5.2.10(v).

If on the other hand $t = 0$, then $m = p$ and $n = q$ and we may assume without loss of generality that $i \wedge j = j$. We get in this case

$$(m, a, n)(n, 1_j, n) = (m, \alpha_{i,j}(a), n) = (m, b, n) = (p, b, q)$$

and thus $(m, a, n) \geq (p, b, q)$. We have shown that for simple inverse ω -semigroups we also have that the σ -classes are chains. We can even write them down. There exist $|G_0|$ σ -classes. For $g \in G_0$ the corresponding σ -class is

$$(0, g, 0) > (0, g1_1, 0) > (0, g1_2, 0) > \dots > (0, g1_{d-1}, 0) > (1, \theta(g), 1) > (1, \theta(g)1_1, 1) > \dots > (1, \theta(g)1_{d-1}, 1) > (2, \theta^2(g), 2) > \dots > (2, \theta^2(g)1_{d-1}, 2) > \dots$$

In particular, we see that each σ -class has a maximum.

□

In 6.1.8 we saw that any free inverse semigroup is ‘almost’ F-inverse. The only thing that might prevent the semigroup from being F-inverse is the absence of an identity element. Indeed, if there is an identity element, it is plainly the maximum element of the σ -class of idempotents. We thus have

Proposition 7.1.5 *Free inverse monoids are F-inverse.*

Using the above result Lawson proved a modification of McAlister’s covering theorem.

Theorem 7.1.6 ([5], Theorem 7.5.7) *Every inverse semigroup has an F-inverse cover.*

Notice that in contrast to 6.0.18 we did not claim that there exists an idempotent-separating covering morphism.

7.2 A McAlister-like construction

We might hope that the construction in 6.2.1 allows some simplification for F-inverse semigroups. We shall determine the consequences of the additional structure of an F-inverse semigroup compared to an E-unitary inverse one on the sets \mathcal{X} and \mathcal{Y} featuring in the McAlister triple.

We start with a lemma:

Lemma 7.2.1 ([7], Lemma VII.5.2) *In an E-unitary inverse semigroup S we have for every σ -class H and every $e \in E_S$:*

$$(i) \ HeH^{-1} = \{heh^{-1} : h \in H\}.$$

$$(ii) \ HH^{-1} = \{hh^{-1} : h \in H\}.$$

Proof

Obviously it suffices to prove that $HeH^{-1} \subseteq \{heh^{-1} : h \in H\}$ and that $HH^{-1} \subseteq \{hh^{-1} : h \in H\}$ respectively.

(i): Let $a, b \in H$. Then $ab^{-1}, b^{-1}a \in E_S$, since in a E-unitary inverse semigroup the compatibility relation and the σ -relation coincide due to 6.0.21(ii). This implies that aeb^{-1} is an idempotent as well. Because of $a\sigma ab^{-1}b$ it follows that

$$\begin{aligned} aeb^{-1} &= (aeb^{-1})(aeb^{-1})^{-1} = aeb^{-1}bea^{-1} = aeb^{-1}bb^{-1}bea^{-1} = \\ &ab^{-1}beeb^{-1}ba^{-1} = (ab^{-1}b)e(ab^{-1}b)^{-1} \in \{heh^{-1} : h \in H\}. \end{aligned}$$

The proof of (ii) is similar.

□

The next lemma is more or less a consequence of 7.2.1. It tells us more about the behaviour of σ -classes in a P-semigroup.

Lemma 7.2.2 (*[7], Lemma VII.5.3*) Let $S = \mathcal{P}(G, X, Y)$ be a P -semigroup and let $e = (\alpha, 1)$ be an idempotent in S . Furthermore let $g \in G$ and denote by H the corresponding σ -class of S . Then

- (i) $H = \{(\beta, g) : g^{-1}\beta \in Y\}$,
- (ii) $HH^{-1} = \{(\beta, 1) : g^{-1}\beta \in Y\}$,
- (iii) $HeH^{-1} = \{(\beta, 1) : \beta \in [g\alpha] \cap Y\}$.

Proof

(i): This is the content of 6.2.8(iii).

(ii): From (i), 6.2.8(v) and 7.2.1(ii) we get

$$HH^{-1} = \{(\beta, g) (g^{-1}\beta, g^{-1}) : g^{-1}\beta \in Y\} = \{(\beta, 1) : g^{-1}\beta \in Y\}.$$

(iii): By 7.2.1(i) and 6.2.8(v) it follows that

$$HeH^{-1} = \{(\gamma, g)(\alpha, 1) (g^{-1}\gamma, g^{-1}) : g^{-1}\gamma \in Y\} = \{(\gamma \wedge g\alpha, 1) : g^{-1}\gamma \in Y\}.$$

Letting $\beta := \gamma \wedge g\alpha$ we obtain $\beta \in [g\alpha] \cap Y$ since $\gamma \in Y$ and Y an order ideal. Hence $\{(\gamma \wedge g\alpha, 1) : g^{-1}\gamma \in Y\} \subseteq \{(\beta, 1) : \beta \in [g\alpha] \cap Y\}$. Conversely, if $\beta \in [g\alpha] \cap Y$, we must find $\gamma \in Y$ such that $g^{-1}\gamma \in Y$ and $\beta = \gamma \wedge g\alpha$. It turns out that β itself does the job:

By assumption $\beta \leq g\alpha$. On the one hand this yields $\beta = \beta \wedge g\alpha$. On the other hand we get $g^{-1}\beta \leq \alpha$. Hence $g^{-1}\beta \in Y$ since Y is an order ideal and $\alpha \in Y$. It follows that $HeH^{-1} = \{(\beta, 1) : [g\alpha] \cap Y\}$, as required.

□

The following lemma will prove useful in our subsequent investigations.

Lemma 7.2.3 Let (X, \wedge) be a semilattice and let $Y \subseteq X$.

- (i) If Y is a principal ideal of X then it is also a retract ideal² of X .
- (ii) Y is a retract ideal of X if and only if $Y \cap [x]$ is a principal ideal of X for all x in X .

Proof

(i): By assumption $Y = [y]$ for some y in X . We need to show that Y is also a retract. Consider the mapping $\phi: X \rightarrow Y$, $x \mapsto x \wedge y$ for this y . This is well-defined since $y \in Y$ and Y is an ideal. It is immediate that $\phi(x) = x$ for all $x \in Y$. Also,

$$\phi(x \wedge z) = x \wedge z \wedge y = (x \wedge y) \wedge (z \wedge y) = \phi(x) \wedge \phi(z) \quad (x, z \in X).$$

Hence Y is a retract ideal.

²For a definition see 2.2.4

(ii): Suppose first that Y is a retract ideal of X with endomorphism $\phi: X \rightarrow Y$ such that $\forall y \in Y: \phi(y) = y$. Let x in X . We claim that $Y \cap [x] = [\phi(x)]$. Let $x \geq y \in Y$. Then

$$y = \phi(y) = \phi(y \wedge x) = \phi(y) \wedge \phi(x) \leq \phi(x).$$

Hence $Y \cap [x] \subseteq [\phi(x)]$. Using that Y is an ideal and that ϕ fixes the elements of Y we get

$$\underbrace{\phi(x)}_{\in Y} \wedge x = \phi(\phi(x) \wedge x) = \underbrace{\phi^2(x)}_{=\phi(x)} \wedge \phi(x) = \phi(x).$$

This implies $[\phi(x)] \subseteq Y \cap [x]$. We have shown that $Y \cap [z]$ is a principal ideal for all $z \in X$.

Conversely, suppose that $Y \cap [x] = [\hat{x}]$ is a principal ideal of X for every $x \in X$. Let $y \in Y, x \in X$. Trivially, y is contained in the ideal $Y \cap [y]$ and thus also $y \wedge x \in Y \cap [y]$. In particular $y \wedge x \in Y$ for every $x \in X$ and so Y is an ideal. It remains to demonstrate that it is a retract as well. By assumption $Y \cap [x] = [\hat{x}]$ for an element \hat{x} in Y . This defines a mapping

$$\begin{aligned} \phi: X &\rightarrow Y \\ x &\mapsto \hat{x}. \end{aligned}$$

Using the fact that Y is an ideal we have for $x \in Y$ that $Y \cap [x] = [x]$ which implies $x = \hat{x}$ in this case. Consequently, the mapping ϕ lets the elements of Y fixed. Next we verify that ϕ is a morphism. We need to show $\hat{x} \wedge \hat{y} = \widehat{x \wedge y}$. Let $x, y \in X$. Obviously we have $a \leq b \implies \hat{a} \leq \hat{b}$ for all $a, b \in X$. Hence $\widehat{x \wedge y} \leq \hat{x}, \hat{y}$ and thus $\widehat{x \wedge y} \leq \hat{x} \wedge \hat{y}$. To show the reverse direction suppose that $z \leq \hat{x}, \hat{y}$ for some z . It follows that $z \in Y \cap [x]$ and $z \in Y \cap [y]$. We conclude that $z \leq x, y$ and thus $z \leq x \wedge y$ with $z \in Y$. But $\widehat{x \wedge y}$ is the largest element in Y such that $\widehat{x \wedge y} \leq x \wedge y$. It follows that $z \leq \widehat{x \wedge y}$. This completes the proof.

□

Proposition 7.2.4 ([7], Proposition VII.5.5) *The following conditions on a P -semigroup $S = \mathcal{P}(G, X, Y)$ are equivalent:*

- (i) HH^{-1} is a retract ideal of E_S for all σ -classes H .
- (ii) $gY \cap Y$ is a retract ideal of Y for all $g \in G$.
- (iii) Let H be a σ -class and let $e = (\alpha, 1) \in E_S$. Then HeH^{-1} is a principal ideal of E_S .
- (iv) $[\alpha] \cap Y$ is a principal ideal of Y for all $\alpha \in X$.
- (v) X is a semilattice and Y is a retract ideal of X .

Proof

(i) \iff (ii): Recall that by 7.2.2(ii),

$$HH^{-1} = \{(\beta, 1) : \beta \in Y, g^{-1}\beta \in Y\} = \{(\beta, 1) : \beta \in gY \cap Y\}.$$

The equivalence of (i) and (ii) then follows from 6.2.8(i).

(iii) \Rightarrow (iv): Let $\alpha \in X$. Since $GY = X$ there exist g in G and $\kappa \in Y$ such that $g\kappa = \alpha$. Denote by H the σ -class corresponding to g and let $e = (\kappa, 1)$. By assumption HeH^{-1} is a principal ideal of E_S and hence, using 6.2.8(ii),

$$HeH^{-1} \stackrel{7.2.2(iii)}{=} \{(\beta, 1) : \beta \in [g\kappa] \cap Y\} = \{(\beta, 1) : \beta \in [\alpha] \cap Y\} = \{(\beta, 1) : \beta \leq \gamma\}$$

for some $\gamma \in Y$. We deduce that $[\alpha] \cap Y$ is a principal ideal of Y .

(iv) \Rightarrow (iii): Let $g \in G$ and $\alpha \in Y$. Put $e := (g^{-1}\alpha, 1) \in E_S$. Let further H be the σ -class corresponding to g . Then by 7.2.2(iii) we get

$$HeH^{-1} = \{(\beta, 1) : \beta \in [gg^{-1}\alpha] \cap Y\} = \{(\beta, 1) : \beta \in [\alpha] \cap Y\}.$$

But $[\alpha] \cap Y = [\gamma]$ by assumption for some $\gamma \in Y$. Hence

$$HeH^{-1} = \{(\beta, 1) : \beta \leq \gamma\} = [(\gamma, 1)].$$

(ii) \Rightarrow (iv): Since $GY = X$ it suffices to show that $[g\alpha] \cap Y$ is a principal ideal of Y for all $g \in G, \alpha \in Y$. In view of the fact that Y is an order ideal we have $[\alpha] \cap Y = [\alpha]$. Using 7.2.3 we thus get

$$g^{-1}Y \cap [\alpha] = (g^{-1}Y \cap Y) \cap [\alpha] = [\beta]$$

for some $\beta \in Y$ since the expression in the brackets is a retract ideal by assumption. Applying 2.4.4 two times we get

$$[g\alpha] \cap Y = g[\alpha] \cap Y = g([\alpha] \cap g^{-1}Y) = g[\beta] = [g\beta],$$

where $g\beta \in Y$. Thus $[g\alpha] \cap Y$ is indeed a principal ideal of Y .

(iv) \Rightarrow (v): We start by showing that X is a semilattice. By $GY = X$ it suffices to verify that $g\alpha \wedge h\beta$ exists for all $g, h \in G, \alpha, \beta \in Y$. By assumption

$$[h^{-1}g\alpha] \cap \underbrace{[\beta]}_{\subseteq Y} = ([h^{-1}g\alpha] \cap Y) \cap [\beta] = [\gamma] \cap [\beta]$$

for some $\gamma \in Y$. But γ and β are both elements of the semilattice Y and thus $[\gamma] \cap [\beta] = [\gamma \wedge \beta]$ by 2.1.8. Consequently $\gamma \wedge \beta$ is the greatest lower bound of $h^{-1}g\alpha$ and β . Since h acts on X as an order automorphism this implies

$$h(\gamma \wedge \beta) = h(h^{-1}g\alpha \wedge \beta) = g\alpha \wedge h\beta$$

and thus X is a semilattice.

By assumption $[\alpha] \cap Y$ is a principal ideal of Y and thus also of X for all $\alpha \in X$. By 7.2.3(ii) we deduce that Y is a retract ideal of X .

(v) \Rightarrow (ii): Suppose that X is a semilattice and Y is a retract ideal of X . Let $\alpha \in Y, g \in G$. As above, $[\alpha] \cap Y = [\alpha]$. Using 7.2.3 we have by assumption,

$$[\alpha] \cap (gY \cap Y) = [\alpha] \cap gY \stackrel{2.4.4}{=} g([\alpha] \cap Y) = g[\alpha] = [g\alpha],$$

for some $\beta \in Y$. Again by 7.2.3 we deduce that $gY \cap Y$ is a retract ideal of Y .

□

The previous proposition gives rise to a definition.

Definition 7.2.5 *An R-inverse semigroup is an inverse semigroup in which HH^{-1} is a retract ideal of E_S for every σ -class H .*

Remark 7.2.6 *In particular for an R-inverse semigroup S , HH^{-1} is an ideal. This is equivalent to S being an E-unitary semigroup by 6.0.21. Hence S can be represented as a P-semigroup. As a consequence R-inverse semigroups are precisely those appearing in 7.2.4. In naming these semigroups R-inverse we follow [7]. The ‘R’ most likely stems from ‘retract ideal’. We use R-inverse semigroups to tackle F-inverse semigroups.*

Next we give a construction method for R-inverse semigroups.

Construction 7.2.7 ([7], Construction VII.5.7) Let G be a group and Y a semi-lattice. Let $(g, \alpha) \mapsto g \cdot \alpha$ be a mapping from $G \times Y$ to Y . For reasons of convenience we shall write $g\alpha$ instead of $g \cdot \alpha$ if no misunderstandings are to be expected. Assume that the following axioms are satisfied for all $g, h \in G, \alpha, \beta \in Y$.

$$(R1) \quad 1\alpha = \alpha,$$

$$(R2) \quad g(\alpha \wedge \beta) = g\alpha \wedge g\beta,$$

$$(R3) \quad \lfloor g(h\alpha) \rfloor = \lfloor (gh)\alpha \rfloor \cap gY.$$

Let

$$S := \{(\alpha, g) \in Y \times G : \alpha \in gY\}$$

with multiplication given by

$$(\alpha, g)(\beta, h) = (\alpha \wedge g\beta, gh). \tag{7.2.1}$$

We shall denote this structure by $R(G, Y)$.

Remark 7.2.8 (i) *A little sloppy we call a mapping obeying axioms (R1) to (R3) a ‘group action of G on Y ’ although it is none in the sense of our earlier definition (compare to 2.4.2).*

(ii) *The first and the second axiom seem very reasonable. The latter demands that the action of G on Y respects the additional structure that the semi-lattice Y carries compared to the set Y . However, at this point the third axiom probably comes as a surprise. It is ‘to blame’ that we lack a group action in the rigid sense.*

(iii) *One might have expected $g^{-1}\alpha \in Y$ instead of $\alpha \in gY$ in the definition of S as has been the case in (6.2.1). There, this is equivalent to the condition $\alpha \in gY$ which means that we are not doing something completely new here.*

As promised above the semigroups thus built are indeed *R*-inverse. This is the content of the next proposition. First we need two lemmas.

Lemma 7.2.9 *Let $S = R(G, Y)$. Then gY is an order ideal of Y for all g in G .*

Proof Let $g \in G, \alpha \in Y$. Using (R1) and (R3) we have

$$\lfloor g\alpha \rfloor = \lfloor g(1\alpha) \rfloor = \lfloor (g1)\alpha \rfloor \cap gY = \lfloor g\alpha \rfloor \cap gY.$$

Hence all elements less than an element in gY are in gY which is therefore an order ideal. □

Lemma 2.4.4 remains true for a group action in the sense of 7.2.7. This is the content of the next lemma.

Lemma 7.2.10 *In $R(G, Y)$ we have the formula*

$$\forall g \in G \forall \beta \in Y: g\lfloor \beta \rfloor = \lfloor g\beta \rfloor. \quad (7.2.2)$$

Proof Let $\alpha \in g\lfloor \beta \rfloor$. Then $\alpha = g\gamma$ for some $\gamma \leq \beta$. Hence, by (R2),

$$\alpha = g\gamma = g(\gamma \wedge \beta) = g\gamma \wedge g\beta \leq \lfloor g\beta \rfloor$$

and thus $g\lfloor \beta \rfloor \subseteq \lfloor g\beta \rfloor$.

Conversely, let $\alpha \in \lfloor g\beta \rfloor$. Then $\alpha = g\gamma$ for some $\gamma \in Y$ since, as we saw in 7.2.9, $\lfloor g\beta \rfloor \subseteq gY$. Hence, by (R2),

$$\alpha = g\gamma = g\beta \wedge g\gamma = g(\beta \wedge \gamma) \in g\lfloor \beta \rfloor.$$

□

Proposition 7.2.11 ([7], Lemma VII.5.8) *The structure $R(G, Y)$ is an *R*-inverse semigroup.*

Proof Let $S = R(G, Y)$. We begin by showing that S is closed under the operation defined in (7.2.1). Let $(\alpha, g), (\beta, h) \in S$. Then $\alpha \in Y \cap gY, \beta \in Y \cap hY$. Thus $\beta = h\gamma$ for some $\gamma \in Y$. We have to show that $\alpha \wedge g\beta \in Y \cap (gh)Y$. This is established by the following chain of equalities where we use (R3) and the fact that $(gh)Y$ is an order ideal.

$$\lfloor \alpha \wedge g\beta \rfloor \stackrel{2.1.8}{=} \lfloor \alpha \rfloor \cap \lfloor g(h\gamma) \rfloor = \lfloor \alpha \rfloor \cap \lfloor (gh)\gamma \rfloor \cap gY \subseteq \lfloor \alpha \rfloor \cap \lfloor (gh)\gamma \rfloor \subseteq Y \cap (gh)Y.$$

Next we verify associativity. On the one hand

$$((\alpha, g)(\beta, h))(\gamma, k) = (\alpha \wedge g\beta, gh)(\gamma, k) = (\alpha \wedge g\beta \wedge (gh)\gamma, ghk),$$

whereas on the other hand

$$(\alpha, g) ((\beta, h)(\gamma, k)) = (\alpha, g)(\beta \wedge h\gamma, hk) = (\alpha \wedge g(\beta \wedge h\gamma), ghk) = (\alpha \wedge g\beta \wedge g(h\gamma), ghk).$$

Therefore, we have proved associativity if we can show $\alpha \wedge g\beta \wedge (gh)\gamma = \alpha \wedge g\beta \wedge g(h\gamma)$. Now (7.2.2) enters the scene enabling us to obtain the desired

$$\begin{aligned} [\alpha \wedge g\beta \wedge (gh)\gamma] &\stackrel{2.1.8}{=} [\alpha] \cap [g\beta] \cap [(gh)\gamma] = [\alpha] \cap g[\beta] \cap gY \cap [(gh)\gamma] = \\ &[\alpha] \cap [g\beta] \cap [g(h\gamma)] = [\alpha \wedge g\beta \wedge g(h\gamma)]. \end{aligned}$$

Hence S is a semigroup. We would like to show that it is also inverse and do this by showing that it is regular and that idempotents commute. Let $(\alpha, g) \in S$. By definition $\alpha \in Y$ and consequently $g^{-1}\alpha \in g^{-1}Y$. Hence $(g^{-1}\alpha, g^{-1}) \in S$. Then $(\alpha, g)(g^{-1}\alpha, g^{-1})(\alpha, g)$ computes to $(\alpha \wedge g(g^{-1}\alpha), g)$, where we would like to cancel out but are not allowed to. We straighten this out with

$$[\alpha \wedge g(g^{-1}\alpha)] = [\alpha] \cap [g(g^{-1}\alpha)] = [\alpha] \cap [(gg^{-1})\alpha] \cap gY = [\alpha], \quad (7.2.3)$$

where $\alpha \in gY$ by the definition of S . It follows that S is regular. The idempotents in S are easily found:

$$(\beta, g) = (\beta, g)(\beta, g) = (\beta \wedge g\beta, g^2)$$

yields that $g = 1$ and conversely every such element is an idempotent by (R1). Thus

$$E_S = \{(\alpha, 1) : \alpha \in Y\}.$$

Obviously then

$$(\alpha, 1)(\beta, 1) = (\alpha \wedge \beta, 1) = (\beta \wedge \alpha, 1) = (\beta, 1)(\alpha, 1).$$

It is also not difficult to see that S is E-unitary since $(\alpha, 1) \leq (\beta, g)$ implies $(\alpha, 1) = (\gamma, 1)(\beta, g) = (\gamma \wedge \beta, g)$ for some $\gamma \in Y$ and thus $(\beta, g) = (\beta, 1)$ is an idempotent. Moreover, it is straightforward to check that

$$(\alpha, g) \leq (\beta, h) \iff g = h, \alpha \leq \beta \quad (7.2.4)$$

and this entails

$$(\alpha, g) \sigma (\beta, h) \iff g = h, \quad (7.2.5)$$

exactly like for P-semigroups.

Finally we can show that $S = R(G, Y)$ is R-inverse. Let H be the σ -class corresponding to $g \in G$. Then by 7.2.1(ii) and (7.2.3) we obtain

$$\begin{aligned} HH^{-1} &= \{(\alpha, g)(g^{-1}\alpha, g^{-1}) : \alpha \in gY\} = \{(\alpha \wedge g(g^{-1}\alpha), 1) : \alpha \in gY\} = \\ &\{(\alpha, 1) : \alpha \in gY\}. \end{aligned}$$

In order to show that HH^{-1} is a retract ideal, by 7.2.3 it suffices to check that $HH^{-1} \cap \llbracket(\gamma, 1)\rrbracket$ is a primitive ideal for all idempotents $(\gamma, 1)$. Now, due to (7.2.4), we have

$$\begin{aligned} HH^{-1} \cap \llbracket(\gamma, 1)\rrbracket &= \{(\alpha, 1) : \alpha \in \llbracket\gamma\rrbracket \cap gY\} = \{(\alpha, 1) : \alpha \in \llbracket(gg^{-1})\gamma\rrbracket \cap gY\} = \\ &= \{(\alpha, 1) : \alpha \in \llbracket g(g^{-1}\gamma)\rrbracket\} = \llbracket(g(g^{-1}\gamma))\rrbracket \end{aligned}$$

and so S is an R -inverse semigroup. □

The construction in 7.2.7 would be merely a nice example lest there was something more to it. Indeed, as with the Rees theorem, the converse is true here as well.

Theorem 7.2.12 ([7], Lemma VII.5.10) *Let a group G act on a semilattice Y in the sense of 7.2.7. Then $R(Y, G)$ is an R -inverse semigroup. Conversely, every R -inverse semigroup is isomorphic to some $R(Y, G)$.*

Proof We have already established the direct half of the assertion. Conversely, let S be an R -inverse semigroup. It is then an E -unitary inverse semigroup as remarked in 7.2.6. Hence S is isomorphic to some P -semigroup $\mathcal{P}(G, X, Y)$. We are going to show that $\mathcal{P}(G, X, Y) = R(G, Y)$, where the action \cdot of G on the semilattice Y is given by

$$\llbracket g \cdot \alpha \rrbracket = \llbracket g\alpha \rrbracket \cap Y.$$

We distinguish here $g \cdot \alpha$ from $g\alpha$. As always, juxtaposition denotes the action of G on X stemming from the P -semigroup $\mathcal{P}(G, X, Y)$. By 7.2.4(v) we have that X is a semilattice and Y is a retract ideal of X . Hence the definition of \cdot is well-defined by virtue of 7.2.3. We now verify the axioms (R1) to (R3). First

$$\llbracket 1 \cdot \alpha \rrbracket = \llbracket 1\alpha \rrbracket \cap Y = \llbracket \alpha \rrbracket,$$

since $\alpha \in Y$ and Y is an order ideal. This implies $1 \cdot \alpha = \alpha$ for all $\alpha \in Y$, i.e. (R1).

Secondly,

$$\begin{aligned} \llbracket g \cdot (\alpha \wedge \beta) \rrbracket &= \llbracket g(\alpha \wedge \beta) \rrbracket \cap Y = \llbracket g\alpha \wedge g\beta \rrbracket \cap Y = \llbracket g\alpha \rrbracket \cap \llbracket g\beta \rrbracket \cap Y = \\ &= (\llbracket g\alpha \rrbracket \cap Y) \cap (\llbracket g\beta \rrbracket \cap Y) = \llbracket g \cdot \alpha \rrbracket \cap \llbracket g \cdot \beta \rrbracket = \llbracket g \cdot \alpha \wedge g \cdot \beta \rrbracket. \end{aligned}$$

It follows that $g \cdot (\alpha \wedge \beta) = g \cdot \alpha \wedge g \cdot \beta$ for all $g \in G$, $\alpha, \beta \in Y$. This is (R2).

Finally we have for all $g, h \in G$, $\alpha \in Y$,

$$\begin{aligned} \llbracket g \cdot (h \cdot \alpha) \rrbracket &= \llbracket g(h \cdot \alpha) \rrbracket \cap Y \stackrel{2.4.4}{=} g \llbracket h \cdot \alpha \rrbracket \cap Y = g(\llbracket h\alpha \rrbracket \cap Y) \cap Y = \\ &= g \llbracket h\alpha \rrbracket \cap gY \cap Y = (\llbracket g(h\alpha) \rrbracket \cap Y) \cap (gY \cap Y) = \\ &= (\llbracket (gh)\alpha \rrbracket \cap Y) \cap (gY \cap Y) = \llbracket (gh) \cdot \alpha \rrbracket \cap g \cdot Y. \end{aligned}$$

This is (R3).

Moreover,

$$(\alpha, g) \in S \iff \alpha \in Y, g^{-1}\alpha \in Y \iff \alpha \in Y \cap gY \iff \alpha \in g \cdot Y$$

and thus $S = R(G, Y)$ as sets. It remains to show that they have the same multiplication as well:

$$\begin{aligned} [\alpha \wedge g\beta] &= [\alpha] \cap [g\beta] = ([\alpha] \cap Y) \cap [g\beta] = [\alpha] \cap ([g\beta] \cap Y) = \\ &= [\alpha] \cap [g \cdot \beta] = [\alpha \wedge g \cdot \beta]. \end{aligned}$$

□

Finally we can move on to F-inverse semigroups.

Proposition 7.2.13 ([7], Proposition VII.5.11) *The following conditions on $S = \mathcal{P}(G, X, Y)$ are equivalent:*

- (i) HH^{-1} is a principal ideal of E_S for all σ -classes H .
- (ii) $gY \cap Y$ is a principal ideal of Y for all $g \in G$.
- (iii) X is a semilattice and Y is a principal ideal of X .

Proof

(i) \iff (ii): The proof of this is similar to the one in 7.2.4. We repeat the essential part: by 7.2.2(ii) we have

$$HH^{-1} = \{(\beta, 1) : \beta \in Y, g^{-1}\beta \in Y\} = \{(\beta, 1) : \beta \in Y \cap gY\}.$$

The equivalence then follows from 6.2.8(i).

(ii) \implies (iii): From the assumption it follows by 7.2.3(i) that $gY \cap Y$ is a retract ideal of Y . We deduce from 7.2.4 that X is a semilattice. For $g = 1$ we obtain that $Y = 1Y \cap Y$ is a principal ideal of Y and hence also of X since Y is an order ideal.

(iii) \implies (ii): By hypothesis Y has a greatest element ε . In a semilattice this means that ε is the identity element. For $g \in G$ we get

$$gY \cap Y = g[\varepsilon] \cap [\varepsilon] = [g\varepsilon] \cap [\varepsilon] = [g\varepsilon \wedge \varepsilon].$$

Hence $gY \cap Y$ is a principal ideal of Y for all $g \in G$.

□

The conditions in the previous proposition will turn out to be necessary and sufficient for a semigroup to be F-inverse. As for R-inverse semigroups we will give a constructive method for building all such semigroups soon.

Proposition 7.2.14 ([7], Proposition VII.5.14) *F-inverse semigroups are precisely those appearing in 7.2.13.*

Proof Let S be an *F*-inverse semigroup. We already know from 6.1.14 that S is *E*-unitary and thus isomorphic to some *P*-semigroup $\mathcal{P}(G, X, Y)$. Let H be a σ -class. By definition H has a greatest element, h say. Then

$$\begin{aligned} HH^{-1} &= [h][h]^{-1} = (E_S h)(E_S h)^{-1} = (E_S h)(h^{-1}E_S) = E_S(hh^{-1})E_S = \\ &E_S E_S h h^{-1} = E_S h h^{-1} = [hh^{-1}] \end{aligned}$$

which means that HH^{-1} is a principal ideal of E_S . This is 7.2.13(i).

For the converse let $S = \mathcal{P}(G, X, Y)$ be a *P*-semigroup and let H be a σ -class. Suppose that condition 7.2.13(i) is satisfied, i.e. $HH^{-1} = [e]$ for some $e \in E_S$. We have $H^{-1}H \subseteq E_S$ as can be shown analogously to 7.2.1(ii) and so for every $a \in H$: $h^{-1}a \in H^{-1}H \subseteq E_S$. Also by 7.2.1(ii) we can find an element $h \in H$ such that $e = hh^{-1}$. Thus, by assumption, $aa^{-1} \leq e = hh^{-1}$. It follows

$$a = (aa^{-1})a \leq (hh^{-1})a = h \underbrace{h^{-1}a}_{\in E_S} \leq h.$$

Hence h is the greatest element of H and S is *F*-inverse. □

Remark 7.2.15 *By virtue of 7.2.13(i) and 7.2.1(ii) the set $\{tt^{-1} : t \in \sigma(s)\}$ contains a maximum element for every s in an *F*-inverse semigroup S . We have here a special case of a so-called Billhardt congruence, that is a congruence ρ on an inverse semigroup S such that for each $s \in S$ the set $\{tt^{-1} : t \in \rho(s)\}$ contains a maximum element. Billhardt congruences can be used to build a more general theory of ω -inverse semigroups than we have done in chapter 5. The key idea is the bicyclic semigroup.*

After we have established several equivalent characterisations of *F*-inverse semigroups we give a general construction method for them.

Construction 7.2.16 ([7], Construction VII.5.15) Let Y in 7.2.7 have an identity element ε . Clearly then $[\varepsilon] = Y$ and so $[g\varepsilon] = g[\varepsilon] = gY$ by (7.2.2). This yields

$$\forall g, h \in G \forall \alpha \in Y: [g(h\alpha)] = [(gh)\alpha] \cap [g\varepsilon] = [(gh)\alpha \wedge g\varepsilon]$$

and we may thus rewrite (R3) as

$$(F3) \quad g(h\alpha) = (gh)\alpha \wedge g\varepsilon \quad (g, h \in G, \alpha \in Y).$$

Axioms (R1) and (R2) remain unchanged apart from the new names (F1) and (F2) we want to give them in this context. We denote the semigroup $R(G, Y)$ by $F(G, Y)$ in this case.

Proposition 7.2.17 *$F(G, Y)$ as defined above is an *F*-inverse semigroup.*

Proof By construction $F(G, Y)$ is an R-inverse semigroup. From (7.2.5) we know that there is a one-one correspondence between group elements in G and σ -classes in $F(G, Y)$. Let H be the σ -class corresponding to $g \in G$. Using 7.2.1(ii) we get

$$HH^{-1} = \{(\beta, g)(g^{-1}\beta, g^{-1}) : \beta \in Y \cap gY\} \stackrel{(7.2.3)}{=} \{(\beta, 1) : \beta \in Y \cap gY\}.$$

Now, $Y \cap gY = [\varepsilon] \cap g[\varepsilon] = [\varepsilon] \cap [g\varepsilon] = [\varepsilon \wedge g\varepsilon]$. Hence HH^{-1} is a principal ideal of $E_S = \{(\beta, 1) : \beta \in Y\}$. This implies that $F(G, Y)$ is F-inverse by 7.2.14. □

At last we have put together everything necessary for a quick proof of

Theorem 7.2.18 ([7], Lemma VII.5.16) *Let a group G act on a semilattice Y obeying the axioms (F1), (F2) and (F3). Then $F(G, Y)$ is an F-inverse semigroup. Conversely, every F-inverse semigroup is isomorphic to some $F(G, Y)$.*

Proof We have just shown the direct part. For the converse let S be an F-inverse semigroup. By 7.2.14, S satisfies the conditions of 7.2.13. In particular the semilattice Y has an identity element ε . Comparing this with 7.2.4 and using 7.2.3(i) we deduce that S is R-inverse. Hence S is isomorphic to some $R(G, Y)$. The observations in 7.2.16 complete the proof. □

8 Appendix

8.1 curriculum vitae

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